Journal of Fractional Calculus and Applications
Vol. 12(2) July 2021, pp. 204-227
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

NEW GENERALIZED BETA FUNCTION ASSOCIATED WITH THE FOX-WRIGHT FUNCTION

U. M. ABUBAKAR

Abstract. Many researchers proposed generalized gamma and beta functions by considering variants kernel such as exponential, Mittag-Leffler, Macdonald, Bessel-Struve, confluent hypergeometric, classical Wright and generalized Wrigh functions. In this research paper, we introduced and investigated new generalized beta function by considering the Fox-Wright function in its kernel. We also gave certain of its properties such as integral representations, differential formulas, difference formulas, Mellin transform, Mellin inversion formula and summation formulas. We also gave some statistical applications by introducing beta distribution and its corresponding mean, variance and moment generating function.

1. Introduction

Throughout this research paper $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$ and ${\mathbb{Z}^-}_0$ represents sets of complex numbers, real numbers, integers, positive integers, negative integers and non-positive integers, respectively. The classical gamma and beta functions are defined by (see [1] and [2]):

$$\Gamma(\omega_1) = \int_0^\infty t^{\omega_1 - 1} exp(-t)dt, \ (Re(\omega_1) > 0).$$

And

$$B(\omega_1, \omega_2) = \begin{cases} \int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1}dt, & (Re(\omega_1) > 0, Re(\omega_2) > 0), \\ \frac{\Gamma(\omega_1)\Gamma(\omega_2)}{\Gamma(\omega_1+\omega_2)}, & (\omega_1, \omega_2 \notin {\mathbb{Z}^-}_0). \end{cases} \tag{1}$$

The following equality holds:

$$B(\omega_1, \omega_2 - \omega_1) = \frac{\omega_2}{\omega_1} B(\omega_1 + 1, \omega_2 - \omega_1), (Re(\omega_2) > Re(\omega_1) > 0). \tag{2}$$

2010 *Mathematics Subject Classification.* 33B15, 33C05, 33C015.

Key words and phrases. Gamma function, beta function, Fox-Wright function, incomplete gama function, Gauss hypergeometric function, confluent hypergeometric function.

Submitted Dec. 11, 2020. Revised Jan. 29, 2021.

The classical Gauss hypergeometric function is defined by (see [3]):

$$
\begin{aligned}
{}_2F_1(p_1,p_2;p_3;x_1) &= \sum_{r_1=0}^{\infty} \frac{(p_1)_{r_1}(p_2)_{r_1}}{(p_3)_{r_1}} \frac{x_1^{r_1}}{r_1!} \\
&= \sum_{r_1=0}^{\infty} (p_1)_{r_1} \frac{B(p_2+r_1,p_3-p_2)}{B(p_2,p_3-p_2)} \frac{x_1^{r_1}}{r_1!},
\end{aligned}
$$

$$(Re(p_3) > Re(p_2) > 0, |x_1| < 1).$$

With the integral representation as follows:

$${}_2F_1(p_1,p_2;p_3;x_1) = \frac{1}{B(p_2,p_3-p_2)} \int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1}(1-tx_1)^{-p_1} dt,$$

$$(Re(p_3) > Re(p_2) > 0, |arg(1-x_1)| < 1).$$

And

$$\Phi(p_2;p_3;x_1) = \sum_{r_1=0}^{\infty} \frac{(p_2)_{r_1}}{(p_3)_{r_1}} \frac{x_1^{r_1}}{r_1!} = \sum_{r_1=0}^{\infty} \frac{B(p_2+r_1,p_3-p_2)}{B(p_2,p_3-p_2)} \frac{x_1^{r_1}}{r_1!},$$

$$(Re(p_2) > 0, Re(p_3) > 0, |x_1| < 1).$$

With the integral representation as follows:

$$\Phi(p_2;p_3;x_1) = \frac{1}{B(p_2,p_3-p_2)} \int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1} exp(tx_1) dt,$$

$$(Re(p_3) > Re(p_2) > 0).$$

Where $(p_1)_{r_1}$ is the classical pochammer symbol defined by (see for example [4], [5] and [6]):

$$(p_1)_{r_1} = \begin{cases} (p_1)(p_1+1)(p_1+2)\cdots(p_1+r_1-1), & (r_1 \geq 0, p_1 \neq 0), \\ 1, & (r_1 = 0). \end{cases}$$

With the following important relations:

$$(p_1)_{r_1} = \frac{\Gamma(p_1+r_1)}{\Gamma(p_1)}.$$

And

$$\sum_{r_1=0}^{\infty} (p_1)_{r_1} \frac{(tx_1)^{r_1}}{r_1!} = {}_1F_0(p_1;-;tx_1) = (1-tx_1)^{-p_1}. \tag{3}$$

In 1994, Chaudhry and Zubair [7] used exponential kernel to proposed the following extension of gamma function:

$$\Gamma_{\rho_1}(\omega_1) = \int_0^{\infty} t^{\omega_1-1} exp\left(-t - \frac{\rho_1}{t}\right) dt,$$

$$(Re(\omega_1) > 0, Re(\rho_1) > 0).$$

In 1997, Chaudhry and Zubair [8] used exponential kernel to introduced the following extension of beta function:

$$B_{\rho_1}(\omega_1,\omega_2) = \int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1} exp\left(-\frac{\rho_1}{(1-t)t}\right) dt, \tag{4}$$

$$(Re(\omega_1) > 0, Re(\omega_2) > 0, Re(\rho_1) > 0).$$

In 2004, Chaudhry et al., [9] established the extension of Gauss and confluent hypergeometric functions by considering the extended beta function in (4) as follows:

$$F_{\rho_1}(p_1,p_2;p_3;x_1)=\sum_{r_1=0}^{\infty}(p_1)_{r_1}\frac{B_{\rho_1}(p_2+r_1,p_3-p_2)}{B(p_2,p_3-p_2)}\frac{x_1^{r_1}}{r_1!},$$

$$(\rho_1\geq 0, Re(p_3)>Re(p_2)>0,|x_1|<1).$$

With the integral representation as follows:

$$F_{\rho_1}(p_1,p_2;p_3;x_1)=\frac{1}{B(p_2,p_3-p_2)}\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1}(1-tx_1)^{-p_1}\times exp\left(-\frac{\rho_1}{(1-t)t}\right)dt,$$

$$(\rho_1>0;\rho_1=0, Re(p_3)>Re(p_2)>0,|arg(1-x_1)|<\pi).$$

And

$$\Phi_{\rho_1}(p_2;p_3;x_1)=\sum_{r_1=0}^{\infty}\frac{B_{\rho_1}(p_2+r_1,p_3-p_2)}{B(p_2,p_3-p_2)}\frac{x_1^{r_1}}{r_1!},$$

$$(\rho_1>0;\rho_1=0, Re(p_3)>Re(p_2)>0).$$

With the integral representation as follows:

$$\Phi_{\rho_1}(p_2;p_3;x_1)=\frac{1}{B(p_2,p_3-p_2)}\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1}exp\left(tx_1-\frac{\rho_1}{(1-t)t}\right)dt,$$

$$(\rho_1>0;\rho_1=0, Re(p_3)>Re(p_2)>0).$$

In 2010, Ozarslan and Ozergin [10] used extended beta function in (4) to proposed the following extension of Appell's and Lauricella's hypergeometric functions:

$$F_{1,\rho_1}(p_1,p_2,p_3;p_4;x_1,x_2)=\sum_{r_1,r_2=0}^{\infty}(p_2)_{r_1}(p_3)_{r_2}\frac{B_{\rho_1}(p_1+r_1+r_2,p_4-p_1)}{B(p_1,p_4-p_1)}\frac{x_1^{r_1}}{r_1!}\frac{x_2^{r_2}}{r_2!},$$

$$(\max\{|x_1|,|x_2|\}<1).$$

$$F_{2,\rho_1}(p_1,p_2,p_3;p_4,p_5;x_1,x_2)=\sum_{r_1,r_2=0}^{\infty}(p_1)_{r_1+r_2}\frac{B_{\rho_1}(p_2+r_1,p_4-p_2)}{B(p_2,p_4-p_2)}\times\frac{B_{\rho_1}(p_3+r_2,p_5-p_3)}{B(p_3,p_3-p_3)}\frac{x_1^{r_1}}{r_1!}\frac{x_2^{r_2}}{r_2!},$$

$$(|x_1|+|x_2|<1).$$

And

$$F^3_{D,\rho_1}(p_1,p_2,p_3;p_4;x_1,x_2,x_3)=\sum_{r_1,r_2,r_3=0}^{\infty}\frac{B_{\rho_1}(p_1+r_1+r_2,p_4-p_1)}{B(p_1,p_4-p_1)}\times(p_2)_{r_1}(p_3)_{r_2}(p_4)_{r_3}\frac{x_1^{r_1}}{r_1!}\frac{x_2^{r_2}}{r_2!}\frac{x_3^{r_3}}{r_3!},$$

