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ON THE REACHABLE SET OF A CLASS OF FRACTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. We consider a fractional differential inclusion defined by Caputo-Fabrizio fractional derivative and we prove that the reachable set of a certain variational inclusion is a derived cone in the sense of Hestenes to the reachable set of the fractional differential inclusion. This result allows to obtain a sufficient condition for local controllability along a reference trajectory.

1. Introduction

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order ([3, 9, 11, 12] etc.). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena. In the fractional calculus there are several fractional derivatives. From them, the fractional derivative introduced by Caputo in [4] allows to use Cauchy conditions which have physical meanings.

Recently, a new fractional order derivative with regular kernel has been introduced by Caputo and Fabrizio [5]. The Caputo-Fabrizio operator is useful for modeling several classes of problems with the dynamics having the exponential decay law. This new definition is able to describe better heterogeneousness, systems with different scales with memory effects, the wave movement on surface of shallow water, the heat transfer model, mass-spring-damper model ([13]) etc.. Another good property of this new definition is that using Laplace transform of the fractional derivative the fractional differential equation turns into a classical differential equation of integer order. Properties of this definition have been studied in [1, 5, 6, 13] etc.. Some recent papers are devoted to qualitative results for fractional differential equations defined by Caputo-Fabrizio fractional derivative [14, 15, 16] etc..

In this paper we study the following problem

$$D^{\sigma}_{CF}x(t) \in F(t,x(t)) \quad a.e.\ ([0,T]), \quad x(0) \in X_0, \quad x'(0) \in X_1, \tag{1.1}$$

where $F(.,.) : [0,T] \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map, D^{σ}_{CF} denotes Caputo-Fabrizio's fractional derivative of order $\sigma \in (1,2)$ and $X_0, X_1 \subset \mathbf{R}$ are closed sets.

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The aim of this paper is to prove that the reachable set of a certain variational fractional differential inclusion is a derived cone in the sense of Hestenes to the reachable set of the problem (1.1). In order to obtain the continuity property in the definition of a derived cone we shall use a continuous version of Filippov's theorem for solutions of fractional differential inclusions (1.1), recently obtained in [8]. As an application of our main result we obtain a sufficient condition for local controllability along a reference trajectory.

The notion of derived cone to an arbitrary subset of a normed space introduced by M.Hestenes in [10] and successfully used to obtain necessary optimality conditions in control theory; moreover, other properties of derived cones may be used to obtain controllability and other results in the qualitative theory of control systems. We note that a similar result for fractional differential inclusions defined by Caputo-Katugampola fractional derivative may be found in our previous paper [7].

The paper is organized as follows: in Section 2 we present the notations and the preliminary results to be used in the sequel and in Section 3 we provide our main results.

2. Preliminaries

In general the reachable set to a control system is, generally, neither a differentiable manifold, nor a convex set, its infinitesimal properties may be characterized only by tangent cones in a generalized sense, extending the classical concepts of tangent cones in differential geometry and convex analysis, respectively.

Definition 2.1. ([10]) A subset $D \subset \mathbf{R}^n$ is said to be a *derived set to* $X \subset \mathbf{R}^n$ at $x \in X$ if for any finite subset $\{w_1, ..., w_k\} \subset D$, there exist $s_0 > 0$ and a continuous mapping $\alpha(.) : [0, s_0]^k \to X$ such that $\alpha(0) = x$ and $\alpha(.)$ is (conically) differentiable at $s = 0$ with the derivative col$[w_1, ..., w_k]$ in the sense that

$$\lim_{R_+^k \ni \theta \to 0} \frac{||\alpha(\theta) - \alpha(0) - \sum_{i=1}^k \theta_i w_i||}{||\theta||} = 0.$$

We shall write in this case that the derivative of $\alpha(.)$ at $s = 0$ is given by

$$D\alpha(0)\theta = \sum_{i=1}^k \theta_j w_j \quad \forall \theta = (\theta_1, ..., \theta_k) \in \mathbf{R}_+^k := [0, \infty)^k.$$

A subset $C \subset \mathbf{R}^n$ is said to be a *derived cone* of X at x if it is a derived set and also a convex cone.

