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COMPOSITION PRINCIPLES FOR ALMOST PERIODIC TYPE FUNCTIONS AND APPLICATIONS

M. KOSTIĆ

Abstract. This paper introduces and investigates the classes of two-parameter uniformly recurrent functions, two-parameter $\odot_g$-almost periodic functions and their Stepanov generalizations. We work in the setting of complex Banach spaces, using the recently introduced notions of lower and upper (Banach) g-densities. We prove many composition principles for the introduced classes and apply our results in the analysis of the existence and uniqueness of solutions for various kinds of the abstract inhomogeneous fractional integro-differential inclusions.

1. Introduction

The notion of almost periodicity was first studied by H. Bohr around 1925 and later generalized by many other authors (cf. the research monographs [3], [8] and [19]-[20] for the basic theory of almost periodic functions). For some applications given, see the research monographs [6], [9]-[10], [14] and [24].

In our recent paper [18], we have systematically analyzed generalized uniformly recurrent functions and generalized $\odot_g$-almost periodic functions. This paper continues the analysis raised in [18] by introducing and investigating the classes of two-parameter (asymptotically) uniformly recurrent functions, two-parameter (asymptotically) $\odot_g$-almost periodic functions and their Stepanov generalizations. Several composition principles are established in this context, which enable one to provide certain applications to the abstract semilinear integro-differential Cauchy problems and inclusions.

The organization of paper is briefly described as follows. After recalling the basic facts about almost periodic type functions considered in the paper, we analyze the lower and upper (Banach) g-densities in Subsection 1.1. The main aim of Subsection 1.2 is to recollect the basic definitions and results from fractional calculus that we will use later on. The main structural results are proved in Section 2, where

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we investigate uniformly recurrent functions and $\odot_g$-almost periodic functions depending on two arguments and related composition principles. The final section of paper is reserved for applications of obtained theoretical results.

We use the standard notation throughout the paper. We will always assume that $(X, \|\cdot\|)$ is a complex Banach space. By $L^p_{loc}(I : X)$, $C(I : X)$, $C_b(I : X)$ and $C_0(I : X)$ we denote the vector spaces consisting of all p-locally integrable functions $f : I \to X$, all continuous functions $f : I \to X$, all bounded continuous functions $f : I \to X$ and all continuous functions $f : I \to X$ satisfying that $\lim_{|t|\to+\infty} \|f(t)\| = 0$, respectively $(1 \leq p < \infty)$. As is well known, $C_0(I : X)$ is a Banach space equipped with the sup-norm, denoted henceforth by $\|\cdot\|_\infty$. If Y is also a complex Banach space, then $L(X, Y)$ stands for the space of all continuous linear mappings from X into Y; $L(X) \equiv L(X, X)$.

Let $I = \mathbb{R}$ or $I = [0, \infty)$. Given $\epsilon > 0$, we call $\tau > 0$ an ϵ-period for $f(\cdot)$ if and only if

$$\|f(t + \tau) - f(t)\| \leq \epsilon, \quad t \in I. \tag{1.1}$$

The set constituted of all ϵ-periods for $f(\cdot)$ is denoted by $\vartheta(f, \epsilon)$. We say that a continuous function $f(\cdot)$ is almost periodic if and only if for each $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is relatively dense in $[0, \infty)$, which means that there exists $l > 0$ such that any subinterval of $[0, \infty)$ of length l meets $\vartheta(f, \epsilon)$. By $AP(I : X)$ we denote the vector space consisting of all almost periodic functions from I into X; accompanied with the sup-norm, $AP(I : X)$ becomes a Banach space. The function $f : I \to X$ is said to be asymptotically almost periodic if and only if there exist an almost periodic function $h : I \to X$ and a function $\phi \in C_0(I : X)$ such that $f(t) = h(t) + \phi(t)$ for all $t \in I$. This is equivalent to saying that, for every $\epsilon > 0$, we can find numbers $l > 0$ and $M > 0$ such that every subinterval of I' of length l contains, at least, one number τ such that $\|f(t + \tau) - f(t)\| \leq \epsilon$ provided $|t|,\ |t + \tau| \geq M$ ([25]).

The notion of recurrence plays an important role in the theory of topological dynamical systems (see the research monograph [5] by J. de Vries for more details on the subject). Following A. Haraux and P. Souplet [11], we say that a continuous function $f(\cdot)$ is uniformly recurrent if and only if there exists a strictly increasing sequence (α_n) of positive real numbers such that $\lim_{n\to+\infty} \alpha_n = +\infty$ and

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \big\|f(t + \alpha_n) - f(t)\big\| = 0. \tag{1.2}$$

It is well known that any almost periodic function is uniformly recurrent, while the converse statement is not true in general; any $\odot_g$-almost periodic function under our consideration is uniformly recurrent. Furthermore, $f(\cdot)$ is uniformly recurrent if and only if for each number $\epsilon > 0$ the set $\vartheta(f, \epsilon)$ is unbounded. The pointwise sums and products of bounded uniformly continuous, uniformly recurrent ($\odot_g$-almost periodic) functions need not be uniformly recurrent ($\odot_g$-almost periodic), in general.

Suppose that $p \in [1, \infty)$. Let us recall that a function $f \in L^p_{loc}(I : X)$ is called Stepanov p-bounded if and only if

$$\|f\|_{S^p} := \sup_{t\in I} \left(\int_t^{t+1} \|f(s)\|^p \, ds \right)^{1/p} < \infty.$$

Equipped with the above norm, the space $L^p_S(I : X)$ consisting of all Stepanov p-bounded functions is a Banach space. A function $f \in L^p_S(I : X)$ is said to be

Stepanov p-almost periodic if and only if the function $\hat{f} : I \to L^p([0,1] : X)$, defined by

$$\hat{f}(t)(s) := f(t+s), \quad t \in I, \; s \in [0,1], \tag{1.3}$$

is almost periodic. Furthermore, a function $f \in L_S^p(I : X)$ is called asymptotically Stepanov p-almost periodic if and only if there exist a Stepanov p-almost periodic function $g \in L_S^p(I : X)$ and a function $q \in L_S^p(I : X)$ such that $f(t) = g(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0,1] : X))$. It is well known that, if $1 \leq p \leq q < \infty$ and $f(\cdot)$ is (asymptotically) Stepanov q-almost periodic, then $f(\cdot)$ is (asymptotically) Stepanov p-almost periodic.

The concepts of (asymptotical) Stepanov p-uniform recurrence and (asymptotical) Stepanov $(p, \odot_g)$-almost periodicity are introduced as follows ([18]):

Definition 1.1. (i) Let $1 \leq p < \infty$. A function $f \in L_{loc}^p(I : X)$ is said to be Stepanov p-uniformly recurrent if and only if the function $\hat{f} : I \to L^p([0,1] : X)$, defined by (1.3), is uniformly recurrent.