$$(|x_1|<1,|x_2|<1,|x_3|<1).$$

In 2011, Lee et al., [11] presented and investigated the following extension of beta function:

$$B_{\rho_1}(\omega_1,\omega_2)=\int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1}exp\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt,$$

$$(Re(\omega_1) > 0, Re(\omega_2) > 0, Re(\rho_1) > 0, Re(\aleph) > 0).$$

They [11] also introduced the following Gauss and confluent hypergeometric function:

$$F^{\aleph}_{\rho_1}(p_1, p_2; p_3; x_1) = \sum_{r_1=0}^{\infty} (p_1)_{r_1} \frac{B^{\aleph}_{\rho_1}(p_2 + r_1, p_3 - p_2)}{B(p_2, p_3 - p_2)} \frac{x_1^{r_1}}{r_1!},$$

$$(\rho_1 \geq 0, Re(p_1) > 0, Re(p_2) > 0, Re(p_3) > 0, Re(\aleph) > 0).$$

With the integral representation as follows:

$$F^{\aleph}_{\rho_1}(p_1, p_2; p_3; x_1) = \frac{1}{B(p_2, p_3 - p_2)} \int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1}(1-tx_1)^{-p_1} \times exp\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right) dt,$$

$$(\rho_1 > 0; \rho_1 = 0, Re(p_3) > Re(p_2) > 0, Re(\aleph) > 0).$$

And

$$\Phi^{\aleph}_{\rho_1}(p_2; p_3; x_1) = \sum_{r_1=0}^{\infty} \frac{B^{\aleph}_{\rho_1}(p_2 + r_1, p_3 - p_2)}{B(p_2, p_3 - p_2)} \frac{x_1^{r_1}}{r_1!},$$

$$(\rho_1 \geq 0, Re(p_2) > 0, Re(p_3) > 0, Re(\aleph) > 0).$$

With the integral representation as follows:

$$\Phi^{\aleph}_{\rho_1}(p_2; p_3; x_1) = \frac{1}{B(p_2, p_3 - p_2)} \int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1} exp\left(tx_1 - \frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right) dt,$$

$$(\rho_1 > 0; \rho_1 = 0, Re(\aleph) > 0, Re(p_3) > Re(p_2) > 0).$$

In 2011, Ozegin and Ozarslan [12] presented the following extension of gamma and beta functions:

$$\Gamma^{(\rho_2,\rho_3)}_{\rho_1}(\omega_1) = \int_0^{\infty} t^{\omega_1-1} \Phi\left(\rho_2, \rho_3, -t - \frac{\rho_1}{t}\right) dt,$$

$$(Re(\omega_1) > 0, Re(\rho_1) > 0, Re(\rho_2) > 0, Re(\rho_3) > 0).$$

And

$$B^{(\rho_2,\rho_3)}_{\rho_1}(\omega_1, \omega_2) = \int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1} \Phi\left(\rho_2, \rho_3, -\frac{\rho_1}{(1-t)t}\right) dt, \tag{5}$$

$$(Re(\omega_1) > 0, Re(\omega_2) > 0, Re(\rho_1) > 0, Re(\rho_2) > 0, Re(\rho_3) > 0).$$

They [12] also introduced the following Gauss and confluent hypergeometric functions:

$$F^{(\rho_2,\rho_3)}_{\rho_1}(p_1, p_2; p_3; x_1) = \sum_{r_1=0}^{\infty} (p_1)_{r_1} \frac{B^{(\rho_2,\rho_3)}_{\rho_1}(p_2 + r_1, p_3 - p_2)}{B(p_2, p_3 - p_2)} \frac{x_1^{r_1}}{r_1!},$$

$$(\rho_1 \geq 0, Re(p_3) > 0, Re(p_2) > 0, Re(\rho_2) > 0, Re(\rho_3) > 0, |x_1| < 1).$$

With the integral representation as follows:

$$F^{(\rho_2,\rho_3)}_{\rho_1}(p_1, p_2; p_3; x_1) = \frac{1}{B(p_2, p_3 - p_2)} \int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1}(1-tx_1)^{-p_1} \times \Phi\left(\rho_2, \rho_3, -\frac{\rho_1}{(1-t)t}\right) dt,$$

$$(\rho_1 > 0; \rho_1 = 0, Re(p_3) > Re(p_2) > 0, |arg(1 - x_1)| < \pi).$$

And

$$\Phi_{\rho_1}^{(\rho_2,\rho_3)}(p_2;p_3;x_1)=\sum_{r_1=0}^{\infty}\frac{B_{\rho_1}^{(\rho_2,\rho_3)}(p_2+r_1,p_3-p_2)}{B(p_2,p_3-p_2)}\frac{x_1^{r_1}}{r_1!},$$

$$(\rho_1\geq 0, Re(p_3)>Re(p_2)>0, Re(\rho_2)>0, Re(\rho_3)>0).$$

With the integral representation as follows:

$$\Phi_{\rho_1}^{(\rho_2,\rho_3)}(p_2;p_3;x_1)=\frac{1}{B(p_2,p_3-p_2)}\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1}exp(tx_1)\times\Phi\left(\rho_2,\rho_3,-\frac{\rho_1}{(1-t)t}\right)dt,$$

$$(\rho_1>0;\rho_1=0, Re(p_3)>Re(p_2)>0).$$

In 2014, Liu [13] used generalized beta function in (5), to introduced extended Appell's and Lauricella's hypergeometric functions as follows:

$$F_{1,\rho_1}^{(\rho_2,\rho_3)}(p_1,p_2,p_3;p_4;x_1,x_2)=\sum_{r_1,r_2=0}^{\infty}\frac{B_{\rho_1}^{(\rho_2,\rho_3)}(p_1+r_1+r_2,p_4-p_1)}{B(p_1,p_4-p_1)}\times(p_2)_{r_1}(p_3)_{r_2}\frac{x_1^{r_1}}{r_1!}\frac{x_2^{r_2}}{r_2!},$$

$$(\max\{|x_1|,|x_2|\}<1).$$

$$F_{2,\rho_1}^{(\rho_2,\rho_3)}(p_1,p_2,p_3;p_4,p_5;x_1,x_2)=\sum_{r_1,r_2=0}^{\infty}(p_1)_{r_1+r_2}\frac{B_{\rho_1}^{(\rho_2,\rho_3)}(p_2+r_1,p_4-p_2)}{B(p_2,p_4-p_2)}\times\frac{B_{\rho_1}^{(\rho_2,\rho_3)}(p_3+r_2,p_5-p_3)}{B(p_3,p_3-p_3)}\frac{x_1^{r_1}}{r_1!}\frac{x_2^{r_2}}{r_2!},$$

$$(|x_1|+|x_2|<1).$$

And

$$F_{D,\rho_1}^{(3;\rho_2,\rho_3)}(p_1,p_2,p_3;p_4;x_1,x_2,x_3)=\sum_{r_1,r_2,r_3=0}^{\infty}\frac{B_{\rho_1}^{(\rho_2,\rho_3)}(p_1+r_1+r_2,p_4-p_1)}{B(p_1,p_4-p_1)}\times(p_2)_{r_1}(p_3)_{r_2}(p_4)_{r_3}\frac{x_1^{r_1}}{r_1!}\frac{x_2^{r_2}}{r_2!}\frac{x_3^{r_3}}{r_3!},$$

$$(|x_1|<1,|x_2|<1,|x_3|<1).$$

In 2013, Parmar [14] provided the following extension of gamma and beta functions:

$$\Gamma_{\rho_1}^{(\rho_2,\rho_3;\aleph)}(\omega_1)=\int_0^{\infty}t^{\omega_1-1}\Phi\left(\rho_2,\rho_3,-t-\frac{\rho_1}{t^{\aleph}}\right)dt,$$

$$(Re(\omega_1)>0, Re(\rho_1)>0, Re(\rho_2)>0, Re(\rho_3)>0, Re(\aleph)>0).$$

And

$$B_{\rho_1}^{(\rho_2,\rho_3;\aleph)}(\omega_1,\omega_2)=\int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1}\Phi\left(\rho_2,\rho_3,-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt, \tag{6}$$

$$(Re(\omega_1)>0, Re(\omega_2)>0, Re(\rho_1)>0, Re(\rho_2)>0, Re(\rho_3)>0, Re(\aleph)>0).$$

He [14] also introduced the following Gauss and confluent hypergeometric functions:

$$F_{\rho_1}^{(\rho_2,\rho_3;\aleph)}(p_1,p_2;p_3;x_1)=\sum_{r_1=0}^{\infty}(p_1)_{r_1}\frac{B_{\rho_1}^{(\rho_2,\rho_3;\aleph)}(p_2+r_1,p_3-p_2)}{B(p_2,p_3-p_2)}\frac{x_1^{r_1}}{r_1!},$$

$$(\rho_1\geq 0, Re(p_3)>Re(p_2)>0, Re(\rho_2)>0, Re(\rho_3)>0, Re(\aleph)>0, |x_1|<1).$$