For the basic properties of derived sets and cones we refer to M.Hestenes [10]; we recall that if D is a derived set then $D \bigcup \{0\}$ as well as the convex cone generated by D, defined by

$$cco(D) = \{\sum_{i=1}^k \lambda_j w_j; \quad \lambda_j \geq 0, \, k \in N, \, w_j \in D, \, j = 1, ..., k\}$$

is also a derived set, hence a derived cone.

The fact that the derived cone is a proper generalization of the classical concepts in differential geometry and convex analysis is illustrated by the following results ([10]): if $X \subset \mathbf{R}^n$ is a differentiable manifold and $T_x X$ is the tangent space in the sense of differential geometry to X at x

$$T_x X = \{w \in \mathbf{R}^n; \, \exists \, c : (-s, s) \to \mathbf{R}^n, \text{ of class } C^1, c(0) = x, c'(0) = w\},$$

then T_xX is a derived cone; also, if $X \subset \mathbf{R}^n$ is a convex subset then the tangent cone in the sense of convex analysis defined by

$$TC_xX = cl\{t(y-x); \quad t \geq 0,\ y \in X\}$$

is also a derived cone. Since any convex subcone of a derived cone is also a derived cone, such an object may not be uniquely associated to a point $x \in X$; moreover, simple examples show that even a maximal with respect to set-inclusion derived cone may not be uniquely defined: if the set $X \subset \mathbf{R}^2$ is defined by

$$X = C_1 \bigcup C_2,\ C_1 = \{(x,x); x \geq 0\},\ C_2 = \{(x,-x), x \leq 0\},$$

then C_1 and C_2 are both maximal derived cones of X at the point $(0,0) \in X$.

At the same time, the up-to-date experience in nonsmooth analysis shows that for some problems, the use of one of the intrinsic tangent cones may be preferable. The most known intrinsic tangent cones in the literature (e.g. [2]) are the contingent, the quasitangent (intermediate) and Clarke's tangent cones, defined, respectively, by

$$\begin{aligned}
&K_xX = \{v \in X; \quad \exists\, s_m \to 0+, \quad \exists\, x_m \to x,\ x_m \in X : \tfrac{x_m - x}{s_m} \to v\},\\
&Q_xX = \{v \in X; \quad \forall s_m \to 0+, \quad \exists\, x_m \to x,\ x_m \in X : \tfrac{x_m - x}{s_m} \to v\},\\
&C_xX = \{v \in X; \quad \forall\, (x_m, s_m) \to (x, 0+), \quad x_m \in X, \exists\, y_m \in X : \ \tfrac{y_m - x_m}{s_m} \to v\}
\end{aligned}$$

The next property of derived cone, obtained by Hestenes ([10], Theorem 4.7.4) and stated in the next lemma is essential in the proof of our main result.

Lemma 2.2. ([10]) *Let $X \subset \mathbf{R}^n$. Then $x \in int(X)$ if and only if $C = \mathbf{R}^n$ is a derived cone at $x \in X$ to X.*

Corresponding to each type of tangent cone, say τ_xX one may introduce (e.g., [2]) a *set-valued directional derivative* of a multifunction $G(.) : X \subset \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ (in particular of a single-valued mapping) at a point $(x,y) \in Graph(G)$ as follows

$$\tau_yG(x;v) = \{w \in \mathbf{R}^n; \quad (v,w) \in \tau_{(x,y)}Graph(G)\}, \quad v \in \tau_xE.$$

We recall that a set-valued map, $A(.) : \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ is said to be a *convex* (respectively, closed convex) *process* if $Graph(A(.)) \subset \mathbf{R}^n \times \mathbf{R}^n$ is a convex (respectively, closed convex) cone. For the basic properties of convex processes we refer to [2], but we shall use here only the above definition.