(ii) Let $1 \leq p < \infty$. A function $f \in L_{loc}^p(I : X)$ is said to be Stepanov $(p, \odot_g)$-almost periodic if and only if the function $\hat{f} : I \to L^p([0,1] : X)$ is $\odot_g$-almost periodic.

Definition 1.2. (i) Let $1 \leq p < \infty$. A function $f \in L_{loc}^p(I : X)$ is said to be asymptotically Stepanov p-uniformly recurrent if and only if there exist a Stepanov p-uniformly recurrent function $h(\cdot)$ and a function $q \in L_S^p(I : X)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0,1] : X))$.

(ii) Let $1 \leq p < \infty$. A function $f \in L_{loc}^p(I : X)$ is said to be asymptotically Stepanov $(p, \odot_g)$-almost periodic if and only if there exist a Stepanov $(p, \odot_g)$-almost periodic function $h(\cdot)$ and a function $q \in L_S^p(I : X)$ such that $f(t) = h(t) + q(t)$, $t \in I$ and $\hat{q} \in C_0(I : L^p([0,1] : X))$.

We continue by observing that A. Haraux and P. Souplet have proved, in [11, Theorem 1.1], that there exists a function $f : \mathbb{R} \to \mathbb{R}$ which is uniformly continuous, uniformly recurrent and Besicovitch unbounded (cf. [14] for the notion). The function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(t) := \sum_{n=1}^{\infty} \frac{1}{n} \sin^2\Big(\frac{t}{2^n}\Big)\, dt, \quad t \in \mathbb{R}. \tag{1.4}$$

Further on, in [11, Theorem 1.2], A. Haraux and P. Souplet have proved that for each real number $c > 0$ the function $h(\cdot) = \min(c, f(\cdot))$, where $f(\cdot)$ is given by (1.4), is bounded uniformly continuous, uniformly recurrent and not asymptotically almost periodic. In [18], we have slightly improved this result by showing that the function $f(\cdot)$ is not asymptotically almost automorphic.

The function constructed by H. Bohr on pp. 113–115 of the first part of his landmark trilogy [4] is also bounded uniformly continuous, uniformly recurrent, not asymptotically (Stepanov) almost automorphic, and not (Stepanov) quasi-asymptotically almost periodic, as proved in [18]. Furthermore, in the same paper, we have revisted the bounded uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ used by J. de Vries in [5, point 6., p. 208]. As shown in [18], this function can serve as a much simpler example of a bounded uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying all clarified properties of the function given by (1.4) and the above-mentioned function of H. Bohr. The reading of paper [11] by A. Haraux and P. Souplet has motivated us to analyze vector-valued uniformly recurrent functions in more detail ([18]). In that paper, we have made an attempt to further profile the sets of ϵ-periods of uniformly recurrent functions by introducing the class of $\odot_g$-almost periodic functions, which is simply defined by using the notions of lower and upper (Banach) densities for the subsets of the non-negative real axis.

1.1. **Lower and upper (Banach) g-densities.** We will always assume that $g : [0,\infty) \to [1,\infty)$ is an increasing mapping satisfying that there exists a finite number $L \geq 1$ such that $x \leq Lg(x)$, $x \geq 0$. For any set $A \subseteq [0,\infty)$ and $a,\ b \geq 0$, put $A(a,b) := \{x \in A\ ;\ x \in [a,b]\}$.

We will use the following densities (cf. [15] for more details):

(i) The lower g-density of A, denoted in short by $\underline{d}_{gc}(A)$, as follows
$$\underline{d}_{gc}(A) := \liminf_{x\to+\infty} \frac{m(A(0,g(x)))}{x};$$
(ii) the upper g-density of A, denoted in short by $\overline{d}_{gc}(A)$, as follows
$$\overline{d}_{gc}(A) := \limsup_{x\to+\infty} \frac{m(A(0,g(x)))}{x},$$

as well as:

(i) the lower $l;gc$-Banach density of A, denoted in short by $\underline{Bd}_{l;gc}(A)$, as follows
$$\underline{Bd}_{l;gc}(A) := \liminf_{x\to+\infty} \liminf_{y\to+\infty} \frac{m(A(y,y+g(x)))}{x};$$
(ii) the lower $u;gc$-Banach density of A, denoted in short by $\underline{Bd}_{u;gc}(A)$, as follows
$$\underline{Bd}_{u;gc}(A) := \limsup_{x\to+\infty} \liminf_{y\to+\infty} \frac{m(A(y,y+g(x)))}{x};$$
(iii) the (upper) $l;gc$-Banach density of A, denoted in short by $\overline{Bd}_{l;gc}(A)$, as follows
$$\overline{Bd}_{l;gc}(A) := \liminf_{x\to+\infty} \limsup_{y\to+\infty} \frac{m(A(y,y+g(x)))}{x};$$
(iv) the (upper) $u;fc$-Banach density of A, denoted in short by $\overline{Bd}_{u;gc}(A)$, as follows
$$\overline{Bd}_{u;gc}(A) := \limsup_{x\to+\infty} \limsup_{y\to+\infty} \frac{m(A(y,y+g(x)))}{x}.$$

In [18, Definition 2.1], we have introduced the notion of an $\odot_g$-almost periodic function in the following way: A continuous function $f : I \to X$ is said that $f(\cdot)$ is $\odot_g$-almost periodic if and only if for each $\epsilon > 0$ we have $\odot_g(\vartheta(f,\epsilon)) > 0$. It is worth noting that any uniformly continuous, uniformly recurrent function is $\odot_g$-almost periodic for a suitable chosen function $g(\cdot)$ and $\odot_g \in \{\underline{d}_{gc}, \overline{d}_{gc}\}$; see [18, Proposition 2.19]. In the remainder of paper, by $\odot_g$ we denote exactly one of the above six densities.

1.2. **Fractional calculus.** Fractional calculus and fractional differential equations are rapidly growing fields of research due to their numerous applications in pure and applied science.

Define $g_\eta(t) := t^{\eta-1}/\Gamma(\eta)$, $t > 0$ $(\eta > 0)$, where $\Gamma(\cdot)$ denotes the Euler Gamma function. Suppose now that $\gamma \in (0,1)$. The Caputo fractional derivative $\mathbf{D}_t^\gamma u(t)$ is defined for those functions $u : [0,T] \to X$ for which $u_{|(0,T]}(\cdot) \in C((0,T] : X)$, $u(\cdot) - u(0) \in L^1((0,T) : X)$ and $g_{1-\gamma} * (u(\cdot) - u(0)) \in W^{1,1}((0,T) : X)$, by

$$\mathbf{D}_t^\gamma u(t) = \frac{d}{dt}\Biggl[g_{1-\gamma} * \Bigl(u(\cdot) - u(0)\Bigr)\Biggr](t), \quad t \in (0,T].$$

Here, $W^{1,1}((0,T) : X)$ denotes the usual Sobolev space of order 1. For more details about the Caputo fractional derivatives, we refer the reader to the doctoral dissertation of E. Bazhlekova [1].