With the integral representation as follows:

$$F_{\rho_1}^{(\rho_2,\rho_3;\aleph)}(p_1,p_2;p_3;x_1)=\frac{1}{B(p_2,p_3-p_2)}\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1}(1-tx_1)^{-p_1}$$
$$\times\Phi\left(\rho_2,\rho_3,-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt,$$

$$(\rho_1>0;\rho_1=0, Re(p_3)>Re(p_2)>0, Re(\aleph)>0, |arg(1-x_1)|<\pi).$$

And

$$\Phi_{\rho_1}^{(\rho_2,\rho_3;\aleph)}(p_2;p_3;x_1)=\sum_{r_1=0}^{\infty}\frac{B_{\rho_1}^{(\rho_2,\rho_3;\aleph)}(p_2+r_1,p_3-p_2)}{B(p_2,p_3-p_2)}\frac{x_1^{r_1}}{r_1!},$$

$$(\rho_1\geq 0, Re(p_3)>Re(p_2)>0, Re(\rho_2)>0, Re(\rho_3)>0, Re(\aleph)>0, |x_1|<1).$$

With the integral representation as follows:

$$\Phi_{\rho_1}^{(\rho_2,\rho_3)}(p_2;p_3;x_1)=\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1}exp(tx_1)\Phi\left(\rho_2,\rho_3,-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt,$$

$$(\rho_1>0;\rho_1=0, Re(p_3)>Re(p_2)>0, Re(\aleph)>0).$$

In 2015, Agarwal et al., [15] used extended beta function in (6), to introduced extended Appell's and Lauricella's hypergeometric functions as follows:

$$F_{1,\rho_1}^{(\rho_2,\rho_3;\aleph)}(p_1,p_2,p_3;p_4;x_1,x_2)=\sum_{r_1,r_2=0}^{\infty}\frac{B_{\rho_1}^{(\rho_2,\rho_3;\aleph)}(p_1+r_1+r_2,p_4-p_1)}{B(p_1,p_4-p_1)}$$
$$\times(p_2)_{r_1}(p_3)_{r_2}\frac{x_1^{r_1}}{r_1!}\frac{x_2^{r_2}}{r_2!},$$

$$(\max\{|x_1|,|x_2|\}<1).$$

$$F_{2,\rho_1}^{(\rho_2,\rho_3;\aleph)}(p_1,p_2,p_3;p_4,p_5;x_1,x_2)=\sum_{r_1,r_2=0}^{\infty}(p_1)_{r_1+r_2}\frac{B_{\rho_1}^{(\rho_2,\rho_3;\aleph)}(p_2+r_1,p_4-p_2)}{B(p_2,p_4-p_2)}$$
$$\times\frac{B_{\rho_1}^{(\rho_2,\rho_3;\aleph)}(p_3+r_2,p_5-p_3)}{B(p_3,p_3-p_3)}\frac{x_1^{r_1}}{r_1!}\frac{x_2^{r_2}}{r_2!},$$

$$(|x_1|+|x_2|<1).$$

And

$$F_{D,\rho_1}^{(3;\rho_2,\rho_3;\aleph)}(p_1,p_2,p_3;p_4;x_1,x_2,x_3)=\sum_{r_1,r_2,r_3=0}^{\infty}\frac{B_{\rho_1}^{(\rho_2,\rho_3;\aleph)}(p_1+r_1+r_2,p_4-p_1)}{B(p_1,p_4-p_1)}$$
$$\times(p_2)_{r_1}(p_3)_{r_2}(p_4)_{r_3}\frac{x_1^{r_1}}{r_1!}\frac{x_2^{r_2}}{r_2!}\frac{x_3^{r_3}}{r_3!},$$

$$(|x_1|<1,|x_2|<1,|x_3|<1).$$

In 2018, Ata [16] proposed the following extension of gamma and beta functions:

$$^{\psi}\Gamma_{\rho_1}^{(\rho_2,\rho_3)}(\omega_1)=\int_0^{\infty}t^{\omega_1-1}{}_1\psi_1\left(\rho_2,\rho_3,-t-\frac{\rho_1}{t}\right)dt,$$

$$(Re(\omega_1)>0,Re(\rho_1)>0,Re(\rho_2)>0,Re(\rho_3)>1).$$

And

$$^{\psi}B_{\rho_1}^{(\rho_2,\rho_3)}(\omega_1,\omega_2)=\int_0^{1}t^{\omega_1-1}(1-t)^{\omega_2-1}{}_1\psi_1\left(\rho_2,\rho_3,-\frac{\rho_1}{(1-t)t}\right)dt,$$

$$(Re(\omega_1)>0,Re(\omega_2)>0,Re(\rho_1)>0,Re(\rho_2)>0,Re(\rho_3)>1).$$

They [16] also introduced the following Gauss and confluent hypergeometric functions:

$$^{\psi}F_{\rho_1}^{(\rho_2,\rho_3)}(p_1,p_2;p_3;x_1)=\sum_{r_1=0}^{\infty}(p_1)_{r_1}\frac{^{\psi}B_{\rho_1}^{(\rho_2,\rho_3)}(p_2+r_1,p_3-p_2)}{B(p_2,p_3-p_2)}\frac{x_1^{r_1}}{r_1!},$$

$$(\rho_1\geq 0,Re(p_1)>0,Re(p_2)>0,Re(p_3)>0,Re(\rho_2)>0,Re(\rho_3)>1,|x_1|<1).$$

With the integral representation as follows:

$$^{\psi}F_{\rho_1}^{(\rho_2,\rho_3)}(p_1,p_2;p_3;x_1)=\frac{1}{B(p_2,p_3-p_2)}\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1}(1-tx_1)^{-p_1}$$
$$\times{}_1\psi_1\left(\rho_2,\rho_3,-\frac{\rho_1}{(1-t)t}\right)dt,$$

$$(\rho_1>0;\rho_1=0,Re(p_3)>Re(p_2)>0,|arg(1-x_1)|<\pi).$$

And

$$^{\psi}\Phi_{\rho_1}^{(\rho_2,\rho_3)}(p_2;p_3;x_1)=\sum_{r_1=0}^{\infty}\frac{^{\psi}B_{\rho_1}^{(\rho_2,\rho_3)}(p_2+r_1,p_3-p_2)}{B(p_2,p_3-p_2)}\frac{x_1^{r_1}}{r_1!},$$

$$(\rho_1\geq 0,Re(p_3)>Re(p_2)>0,Re(\rho_2)>0,Re(\rho_3)>1,|x_1|<1).$$

With the integral representation as follows:

$$^{\psi}\Phi_{\rho_1}^{(\rho_2,\rho_3)}(p_2;p_3;x_1)=\frac{1}{B(p_2,p_3-p_2)}\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1}exp(tx_1)$$
$$\times{}_1\psi_1\left(\rho_2,\rho_3,-\frac{\rho_1}{(1-t)t}\right)dt,$$

$$(\rho_1>0;\rho_1=0,Re(p_3)>Re(p_2)>0).$$

In 2020, Ata and Kiymaz [17] established the following extension of gamma and beta functions:

$$^{\Psi}\Gamma_{\rho_1}(\omega_1)={}^{\Psi}\Gamma_{\rho_1}\left[\begin{array}{cc}(\xi_i,\zeta_i)_{1,\gamma} & \\ & |\ \omega_1\\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right]$$
$$=\int_0^{\infty}t^{\omega_1-1}{}_{\eta_1}\Psi_{\eta_2}\left(-t-\frac{\rho_1}{t}\right)dt, \tag{7}$$

$$(Re(\omega_1)>0,Re(\rho_1)>0).$$

And

$$ {}^{\Psi}B_{\rho_1}^{(\rho_2,\rho_3)}(\omega_1,\omega_2) = {}^{\Psi}B_{\rho_1}\left[\begin{array}{cc}(\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid \omega_1,\omega_2 \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right] $$

$$ = \int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1}{}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)t}\right)dt, $$

$$ (Re(\omega_1)>0, Re(\omega_2)>0, Re(\rho_1)>0). $$

They [17] also introduced the following Gauss and confluent hypergeometric functions:

$$ {}^{\Psi}F_{\rho_1}(p_1,p_2;p_3;x_1) = {}^{\Psi}F_{\rho_1}\left[\begin{array}{cc}(\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid p_1,p_2;p_3;x_1 \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right] $$

$$ = \sum_{r_1=0}^{\infty}(p_1)_{r_1}\frac{{}^{\Psi}B_{\rho_1}(p_2+r_1,p_3-p_2)}{B(p_2,p_3-p_2)}\frac{x_1^{r_1}}{r_1!}, $$