Let $T > 0$, $I := [0,T]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Denote by $\mathcal{P}(\mathbf{R})$ the family of all nonempty subsets of $\mathbf{R}$ and by $\mathcal{B}(\mathbf{R})$ the family of all Borel subsets of $\mathbf{R}$.

As usual, we denote by $C(I,\mathbf{R})$ the Banach space of all continuous functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x(.)|_C = \sup_{t\in I}|x(t)|$ and by $L^1(I,\mathbf{R})$ the Banach space of all (Bochner) integrable functions $x(.) : I \to \mathbf{R}$ endowed with the norm $|x(.)|_1 = \int_0^T |x(t)|\mathrm{d}t$.

In [5] the following notions were introduced.

Definition 2.3. a) Caputo-Fabrizio integral of order $\alpha \in (0,1)$ of a function $f \in AC_{loc}([0,\infty),\mathbf{R})$ (which means that $f'(.)$ is integrable on $[0,T]$ for any $T > 0$) is defined by

$$I^\alpha_{CF}f(t) = (1-\alpha)f(t) + \alpha\int_0^t f(s)ds.$$

b) Caputo-Fabrizio fractional derivative of order $\alpha \in (0,1)$ of f is defined for $t \geq 0$ by

$$D^{\alpha}_{CF} f(t) = \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f'(s) ds.$$

c) Caputo-Fabrizio fractional derivative of order $\sigma = \alpha + n$, $\alpha \in (0,1)$ $n \in \mathbf{N}$ of f is defined by

$$D^{\sigma}_{CF} f(t) = D^{\alpha}_{CF}(D^{n}_{CF} f(t)).$$

In particular, if $\sigma = \alpha + 1$, $\alpha \in (0,1)$ $D^{\sigma}_{CF} f(t) = \frac{1}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f''(s) ds.$

Definition 2.4. A mapping $x(.) \in AC(I, \mathbf{R})$ is called a *solution* of problem (1.1) if there exists a function $f(.) \in L^1(I, \mathbf{R})$ such that $f(t) \in F(t, x(t))$ *a.e.* (I), $D^{\alpha}_{CF} x(t) = f(t)$, $t \in I$ and $x(0) = x_0 \in X_0, x'(0) = x_1 \in X_1$.

In this case we say that $(x(.), v(.))$ is a *trajectory-selection pair* of (1.1).

Hypothesis 2.5. (i) *$F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ measurable.*

(ii) *There exists $L(.) \in L^1(I, (0, \infty))$ such that, for almost all $t \in I, F(t,.)$ is $L(t)$-Lipschitz in the sense that*

$$d_H(F(t,x), F(t,y)) \leq L(t)|x - y| \quad \forall x, y \in \mathbf{R},$$

where $d_H(.,.)$ is the Hausdorff distance

$$d(A,B) = \max\{d^*(A,B), d^*(B,A)\}, d^*(A,B) = \sup\{d(a,B); a \in A\}.$$

Hypothesis 2.6. i) S is a separable metric space and $a(.), b(.) : S \to \mathbf{R}$, $c(.) : S \to (0, \infty)$ are continuous mappings.

ii) There exists the continuous mappings $y(.) : S \to AC(I, \mathbf{R})$ and $p(.) : S \to \mathbf{R}$ such that

$$d(D(y(s))^{\sigma}_{CF}(t), F(t, y(s)(t)) \leq p(s)(t) \quad a.e.\ (I),\ \forall\, s \in S.$$

We use next the notations

$$\begin{aligned} &\xi(s)(t) = Me^{Mm(t)}[tc(s) + |a(s) - y(s)(0)| + T|b(s) - (y(s))'(0)|] \\ &+ \int_0^t p(s)(u) e^{M(m(t)-m(u))} du, \quad m(t) = \int_0^t L(s) ds. \end{aligned}$$

The main tool in the study of reachable sets of our fractional differential inclusion is a certain version of Filippov's theorem for fractional differential inclusion (1.1) in [8].