The Weyl-Liouville fractional derivative $D_{t,+}^\gamma u(t)$ of order $\gamma \in (0,1)$ is defined for those continuous functions $u : \mathbb{R} \to X$ such that $t \mapsto \int_{-\infty}^t g_{1-\gamma}(t-s)u(s)\,ds$, $t \in \mathbb{R}$ is a well-defined continuously differentiable mapping, by

$$D_{t,+}^\gamma u(t) := \frac{d}{dt}\int_{-\infty}^t g_{1-\gamma}(t-s)u(s)\,ds, \quad t \in \mathbb{R}.$$

Set $D_{t,+}^1 u(t) := -(d/dt)u(t)$. For more details about the Weyl-Liouville fractional derivatives, we refer the reader to the paper [22] by J. Mu, Y. Zhoa and L. Peng.

The Wright function $\Phi_\gamma(\cdot)$ is an entire function which can be introduced by the formula

$$\Phi_\gamma(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(1-\gamma-\gamma n)}, \quad z \in \mathbb{C}.$$

Let us recall that $\Phi_\gamma(t) \geq 0$, $t \geq 0$.

For more details about fractional calculus and fractional differential equations, we refer the reader to [1], [12]-[13], [16] and [23].

1.3. **Multivalued linear operators.** In this subsection, we will recall the basic definitions about multivalued linear operators. Let us recall that a multivalued map $\mathcal{A} : X \to P(X)$ is said to be a multivalued linear operator (MLO in X, or simply, MLO) if and only if the following holds:

(i) $D(\mathcal{A}) := \{x \in X : \mathcal{A}x \neq \emptyset\}$ is a linear subspace of X;
(ii) $\mathcal{A}x + \mathcal{A}y \subseteq \mathcal{A}(x+y)$, $x,\ y \in D(\mathcal{A})$ and $\lambda\mathcal{A}x \subseteq \mathcal{A}(\lambda x)$, $\lambda \in \mathbb{C}$, $x \in D(\mathcal{A})$.

It is well known that, for every $x,\ y \in D(\mathcal{A})$ and for every $\lambda,\ \eta \in \mathbb{C}$ with $|\lambda| + |\eta| \neq 0$, we have $\lambda\mathcal{A}x + \eta\mathcal{A}y = \mathcal{A}(\lambda x + \eta y)$. Set $R(\mathcal{A}) := \{\mathcal{A}x : x \in D(\mathcal{A})\}$. It simply follows that $\mathcal{A}^{-1}$ is an MLO in X. We say that $\mathcal{A}$ is closed if and only if for any sequence (x_n) in $D(\mathcal{A})$ and (y_n) in X such that $y_n \in \mathcal{A}x_n$ for all $n \in \mathbb{N}$ we have that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ imply $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$.

Suppose that $\mathcal{A}$ is an MLO in X, as well as that $C \in L(X)$ is injective and $C\mathcal{A} \subseteq \mathcal{A}C$. Then the C-resolvent set of $\mathcal{A}$, $\rho_C(\mathcal{A})$ for short, is defined as the union of those complex numbers $\lambda \in \mathbb{C}$ for which

(i) $R(C) \subseteq R(\lambda - \mathcal{A})$;
(ii) $(\lambda - \mathcal{A})^{-1}C$ is a single-valued linear continuous operator on X.

The operator $\lambda \mapsto (\lambda - \mathcal{A})^{-1}C$ is said to be the C-resolvent of $\mathcal{A}$ $(\lambda \in \rho_C(\mathcal{A}))$; the resolvent set of $\mathcal{A}$ is defined by $\rho(\mathcal{A}) := \rho_I(\mathcal{A})$, $R(\lambda : \mathcal{A}) \equiv (\lambda - \mathcal{A})^{-1}$ $(\lambda \in \rho(\mathcal{A}))$.

For more details about C-resolvent sets of multivalued linear operators, we refer the reader to [7] and [16].

We will use the following condition henceforth:

(P) There exist finite constants $c,\ M>0$ and $\beta\in(0,1]$ such that
$$\Psi:=\Big\{\lambda\in\mathbb{C}:\Re\lambda\geq -c\big(|\Im\lambda|+1\big)\Big\}\subseteq\rho(\mathcal{A})$$
and
$$\|R(\lambda:\mathcal{A})\|\leq M\big(1+|\lambda|\big)^{-\beta},\quad \lambda\in\Psi.$$

2. Uniformly recurrent functions and $\odot_g$-almost periodic functions depending on two arguments

The main aim of this section is to introduce and analyze the class of two-parameter (asymptotically) uniformly recurrent functions, the class of two-parameter (asymptotically) $\odot_g$-almost periodic functions and prove several related composition principles, which will enable us to consider applications to the abstract inhomogeneous fractional integro-differential inclusions in the final section of paper. Since the structural results presented in this section can be deduced by uncomplicated modifications of results known in the existing literature, we have decided to provide the main details of proofs for only two statements, Theorem 2.4 and Theorem 2.6.

Let $I=\mathbb{R}$ or $I=[0,\infty)$. By $C_0(I\times Y:X)$ we denote the space of all continuous functions $H:I\times Y\rightarrow X$ such that $\lim_{|t|\rightarrow+\infty}H(t,y)=0$ uniformly for y in any compact subset of Y. Let us recall that a continuous function $F:I\times Y\rightarrow X$ is said to be uniformly continuous on bounded sets, uniformly for $t\in I$ if and only if for every $\epsilon>0$ and every bounded subset B of Y there exists a number $\delta_{\epsilon,B}>0$ such that $\|F(t,x)-F(t,y)\|\leq\epsilon$ for all $t\in I$ and all $x,\ y\in B$ satisfying that $\|x-y\|\leq\delta_{\epsilon,B}$. If $F:I\times Y\rightarrow X$, then we define $\hat{F}:I\times Y\rightarrow L^p([0,1]:X)$ by $\hat{F}(t,y):=F(t+\cdot,y)$, $t\geq 0$, $y\in Y$.