$$ (\rho_1\geq 0, Re(p_3)>Re(p_2)>0). $$

With the integral representation as follows:

$$ {}^{\Psi}F_{\rho_1}(p_1,p_2;p_3;x_1) = \frac{1}{B(p_2,p_3-p_2)}\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1}(1-tx_1)^{-p_1} $$

$$ \times {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)t}\right)dt, $$

$$ (\rho_1>0;\rho_1=0, Re(p_3)>Re(p_2)>0, |arg(1-x_1)|<\pi). $$

And

$$ {}^{\Psi}\Phi_{\rho_1}(p_2;p_3;x_1) = {}^{\Psi}\Phi_{\rho_1}\left[\begin{array}{cc}(\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid p_2;p_3;x_1 \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right] $$

$$ = \sum_{r_1=0}^{\infty}\frac{{}^{\Psi}B_{\rho_1}(p_2+r_1,p_3-p_2)}{B(p_2,p_3-p_2)}\frac{x_1^{r_1}}{r_1!}, $$

$$ (\rho_1\geq 0, Re(p_3)>Re(p_2)>0, |x_1|<1). $$

With the integral representation as follows:

$$ {}^{\Psi}\Phi_{\rho_1}(p_2;p_3;x_1) = \frac{1}{B(p_2,p_3-p_2)}\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1}exp(tx_1) $$

$$ \times {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)t}\right)dt, $$

$$ (\rho_1>0;\rho_1=0, Re(p_3)>Re(p_2)>0). $$

Other forms and generalizations gamma, beta, hypergeometric and confluent hypergeometric functions, reader can refer to [18]-[47].
Throughout the rest of this research paper, we consider ω_1, ω_2, x_1, x_2, x_3, p_1, p_2, p_3, ρ_1, $\aleph$, ξ_i, ℓ_j, $\in\mathbb{C}$, ζ_i, $\varepsilon_j\in\mathbb{R}$, k, $n\in\mathbb{N}$, $Re(\omega_1)>0$, $Re(\omega_2)>0$, $Re(\rho_1)>0$,

$Re(\aleph) > 0$, $Re(p_1) > 0$, $Re(p_3) > Re(p_2) > 0$.
We defined the following generalization of beta function:

$$
\begin{aligned}
{}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2) &= {}^{\Psi}B^{\aleph}_{\rho_1}\left[\begin{array}{cc} (\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid \omega_1,\omega_2 \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right] \\
&= \int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1}{}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt, \qquad (8)
\end{aligned}
$$

2. Integral representations of the new generalized beta function

In this section, integral representations of the new generalized beta function are investigated.

Theorem 1 The following integral representations hold true.

$$
{}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2) = 2\int_0^{\frac{\pi}{2}} \sin^{2\omega_1-1}\phi\cos^{2\omega_2-1}\phi\ {}_{\eta_1}\Psi_{\eta_2}\left(-\rho_1\sec^{2\aleph}\phi\csc^{2\aleph}\phi\right)d\phi, \qquad (9)
$$

$$
{}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2) = \int_0^{\infty}\frac{t^{\omega_1-1}}{(1-t)^{\omega_2-1}}{}_{\eta_1}\Psi_{\eta_2}\left(-\rho_1(t^{-1}+2+t)^{\aleph}\right)dt, \qquad (10)
$$

$$
\begin{aligned}
{}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2) = (q-p)^{1-\omega_1-\omega_2}\int_p^q (t-p)^{\omega_1-1}(q-t)^{\omega_2-1} \\
\times\ {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1(q-p)^{2\aleph}}{(q-t)^{\aleph}(t-p)^{\aleph}}\right)dt. \qquad (11)
\end{aligned}
$$

Proof. Putting $t=\sin^2\phi$, $t=r(1+r)^{-1}$ and $t=(r-p)(q-p)^{-1}$ in (8) and change of variables, we obtained the desired result in (9), (10) and (11), respectively.

3. Differential formulas of the new generalized beta function

This section discussed differential formulas of the new generalized beta function

Theorem 2 The following differential formula holds.

$$
\frac{d^k}{d\rho_1^k}\ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2) = (-1)^k\ {}^{\Psi}B^{\aleph}_{\rho_1}\left[\begin{array}{cc} (\xi_i n+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid \omega_1-k\aleph,\omega_2-k\aleph \\ (\ell_j n+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right], \qquad (12)
$$

$$
(Re(\omega_1) > k\aleph, Re(\omega_2) > k\aleph).
$$

Proof. Using the principal of mathematical induction, we have

$$
\frac{d}{d\rho_1}\ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2) = \frac{d}{d\rho_1}\left\{\int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1}{}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt\right\}. \qquad (13)
$$

On simplification of (13), we obtain

$$
\frac{d}{d\rho_1}\ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2) = (-1)\ {}^{\Psi}B^{\aleph}_{\rho_1}\left[\begin{array}{cc} (\xi_i+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid \omega_1-\aleph,\omega_2-\aleph \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right]. \qquad (14)
$$

Assuming n^{th} order derivative is hold, then

$$
\frac{d^n}{d\rho_1^n}\ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2) = (-1)^n\ {}^{\Psi}B^{\aleph}_{\rho_1}\left[\begin{array}{cc} (\xi_i n+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid \omega_1-n\aleph,\omega_2-n\aleph \\ (\ell_j n+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right].
$$

The $(n+1)^{th}$ order derivative is as follows

$$\frac{d^{n+1}}{d\rho_1^{n+1}}\ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)=\frac{d}{d\rho_1}\left\{\frac{d^n}{d\rho_1^n}\ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)\right\}. \quad (15)$$

Applying (14) to (15) and simplification, yields

$$\frac{d^{n+1}}{d\rho_1^{n+1}}\ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)=(-1)^{n+1}$$
$$\times\ {}^{\Psi}B^{\aleph}_{\rho_1}\left[\begin{array}{c}(\xi_i n+\zeta_i,\xi_i)_{1,\gamma}\\ \\ (\ell_j n+\varepsilon_j,\ell_j)_{1,\lambda}\end{array}\mid \omega_1-(n+1)\aleph,\omega_2-(n+1)\aleph\right].$$

4. The Mellin transform of the new generalized beta function

The Mellin transform and inverse Mellin transform of the new generalized beta function is given below:

Theorem 3 The following Mellin transform formula hold.

$$\mathbf{M}\left\{{}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)\right\}=B(\omega_1+\aleph s,\omega_2+\aleph s)^{\Psi}\Gamma(s), \quad (16)$$
$$(Re(s)>0, Re(\omega_1+\aleph s)>0, Re(\omega_2+\aleph s)>0).$$

Proof. On using direct substituting

$$\mathbf{M}\left\{{}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)\right\}=\int_0^\infty \rho_1^{s-1}\ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)d\rho_1. \quad (17)$$

Putting (8) into (17), gives

$$\mathbf{M}\left\{{}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)\right\}=\int_0^\infty \rho_1^{s-1}$$
$$\times\left\{\int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1}{}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt\right\}d\rho_1. \quad (18)$$

Interchanging the order of integrations in (18), we have

$$\mathbf{M}\left\{{}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)\right\}=\int_0^1 t^{\omega_1-1}(1-t)^{\omega_2-1}$$
$$\times\left\{\int_0^\infty \rho_1^{s-1}{}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)d\rho_1\right\}dt. \quad (19)$$

Setting $\rho_1=u(1-t)^{\aleph}t^{\aleph}$ to the inner integral of (19), gives

$$\mathbf{M}\left\{{}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)\right\}=\int_0^1 t^{\omega_1+\aleph s-1}(1-t)^{\omega_2+\aleph s-1}$$
$$\times\left\{\int_0^\infty u^{s-1}{}_{\eta_1}\Psi_{\eta_2}(-u)\,d\rho_1\right\}dt. \quad (20)$$

Applying (1) and (7) to (20), we have

$$\mathbf{M}\left\{{}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)\right\}=B(\omega_1+\aleph s,\omega_2+\aleph s)^{\Psi}\Gamma(s).$$

Theorem 4 The following inverse Mellin transform hold true.

$${}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}B(\omega_1+\aleph s,\omega_2+\aleph s)^{\Psi}\Gamma(s)\rho_1^{-s}ds, \quad (21)$$

$$(\sigma > 0, Re(s) > 0, Re(\omega_1 + \aleph s) > 0, Re(\omega_2 + \aleph s) > 0).$$

Proof. On taking Mellin inversion of Theorem 3, we obtain required result in (21).