Theorem 2.7. ([8]) *Assume that Hypotheses 2.5 and 2.6 are satisfied.*

Then there exist a continuous mapping $x(.) : S \to C(I, \mathbf{R})$, such that for any $s \in S$, $x(s)(.)$ is a solution of problem

$$D^{\sigma}_{CF} z(t) \in F(t, z(t)), \quad x(0) = a(s), \quad x'(0) = b(s)$$

and

$$|x(s)(t) - y(s)(t)| \leq \xi(s)(t) \quad \forall (t,s) \in I \times S.$$

3. The main results

We study next the reachable set of (1.1) defined by

$$R_F(T,X_0,X_1):=\{x(T);\quad x(.) \text{ is a solution of } (1.1)\}.$$

We consider a certain variational fractional differential inclusion and we shall prove that the reachable set of this variational inclusion from derived cones $C_0\subset \mathbf{R}$ to X_0 and $C_1\subset \mathbf{R}$ to X_1 at time T is a derived cone to the reachable set $R_F(T,X_0,X_1)$.

Throughout in this section we assume the folowwing hypotheses.

Hypothesis 3.1. i) *Hypothesis 2.5 is satisfied and $X_0,X_1\subset \mathbf{R}$ are closed sets.*

ii) $(z(.),f(.))\in AC(I,\mathbf{R})\times L^1(I,\mathbf{R})$ *is a trajectory-selection pair of (1.1) and a family $A(t,.):\mathbf{R}\to \mathcal{P}(\mathbf{R})$, $t\in I$ of convex processes satisfying the condition*

$$A(t,u)\subset Q_{f(t)}F(t,.)(z(t);u)\quad \forall\, u\in dom(A(t,.)),\ a.e.\ t\in I \tag{3.1}$$

is assumed to be given and defines the variational inclusion

$$D^{\sigma}_{CF}w(t)\in A(t,w(t)). \tag{3.2}$$

Remark 3.2. We mention that for any set-valued map $F(.,.)$, one may find an infinite number of families of convex process $A(t,.)$, $t\in I$, satisfying condition (3.1); in fact any family of closed convex subcones of the quasitangent cones, $\overline{A}(t)\subset Q_{(z(t),f(t))}graph(F(t,.))$, defines the family of closed convex process

$$A(t,u)=\{v\in \mathbf{R};\ (u,v)\in \overline{A}(t)\},\quad u,v\in \mathbf{R},\ t\in I$$

that satisfy condition (3.1). For example, we may take an "intrinsic" family of such closed convex process; namely, Clarke's convex-valued directional derivatives $C_{f(t)}F(t,.)(z(t);.)$.

When $F(t,.)$ is assumed to be Lipschitz a.e. on I an alternative characterization of the quasitangent directional derivative is (e.g., [2])

$$Q_{f(t)}F(t,.)((z(t);u))=\{w\in \mathbf{R};\ \lim_{\theta\to 0+}\frac{1}{\theta}d(f(t)+\theta w,F(t,z(t)+\theta u))=0\}. \tag{3.3}$$

Theorem 3.3. *Assume that Hypothesis 3.1 is satisfied, let $C_0\subset \mathbf{R}$ be a derived cone to X_0 at $z(0)$ and $C_1\subset \mathbf{R}$ be a derived cone to X_1 at $z'(0)$. Then the reachable set $R_A(T,C_0,C_1)$ of (3.2) is a derived cone to $R_F(T,X_0,X_1)$ at $z(T)$.*

Proof. In view of Definition 2.1, let $\{w_1,...,w_m\}\subset R_A(T,C_0,C_1)$, hence such that there exist the trajectory-selection pairs $(v_1(.),g_1(.)),...,(v_m(.),g_m(.))$ of the variational inclusion (3.2) such that

$$v_j(T)=w_j,\quad v_j(0)\in C_0,\quad v_j'(0)\in C_1,\quad j=1,2,...,m \tag{3.4}$$