Remark 2.1. It is worth noting that we can also define the space $C_{0,b}(I\times Y:X)$ consisting of all continuous functions $H:I\times Y\rightarrow X$ such that $\lim_{|t|\rightarrow+\infty}H(t,y)=0$ uniformly for y in any bounded subset of Y. This space, used by T. Diagana in [6, Subsection 3.3.2], can serve us to slightly modify the notion of spaces introduced in Definition 2.2(iii)-(iv) and Definition 2.3(ii) below. Furthermore, we can also consider continuous functions $F:I\times Y\rightarrow X$ which are uniformly continuous on compact sets.

For every $\epsilon>0$ and for every bounded set $B\subseteq Y$, we define $\vartheta(F;\epsilon,B)$ as the set constituted of all numbers $\tau>0$ such that
$$\|F(t+\tau,y)-F(t,y)\|\leq\epsilon,\quad t\in I,\ y\in B.$$

The following definition is crucial in our analysis:

Definition 2.2. (i) A continuous function $F:I\times Y\rightarrow X$ is called uniformly recurrent, resp. $\odot_g$-almost periodic, if and only if for every $\epsilon>0$ and every compact $K\subseteq Y$ there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that
$$\lim_{n\rightarrow+\infty}\sup_{t\in I}\big\|F(t+\alpha_n,y)-F(t,y)\big\|=0,\quad y\in K,\qquad(2.1)$$

resp. if and only if for every $\epsilon > 0$ and every compact $K \subseteq Y$ we have $\odot_g(\vartheta(f;\epsilon,K)) > 0$. The collection of all two-parameter uniformly recurrent functions, resp. $\odot_g$-almost periodic functions, will be denoted by $UR(I\times Y : X)$, resp. $AP_{\odot_g}(I\times Y : X)$.

(ii) A continuous function $F : I \times Y \to X$ is called uniformly recurrent on bounded sets, resp. $\odot_g$-almost periodic on bounded sets, if and only if for every $\epsilon > 0$ and every bounded set $B \subseteq Y$ there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity such that (2.1) holds with $K = B$, resp. if and only if for every $\epsilon > 0$ and every bounded set $B \subseteq Y$ we have $\odot_g(\vartheta(f;\epsilon,B)) > 0$. The collection of all two-parameter uniformly recurrent functions on bounded sets, resp. $\odot_g$-almost periodic functions on bounded sets, will be denoted by $UR_b(I\times Y : X)$, resp. $AP_{\odot_g,b}(I\times Y : X)$.

(iii) A continuous function $F : I \times Y \to X$ is said to be asymptotically uniformly recurrent, resp. asymptotically $\odot_g$-almost periodic, if and only if $f(\cdot)$ admits a decomposition $F = G + Q$, where $G \in UR(I \times Y : X)$, resp. $G \in AP_{\odot_g}(I\times Y : X)$, and $Q \in C_0(I\times Y : X)$. Denote by $AUR(I\times Y : X)$, resp. $AAP_{\odot_g}(I \times Y : X)$, the vector space consisting of all asymptotically uniformly recurrent functions, resp. asymptotically $\odot_g$-almost periodic functions.

(iv) A continuous function $F : I\times Y \to X$ is said to be asymptotically uniformly recurrent on bounded sets, resp. asymptotically $\odot_g$-almost periodic on bounded sets, if and only if $f(\cdot)$ admits a decomposition $F = G+Q$, where $G \in UR_b(I \times Y : X)$, resp. $G \in AP_{\odot_g,b}(I \times Y : X)$, and $Q \in C_0(I \times Y : X)$. Denote by $AUR_b(I \times Y : X)$, resp. $AAP_{\odot_g,b}(I \times Y : X)$, the vector space consisting of all asymptotically uniformly recurrent functions, resp. asymptotically $\odot_g$-almost periodic functions.

In the contrast to the approach of C. Zhang for almost periodic functions depending on the parameter [25] (see also [14, Definition 2.1.4]), we do not assume a priori the boundedness of function $f(\cdot,\cdot)$ in our approach. This is quite reasonable because uniformly recurrent functions and $\odot_g$-almost periodic functions of one real variable need not be bounded, in general. It is worth noticing that introducing parts (ii) and (iv) is motivated by definition of almost periodicity used by T. Diagana in [14, Definition 3.29].

For the Stepanov classes, we will use the following notion (see also [14, Definition 2.2.4, Definition 2.2.5; Lemma 2.2.7]):

Definition 2.3. Let $1 \leq p < \infty$.

(i) A function $F : I\times Y \to X$ is called Stepanov p-uniformly recurrent/Stepanov p-uniformly recurrent on bounded sets (Stepanov $(p,\odot_g)$-almost periodic/ Stepanov $(p,\odot_g)$-almost periodic on bounded sets) if and only if the function $\hat{F} : I \times Y \to L^p([0,1] : X)$ is uniformly recurrent/uniformly recurrent on bounded sets ($\odot_g$-almost periodic/$\odot_g$-almost periodic on bounded sets).

(ii) We say that $F : I \times Y \to X$ is asymptotically Stepanov p-uniformly recurrent/asymptotically Stepanov p-uniformly recurrent on bounded sets (asymptotically Stepanov $(p,\odot_g)$-almost periodic/asymptotically Stepanov $(p,\odot_g)$-almost periodic on bounded sets) if and only if there exist two functions $G : I \times Y \to X$ and $Q : I \times Y \to X$ satisfying that for each $y \in Y$

the functions $G(\cdot, y)$ and $Q(\cdot, y)$ are locally p-integrable, as well as that the following holds:

(a) $\hat{G} : I \times Y \to L^p([0,1] : X)$ is uniformly recurrent/uniformly recurrent on bounded sets ($\odot_g$-almost periodic/$\odot_g$-almost periodic on bounded sets),

(b) $\hat{Q} \in C_0(I \times Y : L^p([0,1] : X))$,

(c) $F(t, y) = G(t, y) + Q(t, y)$ for all $t \in I$ and $y \in Y$.