5. Summation formulas of the new generalized beta function

Summations and other related formulas are introduced in the following theorems:

Theorem 5 The following formula holds.

$$^{\Psi}B^{\aleph}_{\rho_1}(\omega_1, \omega_2) = {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1 + 1, \omega_2) + {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1, \omega_2 + 1). \tag{22}$$

Proof. By direct calculation

$$^{\Psi}B^{\aleph}_{\rho_1}(\omega_1, \omega_2) = \int_0^1 t^{\omega_1}(1-t)^{\omega_2}\left\{t^{-1} + (1-t)^{-1}\right\} {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right) dt. \tag{23}$$

On simplification of (23), we have

$$^{\Psi}B^{\aleph}_{\rho_1}(\omega_1, \omega_2) = \int_0^1 t^{\omega_1}(1-t)^{\omega_2 - 1} {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right) dt + \int_0^1 t^{\omega_1 - 1}(1-t)^{\omega_2} {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right) dt. \tag{24}$$

Applying (8) to (24), we obtain

$$^{\Psi}B^{\aleph}_{\rho_1}(\omega_1, \omega_2) = {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1 + 1, \omega_2) + {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1, \omega_2 + 1).$$

Theorem 6 The following summation formula holds.

$$^{\Psi}B^{\aleph}_{\rho_1}(\omega_1, 1 - \omega_2) = \sum_{r_1=0}^{\infty} \frac{(\omega_2)_{r_1}}{r_1} \; {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1 + r_1, 1), \; (Re(\omega_2) < 1). \tag{25}$$

Proof. By direct calculation

$$^{\Psi}B^{\aleph}_{\rho_1}(\omega_1, 1 - \omega_2) = \int_0^1 t^{\omega_1 - 1}(1-t)^{-\omega_2} {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right) dt. \tag{26}$$

Applying (3) to (26), we get

$$^{\Psi}B^{\aleph}_{\rho_1}(\omega_1, 1 - \omega_2) = \int_0^1 t^{\omega_1 - 1} \sum_{r_1=0}^{\infty} \frac{(\omega_2)_{r_1}}{r_1} t^{r_1} {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right) dt. \tag{27}$$

Interchanging the order of summation and integration in (27), yield

$$^{\Psi}B^{\aleph}_{\rho_1}(\omega_1, 1 - \omega_2) = \sum_{r_1=0}^{\infty} \frac{(\omega_2)_{r_1}}{r_1} \int_0^1 t^{\omega_1 + r_1 - 1} {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right) dt. \tag{28}$$

Applying (8) to (28), we get

$$^{\Psi}B^{\aleph}_{\rho_1}(\omega_1, 1 - \omega_2) = \sum_{r_1=0}^{\infty} \frac{(\omega_2)_{r_1}}{r_1} \; {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1 + r_1, 1).$$

Theorem 7 The following summation formula holds.

$$^{\Psi}B^{\aleph}_{\rho_1}(\omega_1, \omega_2) = \sum_{r_1=0}^{\infty} \; {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1 + r_1, \omega_2 + 1). \tag{29}$$

Proof. By direct calculation

$$ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)=\int_0^1 t^{\omega_1-1}(1-t)^{\omega_2}(1-t)^{-1}{}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt. \tag{30} $$

Applying (3) to (30), we get

$$ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)=\int_0^1 t^{\omega_1-1}(1-t)^{\omega_2}\sum_{r_1=0}^{\infty}t^{r_1}{}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt. \tag{31} $$

Interchanging the order of summation and integration in (31), yield

$$ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)=\sum_{r_1=0}^{\infty}\int_0^1 t^{\omega_1+r_1-1}(1-t)^{\omega_2}\ {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt. \tag{32} $$

Applying (8) to (32), we get

$$ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)=\sum_{r_1=0}^{\infty}\ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1+r_1,\omega_2+1). $$

Theorem 8 The following formula holds.

$$ \begin{aligned} \omega_1\ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2+1)&=2\rho_1\aleph\ {}^{\Psi}B^{\aleph}_{\rho_1}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array}\mid\omega_1+1-\aleph,\omega_2-\aleph\right]\\ &+\omega_2\ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1+1,\omega_2)-2\rho_1\aleph\ {}^{\Psi}B^{\aleph}_{\rho_1}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array}\mid\omega_1-\aleph,\omega_2-\aleph\right], \end{aligned} \tag{33} $$

$$ (Re(\omega_1)>\aleph, Re(\omega_2)>\aleph). $$

Proof. On using the following equality:

$$ {}^{\Psi}B^{\aleph}_{\rho_1}(\omega_1,\omega_2)=\mathbf{M}\left\{f^{\aleph}(t:\omega_2;\rho_1):\omega_1\right\}. $$

Where

$$ f^{\aleph}(t:\omega_2;\rho_1)=(1-t)^{\omega_2-1}H(1-t){}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right). \tag{34} $$

And $H(1-t)$ is Heaviside delta (see for details [47] and [48]).
Differentiating (34) with respect to t, we have

$$ \begin{aligned} \frac{d}{dt}f^{\aleph}(t:\omega_2;\rho_1)&=-(\omega_2-1)(1-t)^{\omega_2-2}H(1-t)\ {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)\\ &-(1-t)^{\omega_2-1}\delta(1-t)\ {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)-\rho_1\aleph(2t-1)t^{-\aleph-1}\\ &\times(1-t)^{\omega_2-\aleph-2}H(1-t)\ {}_{\eta_1}\Psi_{\eta_2}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array}\mid-\tfrac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right], \end{aligned} \tag{35} $$

where $\delta(1-t)$ is Dirac delta (see [17]).
On simplification of (35), we obtain

$$-(\omega_1-1)\ {}^{\Psi}B_{\rho_1}^{\aleph}(\omega_1-1,\omega_2) = {}^{\Psi}B_{\rho_1}^{\aleph}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array} \mid \omega_1-\aleph-1,\omega_2-\aleph-1\right]$$
$$\times\rho_1\aleph - 2\rho_1\aleph\ {}^{\Psi}B_{\rho_1}^{\aleph}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array} \mid \omega_1-\aleph,\omega_2-\aleph\right]$$
$$-(\omega_2-1)\ {}^{\Psi}B_{\rho_1}^{\aleph}(\omega_1,\omega_2-1). \quad (36)$$

On setting $\omega_1 \to \omega_1+1$ and $\omega_2 \to \omega_2+1$ in (36), give the desired result in (33).

6. Beta distribution of the new generalized beta function

Considering the new generalized beta function, we defined the following beta distribution:

$$f(t) = \begin{cases} \frac{1}{{}^{\Psi}B_{\rho_1}^{\aleph}(h_1,h_2)} t^{h_1-1}(1-t)^{h-1}{}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt, & (0<t<1),\\ 0, & \text{otherwise,}\end{cases} \quad (37)$$

$$(h_1,h_2\in\mathbb{R}).$$

The moment of X is as given below:

$$E(X^{\nu}) = \frac{{}^{\Psi}B_{\rho_1}^{\aleph}(h_1+\nu,h_2)}{{}^{\Psi}B_{\rho_1}^{\aleph}(h_1,h_2)},\ (h_1,h_2\in\mathbb{R}). \quad (38)$$

On setting $\nu=1$ in (38), we obtain the mean as follow:

$$\mu = E(X) = \frac{{}^{\Psi}B_{\rho_1}^{\aleph}(h_1+1,h_2)}{{}^{\Psi}B_{\rho_1}^{\aleph}(h_1,h_2)}.$$

The variance is as follow:

$$\delta = E(X^2)-\{E(X)\}^2 = \frac{{}^{\Psi}B_{\rho_1}^{\aleph}(h_1,h_2)\ {}^{\Psi}B_{\rho_1}^{\aleph}(h_1+2,h_2)-\{{}^{\Psi}B_{\rho_1}^{\aleph}(h_1,h_2)\}^2}{\{{}^{\Psi}B_{\rho_1}^{\aleph}(h_1,h_2)\}^2}$$

The moment generating function of the distribution in (37), is as follows:

$$M(t) = \sum_{r_1=0}^{\infty} E(X^{r_1})\frac{t^{r_1}}{r_1!} = \frac{1}{{}^{\Psi}B_{\rho_1}^{\aleph}(h_1,h_2)}\sum_{r_1=0}^{\infty} {}^{\Psi}B_{\rho_1}^{\aleph}(h_1+r_1,h_2)\frac{t^{r_1}}{r_1!}.$$

The commutative distribution of (37), is as follows:

$$F(x) = \frac{{}^{\Psi}B_{\rho_1,z}^{\aleph}(h_1+1,h_2)}{{}^{\Psi}B_{\rho_1}^{\aleph}(h_1,h_2)},$$

where

$${}^{\Psi}B_{\rho_1,z}^{\aleph}(h_1,h_2) = \int_0^z t^{h_1-1}(1-t)^{h_2-1}{}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt,$$

is the new extended incomplete beta function.