Since $C_0\subset \mathbf{R}$ is a derived cone to X_0 at $z(0)$ and $C_1\subset \mathbf{R}$ is a derived cone to X_1 at $z'(0)$, there exist the continuous mappings $\alpha_0:S=[0,\theta_0]^m\to X_0$, $\alpha_1:S\to X_1$ such that

$$\begin{aligned}&\alpha_0(0)=z(0),\quad D\alpha_0(0)s=\textstyle\sum_{j=1}^m s_jv_j(0)\quad \forall\, s\in \mathbf{R}^m_+,\\ &\alpha_1(0)=z'(0),\quad D\alpha_1(0)s=\textstyle\sum_{j=1}^m s_jv_j'(0)\quad \forall\, s\in \mathbf{R}^m_+.\end{aligned} \tag{3.5}$$

For any $s=(s_1,...,s_m)\in S$ and $t\in I$ we set

$$\begin{aligned}y(s)(t)&=z(t)+\textstyle\sum_{j=1}^m s_jv_j(t),\\ g(s)(t)&=f(t)+\textstyle\sum_{j=1}^m s_jg_j(t),\\ p(s)(t)&=d(g(s)(t),F(t,y(s)(t)))\end{aligned} \tag{3.6}$$

and prove that $y(.)$, $p(.)$ satisfy the hypothesis of Theorem 2.7.

From the lipschitzianity of $F(t,.,.)$ we have that for any $s\in S$, the measurable function $p(s)(.)$ in (3.6) it is also integrable.

$$p(s)(t)=d(g(s)(t),F(t,y(s)(t)))\leq \sum_{j=1}^m s_j|g_j(t)|+d_H(F(t,z(t)),\\ F(t,y(s)(t)))\leq \sum_{j=1}^m s_j|g_j(t)|+L(t)\sum_{j=1}^m s_j|v_j(t)|.$$

At the same time, the mapping $s\to p(s)(.)\in L^1(I,\mathbf{R})$ is Lipschitzian (and, in particular, continuous) since for any $s,s'\in S$ one may write

$$|p(s)(.)-p(s')(.)|_1=\int_0^T|p(s)(t)-p(s')(t)|dt\leq \int_0^T[|g(s)(t)-g(s')(t)|\\ +d_H(F(t,y(s)(t)),F(t,y(s')(t))))]dt\leq ||s-s'||(\sum_{j=1}^m\int_0^T[|g_j(t)|+\\ +L(t)|v_j(t)|]dt)$$

Define $S_1:=S\backslash\{(0,\ldots,0)\}$ and $c(.):S_1\to(0,\infty)$, $c(s):=||s||^2$. It follows from Theorem 2.7 the existence of a continuous function $x(.):S_1\to C(I,\mathbf{R})$ such that for any $s\in S_1$, $x(s)(.)$ is a solution of (1.1) with the property (2.1).

For $s=0$ we define $x(0)(t)=y(0)(t)=z(t)$ $\forall t\in I$. Obviously, $x(.):S\to C(I,\mathbf{R})$ is also continuous.

Finally, we define the function $\alpha(.):S\to R_F(T,X_0,X_1)$ by

$$\alpha(s)=x(s)(T)\quad \forall s\in S.$$

Obviously, $\alpha(.)$ is continuous on S and verifies $\alpha(0)=z(T)$.