The serious difficulty in our investigations presents the fact that for two given uniformly recurrent functions $f : I \to X$ and $g : I \to X$, the sequence (α_n) for which (1.2) holds need not have a subsequence (α_{n_k}) for which

$$\lim_{k\to\infty} \sup_{t\in\mathbb{R}} \big\| g(t + \alpha_{n_k}) - g(t) \big\| = 0;$$

moreover, for given two $\odot_g$-almost periodic functions $f : I \to X$ and $g : I \to X$, the set consisting of their joint ϵ-periods can be bounded. This cannot be the case for almost periodic functions; so, if one wants to slightly improves [14, Theorem 3.30] for uniformly recurrent functions and $\odot_g$-almost periodic functions, it is necessary to impose some extra conditions coming naturally from the above analysis:

Theorem 2.4. *Suppose that $f : I \to Y$ is uniformly recurrent ($\odot_g$-almost periodic) and the range of $f(\cdot)$ is relatively compact, resp. bounded. If $F : I \times Y \to X$ is uniformly recurrent ($\odot_g$-almost periodic), resp. uniformly recurrent on bounded sets ($\odot_g$-almost periodic on bounded sets), and there exists a finite constant $L > 0$ such that*

$$\|F(t, x) - F(t, y)\| \le L\|x - y\|_Y, \quad t \in I, \ x, \ y \in Y, \tag{2.2}$$

then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ is uniformly recurrent ($\odot_g$-almost periodic), providing additionally the following condition: there exists a strictly increasing sequence (α_n) of positive reals tending to plus infinity for which (1.2) holds and (2.1) holds with $K = \overline{\{f(t) : t \in I\}}$, resp. for each $\epsilon > 0$ we have that $\odot_g(\vartheta(F; \epsilon, \overline{\{f(t) : t \in I\}}) \cap \vartheta(f, \epsilon)) > 0$.

Proof. The proof of theorem is very similar to the proof of [14, Theorem 3.30] and we will only outline the main details for $\odot_g$-almost periodic functions. Let $\epsilon > 0$ be given, and let $\tau \in \vartheta(F; \epsilon/2(1+L), \overline{\{f(t) : t \in I\}}) \cap \vartheta(f, \epsilon/2(1+L))$. Then $\|f(t+\tau) - f(t)\| \le \epsilon/2(1+L)$, $t \in I$ and we have

$$\|\mathcal{F}(t+\tau) - \mathcal{F}(t)\| \le L\|f(t+\tau) - f(t)\|_Y + \|F(t+\tau, f(t)) - F(t+\tau, f(t))\|, \quad t \in I.$$

Hence,

$$\|\mathcal{F}(t+\tau) - \mathcal{F}(t)\| \le [L\epsilon/2(1+L)] + \epsilon/2(1+L) < \epsilon, \quad t \in I,$$

which completes the proof. □

Similarly we can prove the following slight extension of [14, Theorem 3.31]:

Theorem 2.5. *Suppose that $f : I \to Y$ is a bounded uniformly recurrent function (bounded $\odot_g$-almost periodic function). If $F : I \times Y \to X$ is uniformly recurrent on bounded sets ($\odot_g$-almost periodic on bounded sets) and uniformly continuous on bounded sets, uniformly for $t \in I$, then the mapping $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ is uniformly recurrent ($\odot_g$-almost periodic), providing additionally the following condition: there exists a strictly increasing sequence (α_n) of positive reals tending* *to plus infinity for which* (1.2) *holds and* (2.1) *holds with* $K = \overline{\{f(t) : t \in I\}}$, *resp. for each* $\epsilon > 0$ *we have that* $\odot_g(\vartheta(F; \epsilon, \overline{\{f(t) : t \in I\}}) \cap \vartheta(f, \epsilon)) > 0$.

Before proceeding further, it should be observed that the statement of [14, Theorem 3.32] (see also the proof of [8, Theorem 2.11]) can be formulated and slightly extended for uniformly recurrent ($\odot_g$-almost periodic) functions with relatively compact range.

Composition principles for asymptotically almost periodic functions have been analyzed in a great number of research papers. With regards to this question, we will state and give the main details of proof for the following slight extension of [6, Theorem 3.49], only (observe, however, that we can similarly reconsider and slightly extend the statements of [6, Theorem 3.50-Theorem 3.52]).

Theorem 2.6. *Suppose that* $h : I \to Y$ *is uniformly recurrent (*$\odot_g$*-almost periodic), the range of* $h(\cdot)$ *is relatively compact, resp. bounded,* $q \in C_0(I : X)$ *and* $f(t) = h(t) + q(t)$ *for all* $t \in I$. *Suppose, further,* $H : I \times Y \to X$ *is uniformly recurrent (*$\odot_g$*-almost periodic), resp. uniformly recurrent on bounded sets (*$\odot_g$*-almost periodic on bounded sets), there exists a finite constant* $L > 0$ *such that* (2.2) *holds with the function* $F(\cdot, \cdot)$ *replaced therein with the function* $H(\cdot, \cdot)$, *and there exists a strictly increasing sequence* (α_n) *of positive reals tending to plus infinity for which* (1.2) *holds with the function* $f(\cdot)$ *replaced therein with the function* $h(\cdot)$ *and* (2.1) *holds with the function* $f(\cdot)$ *replaced therein with the function* $h(\cdot)$ *and set* $K = \overline{\{h(t) : t \in I\}}$, *resp. for each* $\epsilon > 0$ *we have that* $\odot_g(\vartheta(H; \epsilon, \overline{\{h(t) : t \in I\}}) \cap \vartheta(h, \epsilon)) > 0$. *If* $f(\cdot)$ *has a relatively compact range,* $Q \in C_0(I \times Y : X)$ *and* $F(t, y) = H(t, y) + Q(t, y)$ *for all* $t \in I$ *and* $y \in Y$, *then the mapping* $\mathcal{F}(t) := F(t, f(t))$, $t \in I$ *is asymptotically uniformly recurrent (asymptotically* $\odot_g$*-almost periodic).*

Proof. Due to Theorem 2.4, we have that the mapping $t \mapsto H(t, h(t))$, $t \in I$ is uniformly recurrent ($\odot_g$-almost periodic). Furthermore, we have the decomposition

$$F(t, f(t)) = H(t, h(t)) + [H(t, f(t)) - H(t, h(t))] + Q(t, f(t)), \quad t \in I.$$

Since the function $H(\cdot, \cdot)$ satisfies (2.2), we have

$$\|H(t, f(t)) - H(t, h(t))\| \leq L\|f(t) - h(t)\|_Y \leq L\|q(t)\|_Y \to 0 \text{ as } |t| \to +\infty.$$

The proof of theorem completes the observation that $\lim_{|t|\to+\infty} \|Q(t, f(t))\| = 0$, which follows from definition of space $C_0(I \times Y : X)$ and our assumption that $f(\cdot)$ has a relatively compact range. □

Remark 2.7. The assumption [6, (3.13)] is superfluous. Furthermore, we note that the assumption that the range of $h(\cdot)$ is relatively compact, resp. bounded, implies that $f(\cdot)$ is bounded; therefore, if we use the space $C_{0,b}(I \times Y : X)$ in place of $C_0(I \times Y : X)$ here, the assumption that $f(\cdot)$ has a relatively compact range is superfluous, as well.