7. Gauss and confluent hypergeometric functions

The newly introduced generalized Gauss and confluent hypergeometric functions are given as follow:

$$ {}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = {}^{\Psi}F_{\rho_1}^{\aleph}\left[\begin{array}{cc} (\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid p_1,p_2;p_3;x_1 \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right] = \sum_{r_1=0}^{\infty}(p_1)_{r_1}\frac{{}^{\Psi}B_{\rho_1}^{\aleph}(p_2+r_1,p_3-p_2)}{B(p_2,p_3-p_2)}\frac{x_1^{r_1}}{r_1!}. \tag{39} $$

And

$$ {}^{\Psi}\Phi_{\rho_1}^{\aleph}(p_2;p_3;x_1) = {}^{\Psi}\Phi_{\rho_1}^{\aleph}\left[\begin{array}{cc} (\xi_i,\zeta_i)_{1,\gamma} & \\ & \mid p_2;p_3;x_1 \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right] = \sum_{r_1=0}^{\infty}\frac{{}^{\Psi}B_{\rho_1}^{\aleph}(p_2+r_1,p_3-p_2)}{B(p_2,p_3-p_2)}\frac{x_1^{r_1}}{r_1!}, \tag{40} $$

Theorem 9 The following integral formula holds.

$$ {}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = \frac{1}{B(p_2,p_3-p_2)}\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1} \times (1-tx_1)^{-p_1}\ {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt. \tag{41} $$

Proof. By direct calculation using (39), we have

$$ {}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = \sum_{r_1=0}^{\infty}\left\{\int_0^1 t^{p_2+r_1-1}(1-t)^{p_3-p_2-1}\right\} \times \frac{1}{B(p_2,p_3-p_2)}(p_1)_{r_1}\frac{x_1^{r_1}}{r_1!}. \tag{42} $$

Interchanging the order of summation and integration in (42), yields

$$ {}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = \frac{1}{B(p_2,p_3-p_2)}\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1} \times \sum_{r_1=0}^{\infty}(p_1)_{r_1}\frac{x_1^{r_1}}{r_1!}\ {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt. \tag{43} $$

Applying (3) to (43), gives

$$ {}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = \frac{1}{B(p_2,p_3-p_2)}\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1} \times (1-tx_1)^{-p_1}\ {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt. $$

Theorem 10 The following integral formulas hold.

$$ {}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = \frac{2}{B(p_2,p_3-p_2)}\int_0^1 \frac{\sin^{2p_2-1}\phi\cos^{2(p_3+p_2)-1}\phi}{(1-x_1\sin^2\phi)^{p_1}} \times {}_{\eta_1}\Psi_{\eta_2}\left(-\rho_1\sec^{2\aleph}\phi\csc^{2\aleph}\phi\right)dt, \quad (44)$$

and

$$ {}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = \frac{1}{B(p_2,p_3-p_2)}\int_0^\infty \frac{t^{p_2-1}}{(1+t)^{p_3-p_1}} \times \{1+t(1-x_1)\}^{-p_1}\,{}_{\eta_1}\Psi_{\eta_2}\left(-\rho_1(t^{-1}+2+t)^{\aleph}\right)dt. \quad (45)$$

Proof. Setting $t=sin^2\phi$, and $t=s(1+s)^{-1}$ in (41) and change of variables, we obtain the desired result in (44) and (45), respectively.
Theorem 11 The following integral formulas hold.

$$ {}^{\Psi}\Phi_{\rho_1}^{\aleph}(p_2;p_3;x_1) = \frac{1}{B(p_2,p_3-p_2)}\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1}exp(tx_1) \times {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt, \quad (46)$$

and

$$ {}^{\Psi}\Phi_{\rho_1}^{\aleph}(p_2;p_3;x_1) = \frac{exp(x_1)}{B(p_2,p_3-p_2)}\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1}exp(tx_1) \times {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt. \quad (47)$$

Proof. Equation (46) and (47) follows from Theorem 10.

8. Differential formulas for the new generalized Gauss and confluent hypergeometric functions

This section considered differential formulas for the generalized Gauss and confluent hypergeometric functions.
Theorem 12 The following differential formulas hold.

$$\frac{d}{dx_1}\,{}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = \frac{p_1p_2}{p_3}\,{}^{\Psi}F_{\rho_1}^{\aleph}(p_1+1,p_2+1;p_3+1;x_1), \quad (48)$$

and

$$\frac{d^k}{dx_1^k}\,{}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = \frac{(p_1)_k(p_2)_k}{(p_3)_k}\,{}^{\Psi}F_{\rho_1}^{\aleph}(p_1+k,p_2+k;p_3+k;x_1). \quad (49)$$

Proof. Using (39), we have

$$\frac{d}{dx_1}\,{}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = \sum_{r_1=1}^{\infty}(p_1)_{r_1}\frac{{}^{\Psi}B_{\rho_1}^{\aleph}(p_2+r_1,p_3-p_2)}{B(p_2,p_3-p_2)}\frac{x_1^{(r_1-1)}}{(r_1-1)!}. \quad (50)$$

On setting $r_1 \to r_1+1$, in (51) and applying (2), we get the desired result in (48). On successive differentiation of (48), we obtained (49).
Theorem 13 The following differential formulas hold.

$$\frac{d}{dx_1}\,{}^{\Psi}\Phi_{\rho_1}^{\aleph}(p_2;p_3;x_1) = \frac{p_2}{p_3}\,{}^{\Psi}\Phi_{\rho_1}^{\aleph}(p_2+1;p_3+1;x_1), \quad (51)$$

and

$$\frac{d^k}{dx_1^k}\,{}^{\Psi}\Phi^{\aleph}_{\rho_1}(p_2;p_3;x_1)=\frac{(p_2)_k}{(p_3)_k}\;{}^{\Psi}\Phi^{\aleph}_{\rho_1}(p_2+k;p_3+k;x_1). \tag{52}$$

Proof. Following similar argument as in Theorem 12, we have the required results in (51) and (52) .

Theorem 14 The following equality hold true.

$$\begin{aligned}(p_2-1)B(p_2-1,p_3-p_2+1)\;{}^{\Psi}F^{\aleph}_{\rho_1}(p_1,p_2-1;p_3;x_1)&=(p_3-p_2-1)B(p_2,p_3-p_2-1)\\ \times\;{}^{\Psi}F^{\aleph}_{\rho_1}(p_1,p_2-1;p_3;x_1)-p_1x_1B(p_2,p_3-p_2)\;{}^{\Psi}F^{\aleph}_{\rho_1}(p_1+1,p_2;p_3;x_1)-\rho_1\aleph\times\\ B(p_2-\aleph-1,p_3-p_2-\aleph-1)\;{}^{\Psi}F^{\aleph}_{\rho_1}\left[\begin{array}{ll}(\xi_i+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid p_1,p_2-\aleph-1;p_3-2\aleph-2;x_1\\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right]\\ +B(p_2-\aleph,p_3-p_2-\aleph-1)\,{}^{\Psi}F^{\aleph}_{\rho_1}\left[\begin{array}{ll}(\xi_i+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid p_1,p_2-\aleph-1;p_3-2\aleph-1;x_1\\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right]\\ \times 2\rho_1\aleph,\end{aligned} \tag{53}$$

$$(Re(p_2)>Re(\aleph+1),Re(p_3)>Re(p_2+\aleph+1)).$$

Proof. Using the following formula

$$B(p_2,p_3-p_2)\,{}^{\Psi}F^{\aleph}_{\rho_1}(p_1,p_2;p_3;x_1)=\mathbf{M}\left\{f^{\aleph}_{p_1,p_2,p_3}(t:x_1;\rho_1):p_2\right\}.$$

Where

$$\begin{aligned}f^{\aleph}_{p_1,p_2,p_3}(t:x_1;\rho_1)&=(1-t)^{p_3-p_2-1}(1-tx_1)^{-p_1}H(1-t)\\ &\times\;{}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right).\end{aligned} \tag{54}$$

Differentiating (54) with respect to t, we obtain

$$\begin{aligned}\frac{d}{dt}f^{\aleph}_{p_1,p_2,p_3}(t:x_1;\rho_1)&=-(p_3-p_2-1)(1-t)^{p_3-p_2-2}(1-tx_1)^{-p_1}H(1-t)\\ &\times\;{}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)+p_1x_1(1-t)^{p_3-p_2-1}(1-tx_1)^{-p_1-1}H(1-t)\\ &\times\;{}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)-(1-t)^{p_3-p_2-1}(1-tx_1)^{-p_1}\delta(1-t)\\ &\times\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)+\rho_1\aleph t^{-\aleph-1}(1-t)^{p_3-p_2-\aleph-2}(1-tx_1)^{-p_1}H(1-t)\\ &\times\;{}_{\eta_1}\Psi_{\eta_2}\left[\begin{array}{ll}(\xi_i+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid -\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right]-2\rho_1\aleph t^{-\aleph}(1-t)^{p_3-p_2-\aleph-1}\\ &\times(1-tx_1)^{-p_1}H(1-t)\left[\begin{array}{ll}(\xi_i+\zeta_i,\xi_i)_{1,\gamma} & \\ & \mid -\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda} & \end{array}\right].\end{aligned} \tag{55}$$

On simplification of (55), we get

$$-(p_2-1)B(p_2-1,p_3-p_2+1)\ {}^{\Psi}F^{\aleph}_{\rho_1}(p_1,p_2-1;p_3;x_1) = -(p_3-p_2-1)B(p_2,p_3-p_2-1)$$
$$\times\ {}^{\Psi}F^{\aleph}_{\rho_1}(p_1,p_2-1;p_3;x_1) + p_1x_1B(p_2,p_3-p_2)\ {}^{\Psi}F^{\aleph}_{\rho_1}(p_1+1,p_2;p_3;x_1) + \rho_1\aleph\times$$
$$B(p_2-\aleph-1,p_3-p_2-\aleph-1)\ {}^{\Psi}F^{\aleph}_{\rho_1}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i\varrho)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array} \mid p_1,p_2-\aleph-1;p_3-2\aleph-2;x_1\right]$$
$$-B(p_2-\aleph,p_3-p_2-\aleph-1)^{\Psi}F^{\aleph}_{\rho_1}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i\varrho)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array} \mid p_1,p_2-\aleph-1;p_3-2\aleph-1;x_1\right]$$
$$\times 2\rho_1\aleph. \quad (56)$$

On simplifying (56), we get the desired result in (53).