In order to finish the proof we must show that $\alpha(.)$ is differentiable at $s_0=0\in S$ and its derivative is given by

$$D\alpha(0)(s)=\sum_{j=1}^m s_jw_j\quad \forall\, s\in\mathbf{R}_+^m$$

which is equivalent with the fact that:

$$\lim_{s\to 0}\frac{1}{||s||}(|\alpha(s)-\alpha(0)-\sum_{j=1}^m s_jw_j|)=0. \tag{3.7}$$

Taking into account (2.6) we obtain

$$\frac{1}{||s||}|\alpha(s)-\alpha(0)-\sum_{j=1}^m s_jw_j|\leq \frac{1}{||s||}|x(s)(T)-y(s)(T)|\leq\\ MTe^{Mm(T)}||s||+Me^{Mm(T)}\frac{1}{||s||}|\alpha_0(s)-z(0)-\sum_{j=1}^m s_jv_j(0)|+\\ MTe^{Mm(T)}\frac{1}{||s||}|\alpha_1(s)-z'(0)-\sum_{j=1}^m s_jv_j'(0)|+MTe^{Mm(T)}\int_0^T\frac{p(s)(u)}{||s||}du$$

and therefore in view of (3.5), relation (3.7) is implied by the following property of the mapping $p(.)$ in (3.6)

$$\lim_{s\to 0}\frac{p(s)(t)}{||s||}=0\quad a.e.\,(I). \tag{3.8}$$

In order to prove the last property we note since $A(t,.)$ is a convex process for any $s\in S$ one has

$$\sum_{j=1}^m\frac{s_j}{||s||}g_j(t)\in A(t,\sum_{j=1}^m\frac{s_j}{||s||}u_j(t))\subset Q_{f(t)}F(t,.)(z(t);\sum_{j=1}^m\frac{s_j}{||s||}u_j(t))\quad a.e.\,(I).$$

Therefore, by (3.3) we obtain

$$\lim_{h\to 0+}\frac{1}{h}d(f(t)+h\sum_{j=1}^{m}\frac{s_j}{||s||}g_j(t),F(t,z(t)+h\sum_{j=1}^{m}\frac{s_j}{||s||}v_j(t)))=0. \qquad (3.9)$$

Finally, in order to prove that (3.9) implies (3.8) we take the compact metric space $\Sigma_+^{m-1}=\{\sigma\in\mathbf{R}_+^m;||\sigma||=1\}$ and the real function $\psi_t(.,.):(0,\theta_0]\times\Sigma_+^{m-1}\to\mathbf{R}_+$ defined by

$$\psi_t(h,\sigma)=\frac{1}{h}d(f(t)+h\sum_{j=1}^{m}\sigma_j g_j(t),F(t,z(t)+h\sum_{j=1}^{m}\sigma_j v_j(t))), \qquad (3.10)$$

where $\sigma=(\sigma_1,...,\sigma_m)$ and which according to (3.9) has the property

$$\lim_{\theta\to 0+}\psi_t(\theta,\sigma)=0 \quad \forall\,\sigma\in\Sigma_+^{m-1}\ a.e.\,(I) \qquad (3.11)$$

Using the fact that $\psi_t(\theta,.)$ is Lipschitzian and the fact that Σ_+^{m-1} is a compact metric space, from (3.11) it follows easily that

$$\lim_{\theta\to 0+}\max_{\sigma\in\Sigma_+^{m-1}}\psi_t(\theta,\sigma)=0$$

which implies the fact that

$$\lim_{s\to 0}\psi_t(||s||,\frac{s}{||s||})=0 \quad a.e.\,(I)$$

and the proof is complete.

We apply now Theorem 3.3 in order to obtain a sufficient condition for local controllability of the fractional differential inclusion (1.1) along a reference trajectory, $z(.)$ at time T, in the sense that

$$z(T)\in Int(R_F(T,X_0,X_1)).$$

Theorem 3.4. *Let $z(.),F(.,.)$ and $A(.,.)$ satisfy Hypothesis 3.1, let $C_0\subset\mathbf{R}$ be a derived cone to X_0 at $z(0)$ and $C_1\subset\mathbf{R}$ be a derived cone to X_1 at $z'(0)$. If, the variational fractional differential inclusion in (3.2) is controllable at T in the sense that $R_A(T,C_0,C_1)=\mathbf{R}$, then the differential inclusion (1.1) is locally controllable along $z(.)$ at time T.*

Proof. The proof follows from Lemma 2.2 and Theorem 3.3.

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