Remark 2.8. Consider, for simplicity, asymptotically uniformly recurrent functions. The principal part $\mathbf{f}(\cdot)$ of function $\mathcal{F}(t) = F(t, f(t))$, $t \in I$ satisfies (1.2) with the same sequence (α_n) and the function $\mathbf{f}(\cdot)$ in place of $f(\cdot)$. This holds for all remaining results established in this paper, and this fact will be of some importance for applications made in the subsequent section.

Concerning the composition principles for Stepanov almost periodic functions, the most influential paper written by now is the paper [21] by W. Long and H.-S. Ding. Repating almost verbatim the arguments given in the proof of [21, Lemma 2.1, Theorem 2.2], we can deduce the following result (we feel it is our duty to say that the previously proved results are more appropriate for applications in infinite-dimensional spaces because condition on relative compactness of range of function $f(\cdot)$ is almost inevitable to be used; see condition (ii) below):

Theorem 2.9. *Let $I=\mathbb{R}$ or $I=[0,\infty)$. Suppose that the following conditions hold:*

(i) *The function $F: I\times Y\rightarrow X$ is Stepanov p-uniformly recurrent, resp. Stepanov $(p,\odot_g)$-almost periodic, with $p>1$, and there exist a number $r\geq \max(p,p/p-1)$ and a function $L_F\in L_S^r(I)$ such that:*

$$\|F(t,x)-F(t,y)\|\leq L_F(t)\|x-y\|_Y,\quad t\in I,\ x,\ y\in Y. \tag{2.3}$$

(ii) *The function $f: I\rightarrow Y$ is Stepanov p-uniformly recurrent, resp. Stepanov $(p,\odot_g)$-almost periodic, and there exists a set $\mathrm{E}\subseteq I$ with $m(\mathrm{E})=0$ such that $K:=\{f(t): t\in I\setminus \mathrm{E}\}$ is relatively compact in Y.*

(iii) *For every compact set $K\subseteq Y$, there exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that*

$$\lim_{n\rightarrow+\infty}\sup_{t\in I}\sup_{u\in K}\int_0^1\|F(t+s+\alpha_n,u)-F(t+s,u)\|^p\,ds=0 \tag{2.4}$$

and (1.2) holds with the function $f(\cdot)$ and the norm $\|\cdot\|$ replaced respectively by the function $\hat{f}(\cdot)$ and the norm $\|\cdot\|_{L^p([0,1]:X)}$ therein, resp. for every number $\epsilon>0$ and for every compact set $K\subseteq Y$, the set consisting of all positive real numbers $\tau>0$ such that

$$\sup_{t\in I}\sup_{u\in K}\int_0^1\|F(t+s+\tau,u)-F(t+s,u)\|^p\,ds<\epsilon^p \tag{2.5}$$

and (1.1) holds with the function $f(\cdot)$ and the norm $\|\cdot\|$ replaced respectively by the function $\hat{f}(\cdot)$ and the norm $\|\cdot\|_{L^p([0,1]:X)}$ therein.

Then $q:=pr/p+r\in[1,p)$ and $F(\cdot,f(\cdot))$ is Stepanov q-uniformly recurrent, resp. Stepanov $(q,\odot_g)$-almost periodic. Furthermore, the assumption that $F(\cdot,0)$ is Stepanov q-bounded implies that the function $F(\cdot,f(\cdot))$ is Stepanov q-bounded, as well.

In [14, Theorem 2.7.2], we have also considered the value $p=1$ in Theorem 2.9 and the usual condition regarding the existence of a Lipschitz constant $L>0$ such that

$$\|F(t,x)-F(t,y)\|\leq L\|x-y\|_Y,\quad t\in I,\ x,\ y\in Y. \tag{2.6}$$

Using the foregoing arguments, we can simply deduce the following extension of the above-mentioned theorem:

Theorem 2.10. *Let $I=\mathbb{R}$ or $I=[0,\infty)$. Suppose that the following conditions hold:*

(i) *The function $F: I\times Y\rightarrow X$ is Stepanov p-uniformly recurrent, resp. Stepanov $(p,\odot_g)$-almost periodic with $p\geq 1$, $L>0$ and (2.6) holds.*

(ii) *The same as condition (ii) of Theorem 2.9.*

(iii) *The same as condition (iii) of Theorem 2.9.*

Then the function $F(\cdot, f(\cdot))$ is Stepanov p-uniformly recurrent, resp. Stepanov $(p, \odot_g)$-almost periodic. Furthermore, the assumption that $F(\cdot, 0)$ is Stepanov p-bounded implies that the function $F(\cdot, f(\cdot))$ is Stepanov p-bounded, as well.

Following the analysis of F. Bedouhene, Y. Ibaouene, O. Mellah and P. Raynaud de Fitte [2, Theorem 3] for the class of equi-Weyl p-almost periodic functions and the analysis of W. Long and H.-S. Ding [21], in [17, Theorem 2.1] we have established a new composition principle for the class of Stepanov p-almost periodic functions that is not comparable with [21, Theorem 2.2]. Using the proof of the last-mentioned theorem and the proof of [17, Theorem 2.1], we can deduce the following generalization of Theorem 2.10:

Theorem 2.11. *Suppose that $p,\ q \in [1, \infty)$, $r \in [1, \infty]$, $1/p = 1/q + 1/r$ and the following conditions hold:*

(i) *The function $F : I \times Y \to X$ is Stepanov p-uniformly recurrent, resp. Stepanov $(p, \odot_g)$-almost periodic, and there exists a function $L_F \in L^r_S(I)$ such that (2.3) holds.*

(ii) *The same as condition (ii) of Theorem 2.9, with the number p replaced with the number q therein.*

(iii) *For every compact set $K \subseteq Y$, there exists a strictly increasing sequence (α_n) of positive real numbers tending to plus infinity such that (2.4) holds and (1.2) holds with the function $f(\cdot)$ and the norm $\|\cdot\|$ replaced respectively by the function $\hat{f}(\cdot)$ and the norm $\|\cdot\|_{L^q([0,1]:X)}$ therein, resp. for every number $\epsilon > 0$ and for every compact set $K \subseteq Y$, the set consisting of all positive real numbers $\tau > 0$ such that (2.5) holds and (1.1) holds with the function $f(\cdot)$ and the norm $\|\cdot\|$ replaced respectively by the function $\hat{f}(\cdot)$ and the norm $\|\cdot\|_{L^q([0,1]:X)}$ therein.*

Then the function $F(\cdot, f(\cdot))$ is Stepanov p-uniformly recurrent, resp. Stepanov $(p, \odot_g)$-almost periodic. Furthermore, the assumption that $F(\cdot, 0)$ is Stepanov p-bounded implies that the function $F(\cdot, f(\cdot))$ is Stepanov p-bounded, as well.