Theorem 15 The following equality hold true.

$$(p_2-1)B(p_2-1,p_3-p_2+1)\ {}^{\Psi}\Phi^{\aleph}_{\rho_1}(p_2-1;p_3;x_1) = (p_3-p_2-1)B(p_2,p_3-p_2-1)$$
$$\times\ {}^{\Psi}\Phi^{\aleph}_{\rho_1}(p_2-1;p_3;x_1) - x_1B(p_2,p_3-p_2)\ {}^{\Psi}\Phi^{\aleph}_{\rho_1}(p_2;p_3;x_1) - \rho_1\aleph\times$$
$$B(p_2-\aleph-1,p_3-p_2-\aleph-1)\ {}^{\Psi}\Phi^{\aleph}_{\rho_1}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array} \mid p_2-\aleph-1;p_3-2\aleph-2;x_1\right]$$
$$+B(p_2-\aleph,p_3-p_2-\aleph-1)^{\Psi}\Phi^{\aleph}_{\rho_1}\left[\begin{array}{c}(\xi_i+\zeta_i,\xi_i)_{1,\gamma}\\ \\ (\ell_j+\varepsilon_j,\ell_j)_{1,\lambda}\end{array} \mid p_2-\aleph-1;p_3-2\aleph-1;x_1\right]$$
$$\times 2\rho_1\aleph, \quad (57)$$

$$(Re(p_2) > Re(\aleph+1), Re(p_3) > Re(p_2+\aleph+1)).$$

Proof. On using the following formula

$$B(p_2,p_3-p_2)^{\Psi}\Phi^{\aleph}_{\rho_1}(p_2;p_3;x_1) = \mathbf{M}\left\{f^{\aleph}_{p_2,p_3}(t:x_1;\rho_1):p_2\right\}. \quad (58)$$

Where

$$f^{\aleph}_{p_2,p_3}(t:x_1;\rho_1) = (1-t)^{p_3-p_2-1}exp(tx_1)H(1-t)_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right). \quad (59)$$

Differentiating (59) with respect t and following similar simplification as in Theorem 14, we get the desired result in (57).

9. The Mellin transform for the extended Gauss and confluent hypergeometric functions

This section investigated the Mellin transform and inverse Mellin transform for the generalized Gauss and confluent hypergeometric functions.

Theorem 16 The following Mellin transform formula hold.

$$\mathbf{M}\left\{{}^{\Psi}F^{\aleph}_{\rho_1}(p_1,p_2;p_3;x_1)\right\} = \frac{B(p_2+\aleph s,p_3+\aleph s-p_2)^{\Psi}\Gamma(s)}{B(p_2,p_3-p_2)}$$
$$\times\ {}_2F_1(p_1,p_2+\aleph s;p_3+2\aleph s;x_1), \quad (60)$$

$$(Re(s)>0, Re(p_2+\aleph s)>0, Re(p_3+\aleph s-p_2)>0).$$

Proof. By direct calculation

$$\mathbf{M}\left\{{}^{\Psi}F^{\aleph}_{\rho_1}(p_1,p_2;p_3;x_1)\right\}=\int_0^{\infty}\rho_1^{s-1}\ {}^{\Psi}F^{\aleph}_{\rho_1}(p_1,p_2;p_3;x_1)d\rho_1. \tag{61}$$

Putting (39) into (61), gives

$$\mathbf{M}\left\{{}^{\Psi}F^{\aleph}_{\rho_1}(p_1,p_2;p_3;x_1)\right\}=\frac{1}{B(p_2,p_3-p_2)}\int_0^{\infty}\rho_1^{s-1}\sum_{r_1=0}^{\infty}(p_1)_{r_1}\frac{x_1^{r_1}}{r_1!}$$
$$\times\left\{\int_0^1 t^{p_2+r_1-1}(1-t)^{p_3-p_2-1}{}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt\right\}d\rho_1. \tag{62}$$

Interchanging the order of integrations in (64) and simplification, we have

$$\mathbf{M}\left\{{}^{\Psi}F^{\aleph}_{\rho_1}(p_1,p_2;p_3;x_1)\right\}=\frac{1}{B(p_2,p_3-p_2)}\int_0^{1}t^{p_2-1}(1-t)^{p_3-p_2-1}(1-tx_1)^{-p_1}$$
$$\times\left\{\int_0^{\infty}\rho_1^{s-1}{}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)d\rho_1\right\}dt. \tag{63}$$

Setting $\rho_1=u(1-t)^{\aleph}t^{\aleph}$ in (63), gives

$$\mathbf{M}\left\{{}^{\Psi}F^{\aleph}_{\rho_1}(p_1,p_2;p_3;x_1)\right\}=\frac{1}{B(p_2,p_3-p_2)}\int_0^{1}t^{p_2+\aleph s-1}(1-t)^{p_3+\aleph s-p_2-1}$$
$$\times(1-tx_1)^{p_1}\left\{\int_0^{\infty}u^{s-1}\ {}_{\eta_1}\Psi_{\eta_2}\left(-u\right)d\rho_1\right\}dt. \tag{64}$$

Applying (1) and (7) to (64), we have

$$\mathbf{M}\left\{{}^{\Psi}F^{\aleph}_{\rho_1}(p_1,p_2;p_3;x_1)\right\}=\frac{B(p_2+\aleph s,p_3+\aleph s-p_2)^{\Psi}\Gamma(s)}{B(p_2,p_3-p_2)}$$
$$\times F(p_1,p_2+\aleph s,p_3+2\aleph s,x_1),$$

Theorem 17 The following inverse Mellin transform formula hold true.

$${}^{\Psi}F^{\aleph}_{\rho_1}(p_1,p_2;p_3;x_1)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{B(p_2+\aleph s,p_3+\aleph s-p_2)^{\Psi}\Gamma(s)}{B(p_2,p_3-p_2)}$$
$$\times{}_2F_1(p_1,p_2+\aleph s;p_3+2\aleph s;x_1)\rho_1^{-s}ds, \tag{65}$$

$$(\sigma>0,Re(s)>0,Re(p_2+\aleph s)>0,Re(p_3+\aleph s-p_2)>0).$$

Proof. Applying inverse Mellin transform to Theorem 16, give the required result in (65).

Theorem 18 The following Mellin transform formula hold.

$$\mathbf{M}\left\{{}^{\Psi}\Phi^{\aleph}_{\rho_1}(p_2;p_3;x_1)\right\}=\frac{B(p_2+\aleph s,p_3+\aleph s-p_2)^{\Psi}\Gamma(s)}{B(p_2,p_3-p_2)}$$
$$\times\Phi(p_2+\aleph s;p_3+2\aleph s;x_1), \tag{66}$$

$$(Re(s)>0,Re(p_2+\aleph s)>0,Re(p_3+\aleph s-p_2)>0).$$

Proof. Following similar argument as in Theorem 16, we get the required result.
Theorem 19 The following inverse Mellin transform hold true.

$$ {}^{\Psi}\Phi_{\rho_1}^{\aleph}(p_2;p_3;x_1) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{B(p_2+\aleph s, p_3+\aleph s-p_2)^{\Psi}\Gamma(s)}{B(p_2,p_3-p_2)} \times \Phi(p_2+\aleph s;p_3+2\aleph s;x_1)\rho_1^{-s}ds, \quad (67) $$

$$ (\sigma > 0, Re(s) > 0, Re(p_2+\aleph s) > 0, Re(p_3+\aleph s-p_2) > 0). $$

Proof. Taking inverse Mellin transform to Theorem 18, give the required result in (67).