Keeping in mind Theorem 2.9-Theorem 2.10, resp. Theorem 2.11, it is straightforward to reformulate the statements of [14, Proposition 2.7.3-Proposition 2.7.4], resp. [17, Proposition 2.1], for the asymptotical Stepanov p-uniform recurrence and the asymptotical Stepanov $(p, \odot_g)$-almost periodicity. Details can be left to the interested readers.

3. Applications to abstract semilinear fractional integro-differential inclusions

In this section, we will present two interesting applications of established theoretical results in the analysis of the existence and uniqueness of uniformly recurrent type solutions of abstract semilinear fractional integro-differential inclusions.

1. In the first application, we will consider the finite-dimensional space $X := \mathbb{C}^n$, where $n \geq 2$. Suppose that $c > 0$, $A,\ B \in \mathbb{C}^{n,n}$ (the space of all complex matrices of format $n \times n$), the matrix B is not invertible, as well as that the degree of complex polynomial $P(\lambda) := \det(\lambda B - A)$, $\lambda \in \mathbb{C}$ is equal to n and its roots lie in the region $\{\lambda \in \mathbb{C} : \Re\lambda < -c(|\Im\lambda|+1)\}$. Due to [16, Proposition 2.1.2], we have that the region

Ψ from the formulation of condition (P) belongs to the resolvent set of multivalued linear operator $\mathcal{A} = AB^{-1}$ as well as that

$$\big(\lambda - AB^{-1}\big)^{-1} = B\big(\lambda B - A\big)^{-1}, \quad \lambda \in \Psi.$$

Since the degree of complex polynomial $P(\cdot)$ is equal to n, the above formula simply implies that there exists a positive real constant $M > 0$ such that condition (P) holds with $\beta = 1$, so that the operator $\mathcal{A}$ generates an exponentially decaying strongly continuous degenerate semigroup $(T(t))_{t\geq 0}$ which can be analytically extented to a sector around positive real axis (cf. [16] for the notion).

Suppose now that $0 < \gamma < 1$ and $\nu > -1$. Define

$$T_{\gamma,\nu}(t)x := t^{\gamma\nu} \int_0^\infty s^\nu \Phi_\gamma(s) T\big(st^\gamma\big)x\, ds, \quad t > 0,\ x \in X,$$

$$S_\gamma(t) := T_{\gamma,0}(t) \text{ and } P_\gamma(t) := \gamma T_{\gamma,1}(t)/t^\gamma, \quad t > 0.$$

Recall [16] that, in general case $\beta \in (0, 1]$, there exists a finite constant $M_1 > 0$ such that

$$\big\|S_\gamma(t)\big\| + \big\|P_\gamma(t)\big\| \leq M_1 t^{\gamma(\beta-1)}, \quad t > 0, \tag{3.1}$$

as well as

$$\big\|S_\gamma(t)\big\| \leq M_1 t^{-\gamma},\ t \geq 1 \quad \text{and} \quad \big\|P_\gamma(t)\big\| \leq M_2 t^{-2\gamma},\ t \geq 1. \tag{3.2}$$

Set $R_\gamma(t) := t^{\gamma-1} P_\gamma(t)$, $t > 0$. Then (3.1)-(3.2) yield

$$\|R_\gamma(t)\| = O\big(t^{\gamma-1} + t^{-\gamma-1}\big), \quad t > 0. \tag{3.3}$$

Consider now the following abstract fractional inclusion

$$D_+^\gamma \vec{u}(t) \in -\mathcal{A}\vec{u}(t) + F(t, \vec{u}(t)), \quad t \in \mathbb{R}, \tag{3.4}$$

where $D_+^\gamma u(t)$ denotes the Weyl-Liouville fractional derivative of order γ and $F : \mathbb{R} \times X \to X$; after the usual substitution $\vec{v}(t) \in B^{-1}\vec{u}(t)$, $t \in \mathbb{R}$, this inclusion becomes

$$D_+^\gamma \big[B\vec{v}(t)\big] = -A\vec{v}(t) + F\big(t, B\vec{v}(t)\big), \quad t \in \mathbb{R}.$$

Following J. Mu, Y. Zhoa and L. Peng [22], it will be said that a continuous function $u : \mathbb{R} \to X$ is a mild solution of (3.4) if and only if

$$\vec{u}(t) = \int_{-\infty}^t R_\gamma(t-s) F\big(s, \vec{u}(s)\big)\, ds, \quad t \in \mathbb{R}.$$

For the sequel, fix a strictly increasing sequence (α_n) of positive reals tending to plus infinity. Denote

$$BUR_{(\alpha_n)}(\mathbb{R} : X) := \big\{\vec{u} \in UR(\mathbb{R} : X)\, ;\, \vec{u}(\cdot) \text{ is bounded and (1.2) holds with } f = \vec{u}\big\}.$$

Equipped with the metric $d(\cdot, \cdot) := \| \cdot - \cdot \|_\infty$, $BUR_{(\alpha_n)}(\mathbb{R} : X)$ becomes a complete metric space.

Now we are able to state the following result:

Theorem 3.1. *Suppose that the function $F : \mathbb{R} \times X \to X$ satisfies that for each bounded subset B of X there exists a finite real constant $M_B > 0$ such that $\sup_{t\in\mathbb{R}} \sup_{y\in B} \|F(t, y)\| \leq M_B$. Suppose, further, that the function $F : \mathbb{R} \times X \to X$ is Stepanov p-uniformly recurrent with $p > 1$, and there exist a number $r \geq$*

$\max(p, p/p-1)$ *and a function* $L_F \in L_S^r(I)$ *such that* $q := pr/(p+r) > 1$ *and* (2.3) *holds with* $I = \mathbb{R}$. *If*

$$\frac{(\gamma-1)q}{q-1} > -1, \tag{3.5}$$

there exists an integer $n \in \mathbb{N}$ *such that* $M_n < 1$, *where*

$$M_n := \sup_{t \geq 0} \int_{-\infty}^{t} \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_2} \Big\| R_\gamma(t-x_n) \Big\| \times \prod_{i=2}^{n} \Big\| R_\gamma(x_i - x_{i-1}) \Big\| \prod_{i=1}^{n} L_F(x_i)\, dx_1\, dx_2 \cdots dx_n,$$

and for every compact set $K \subseteq Y$, (2.4) *holds, then the abstract fractional Cauchy inclusion* (3.4) *has a unique bounded uniformly recurrent solution.*

Proof. Define $\Upsilon : BUR_{(\alpha_n)}(\mathbb{R} : X) \to BUR_{(\alpha_n)}(\mathbb{R} : X)$ by

$$(\Upsilon \vec{u})(t) := \int_{-\infty}^{t} R_\gamma(t-s) F(s, \vec{u}(s))\, ds, \quad t \in \mathbb{R}.$$

Let us firstly show that the mapping $\Upsilon(\cdot)$ is well defined. Suppose that $\vec{u} \in BUR_{(\alpha_n)}(\mathbb{R} : X)$. Then $R(\vec{u}) = B$ is a bounded set so that the mapping $t \mapsto F(t, \vec{u}(t))$, $t \in \mathbb{R}$ is bounded due to the prescribed assumption. Applying Theorem 2.9, we have that the function $F(\cdot, \vec{u}(\cdot))$ is Stepanov q-uniformly recurrent. Define $q' := q/(q-1)$. Then (3.3) and (3.5) together imply that $\|R_\gamma(\cdot)\| \in L^{q'}[0,1]$ and $\sum_{k=0}^{\infty} \|R_\gamma(\cdot)\|_{L^{q'}[k,k+1]} < \infty$ due to our analysis from [14, Remark 2.6.12]. Applying [18, Proposition 3.1], we get that the function $t \mapsto \int_{-\infty}^{t} R_\gamma(t-s) F(s, \vec{u}(s))\, ds$, $t \in \mathbb{R}$ is bounded, continuous and uniformly recurrent (cf. also the proof of [14, Proposition 2.6.11, Proposition 3.5.3]), which yields that $\Upsilon \vec{u} \in BUR_{(\alpha_n)}(\mathbb{R} : X)$, as claimed. Furthermore, a simple calculation shows that

$$\Big\| \big(\Upsilon^n \vec{u}_1\big) - \big(\Upsilon^n \vec{u}_2\big) \Big\|_\infty \leq M_n \big\| \vec{u_1} - \vec{u_2} \big\|_\infty, \quad \vec{u_1},\ \vec{u_2} \in BUR_{(\alpha_n)}(\mathbb{R} : X),\ n \in \mathbb{N}.$$

Since we have assumed that there exists an integer $n \in \mathbb{N}$ such that $M_n < 1$, the well known extension of the Banach contraction principle shows that the mapping $\Upsilon(\cdot)$ has a unique fixed point, finishing the proof of the theorem. □

2. Suppose that a closed multivalued linear operator $\mathcal{A}$ satisfies condition (P) in X, which can be finite-dimensional or infinite-dimensional, with general exponent $\beta \in (0,1]$. Consider the abstract semilinear fractional differential inclusion

$$(\mathrm{DFP})_{f,\gamma,s} : \begin{cases} \mathbf{D}_t^\gamma u(t) \in \mathcal{A}u(t) + F(t, u(t)),\ t > 0, \\ u(0) = x_0, \end{cases}$$

where $\mathbf{D}_t^\gamma$ denotes the Caputo fractional derivative of order γ, $x_0 \in X$ and $F : [0,\infty) \times X \to X$. By a mild solution of $(\mathrm{DFP})_{f,\gamma,s}$, we mean any function $u \in C([0,\infty) : X)$ satisfying that

$$u(t) = S_\gamma(t)x_0 + \int_0^t R_\gamma(t-s) F(s, u(s))\, ds, \quad t \geq 0.$$

In what follows, we will assume that $\lim_{t \to 0+} S_\gamma(t)x_0 = x_0$ so that the mapping $t \mapsto S_\gamma(t)x_0$, $t \geq 0$ belongs to the space $C_0([0,\infty) : X)$; see the estimate (3.1). Arguing as in the proof of [6, Theorem 3.46], we may conculde that $\mathcal{X} := BUR_{(\alpha_n)}([0,\infty) :$

$X) \oplus C_0([0,\infty) : X)$ is a complete metric space equipped with the distance $d(\cdot,\cdot)$ used above. Set, for every $u \in \mathcal{X}$ and $n \in \mathbb{N}$,

$$\big(\Upsilon_A u\big)(t) := S_\gamma(t)x_0 + \int_0^t R_\gamma(t-s)F(s,u(s))\,ds, \quad t \geq 0;$$

$$A_n := \sup_{t\geq 0} \int_0^t \int_0^{x_n} \cdots \int_0^{x_2} \big\| R_\gamma(t-x_n) \big\| \times \prod_{i=2}^n \big\| R_\gamma(x_i - x_{i-1}) \big\| \prod_{i=1}^n L_F(x_i)\,dx_1\,dx_2 \cdots dx_n.$$

Then a simple calculation shows that

$$\Big\| \big(\Upsilon_A^n u\big) - \big(\Upsilon_A^n v\big) \Big\|_\infty \leq A_n \big\| u - v \big\|_\infty, \quad u,\ v \in \mathcal{X},\ n \in \mathbb{N}.$$

Keeping in mind [18, Proposition 3.1], Theorem 2.6, Remark 2.7-Remark 2.8 and the proof of [14, Lemma 2.6.3], we can similarly clarify the following result:

Theorem 3.2. *Suppose that the function $F : [0,\infty) \times X \to X$ is continuous and satisfies that for each bounded subset B of X there exists a finite real constant $M_B > 0$ such that $\sup_{t\geq 0} \sup_{y\in B} \|F(t,y)\| \leq M_B$. Suppose, further, that $H : [0,\infty) \times X \to X$ is uniformly recurrent on bounded sets, there exists a finite constant $L > 0$ such that (2.2) holds with the function $F(\cdot,\cdot)$ replaced therein with the function $H(\cdot,\cdot)$ and $I = [0,\infty)$. Let (2.1) hold with any bounded set $B = K$, and let there exist an integer $n \in \mathbb{N}$ such that $A_n < 1$. If $Q \in C_{0,b}(I \times Y : X)$ and $F(t,y) = H(t,y) + Q(t,y)$ for all $t \geq 0$ and $y \in Y$, then the abstract fractional Cauchy inclusion (DFP)$_{f,\gamma,s}$ has a unique mild solution.*

Let Ω be a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$, $1 < p < \infty$ and $X := L^p(\Omega)$. Suppose that the operator $A := \Delta - b$ acts on X with the Dirichlet boundary conditions, and that B is the multiplication operator by the function $m(x)$. As explained in [14], we can apply Theorem 3.2 with $\mathcal{A} = AB^{-1}$ in the study of existence and uniqueness of asymptotically uniformly recurrent solutions of the semilinear fractional Poisson heat equation

$$\begin{cases} \mathbf{D}_t^\gamma[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t, m(x)v(t,x)), \quad t \geq 0,\ x \in \Omega; \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega. \end{cases}$$

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Marko Kostić

Faculty of Technical Sciences, University of Novi Sad, Trg D. Obradovića 6, 21125 Novi Sad, Serbia

E-mail address: marco.s@verat.net