10. Transformation formulas for the generalized Gauss and confluent hypergeometric functions

In this section difference formulas for the generalized Gauss and confluent hypergeometric functions are formulated.
Theorem 20 The following transformation formulas hold.

$$ {}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = (1-x_1)^{-p_1}\ {}^{\Psi}F_{\rho_1}^{\aleph}\left(p_1,p_2;p_3;\frac{x_1}{x_1-1}\right), \quad (68) $$

$$ {}^{\Psi}\Phi_{\rho_1}^{\aleph}(p_2;p_3;x_1) = exp(x_1)^{\Psi}\Phi_{\rho_1}^{\aleph}(p_3-p_2;p_3;-x_1). \quad (69) $$

Proof. On setting $t \to 1+t$ in (41), (46) and algebraic simplifications, we obtain the required results in (69) and (70), respectively.

11. Difference formulas for the generalized Gauss and confluent hypergeometric functions

In this section difference formulas for the generalized Gauss and confluent hypergeometric functions are formulated.
Theorem 21 The following difference formulas hold.

$$ \Delta_{p_1}{}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = \frac{p_2x_1}{p_3}\ {}^{\Psi}F_{\rho_1}^{\aleph}(p_1+1,p_2+1;p_3+1;x_1), \quad (70) $$

$$ p_1\Delta_{p_1}{}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = x_1\frac{d}{dx_1}\ {}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1), \quad (71) $$

$$ p_2\Delta_{p_2}\ {}^{\Psi}\Phi_{\rho_1}^{\aleph}(p_2;p_3+1;x_1) = -p_3\Delta_{p_3}{}^{\Psi}\Phi_{\rho_1}^{\aleph}(p_2;p_3;x_1), \quad (72) $$

$$ \frac{d}{dx_1}\ {}^{\Psi}\Phi_{\rho_1}^{\aleph}(p_2;p_3;x_1) = \frac{p_2}{p_3}\ {}^{\Psi}\Phi_{\rho_1}^{\aleph}(p_2;p_3+1;x_1) - \Delta_{p_3}{}^{\Psi}\Phi_{\rho_1}^{\aleph}(p_2;p_3;x_1). \quad (73) $$

Proof. By direct calculation

$$ \Delta_{p_1}\ {}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = {}^{\Psi}F_{\rho_1}^{\aleph}(p_1+1,p_2;p_3;x_1) - {}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1). \quad (74) $$

On simplification of (74), yields

$$ \Delta_{p_1}\ {}^{\Psi}F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = \frac{x_1}{B(p_2,p_3-p_2)}\int_0^1 t^{p_2-1}(1-t)^{p_3-p_2-1} \times (1-tx_1)^{-p_1}\ {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt. \quad (75) $$

And

$$\Delta_{p_1}\ {}^{\Psi}F^{\aleph}_{\rho_1}(p_1+1,p_2+1;p_3+1;x_1) = \frac{x_1}{B(p_2+1,p_3-p_2)}\int_0^1 t^{p_2}(1-t)^{p_3-p_2-1} \times (1-tx_1)^{-p_1-1}\ {}_{\eta_1}\Psi_{\eta_2}\left(-\frac{\rho_1}{(1-t)^{\aleph}t^{\aleph}}\right)dt. \quad (76)$$

Using equation (2) and (76) in (75), we obtain (70). Using differential formula in (51), we get (71). Applying differential operator to (46), we obtain (72). Using differential formula in (51) and (7), we have (75).

12. Conclusion and recommendation

In this research paper, we introduced and investigated new generalized beta function, we also gave certain of its properties such as integral representations, differential formulas, difference formulas, Mellin transform, Mellin inversion formula and summation formulas. We also gave some statistical applications by introducing beta distribution and its corresponding mean, variance and moment generating function. The following particular cases can be drawn from the new introduced generalized beta, Gauss and confluent hypergeometric functions if the parameters are replaced appropriately:

For $\rho_1 = 0$ and $\aleph = 1$, then

$$B(\omega_1,\omega_2) = {}^{\Psi}B^{1}_{0}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & \mid \omega_1,\omega_2 \\ (1,1)_{1,1} & \end{array}\right],$$

$${}_2F_1(p_1,p_2;p_3;x_1) = {}^{\Psi}F^{1}_{0}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & \mid p_1,p_2;p_3;x_1 \\ (1,1)_{1,1} & \end{array}\right],$$

And

$$\Phi(p_2;p_3;x_1) = {}^{\Psi}\Phi^{1}_{0}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & \mid p_2;p_3;x_1 \\ (1,1)_{1,1} & \end{array}\right].$$

For $\rho_1 \neq 0$ and $\aleph = 1$, then

$$B_{\rho_1}(\omega_1,\omega_2) = {}^{\Psi}B^{1}_{\rho_1}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & \mid \omega_1,\omega_2 \\ (1,1)_{1,1} & \end{array}\right],$$

$$F_{\rho_1}(p_1,p_2;p_3;x_1) = {}^{\Psi}F^{1}_{\rho_1}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & \mid p_1,p_2;p_3;x_1 \\ (1,1)_{1,1} & \end{array}\right],$$

And

$$\Phi_{\rho_1}(p_2;p_3;x_1) = {}^{\Psi}\Phi^{1}_{\rho_1}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & \mid p_2;p_3;x_1 \\ (1,1)_{1,1} & \end{array}\right].$$

For $\rho_1 \neq 0$ and $\aleph \neq 1$, then

$$B^{\aleph}_{\rho_1}(\omega_1,\omega_2) = {}^{\Psi}B^{\aleph}_{\rho_1}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & \mid \omega_1,\omega_2 \\ (1,1)_{1,1} & \end{array}\right],$$

$$F_{\rho_1}^{\aleph}(p_1,p_2;p_3;x_1) = {}^{\Psi}F_{\rho_1}^{\aleph}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & | \; p_1,p_2;p_3;x_1 \\ (1,1)_{1,1} & \end{array}\right],$$

And

$$\Phi_{\rho_1}^{\aleph}(p_2;p_3;x_1) = {}^{\Psi}\Phi_{\rho_1}^{\aleph}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & | \; p_2;p_3;x_1 \\ (1,1)_{1,1} & \end{array}\right].$$

Again, if $\rho_1 \neq 0$ and $\aleph = 1$, then

$$B_{\rho_1}^{(\rho_2,\rho_3)}(\omega_1,\omega_2) = \frac{\Gamma(\rho_3)}{\Gamma(\rho_2)}{}^{\Psi}B_{\rho_1}^{1}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & | \; \omega_1,\omega_2 \\ (1,1)_{1,1} & \end{array}\right],$$

$$F_{\rho_1}^{(\rho_2,\rho_3)}(p_1,p_2;p_3;x_1) = \frac{\Gamma(\rho_3)}{\Gamma(\rho_2)}{}^{\Psi}F_{\rho_1}^{1}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & | \; p_1,p_2;p_3;x_1 \\ (1,1)_{1,1} & \end{array}\right],$$

And

$$\Phi_{\rho_1}^{(\rho_2,\rho_3)}(p_2;p_3;x_1) = \frac{\Gamma(\rho_3)}{\Gamma(\rho_2)}{}^{\Psi}\Phi_{\rho_1}^{1}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & | \; p_2;p_3;x_1 \\ (1,1)_{1,1} & \end{array}\right].$$

Similarly, if $\rho_1 \neq 0$ and $\aleph \neq 1$, then

$$B_{\rho_1}^{(\rho_2,\rho_3,\aleph)}(\omega_1,\omega_2) = \frac{\Gamma(\rho_3)}{\Gamma(\rho_2)}{}^{\Psi}B_{\rho_1}^{\aleph}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & | \; \omega_1,\omega_2 \\ (1,1)_{1,1} & \end{array}\right],$$

$$F_{\rho_1}^{(\rho_2,\rho_3,\aleph)}(p_1,p_2;p_3;x_1) = \frac{\Gamma(\rho_3)}{\Gamma(\rho_2)}{}^{\Psi}F_{\rho_1}^{\aleph}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & | \; p_1,p_2;p_3;x_1 \\ (1,1)_{1,1} & \end{array}\right],$$

And

$$\Phi_{\rho_1}^{(\rho_2,\rho_3,\aleph)}(p_2;p_3;x_1) = \frac{\Gamma(\rho_3)}{\Gamma(\rho_2)}{}^{\Psi}\Phi_{\rho_1}^{1}\left[\begin{array}{cc} (1,0)_{1,1} & \\ & | \; p_2;p_3;x_1 \\ (1,1)_{1,1} & \end{array}\right].$$

Lastly, if $\rho_1 \neq 0$ and $\aleph = 1$, then

$${}^{\Psi}B_{\rho_1}(\omega_1,\omega_2) = {}^{\Psi}B_{\rho_1}^{1}\left[\begin{array}{cc} (\xi_i,\zeta_i)_{1,\gamma} & \\ & | \; \omega_1,\omega_2 \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right],$$

$${}^{\Psi}F_{\rho_1}(p_1,p_2;p_3;x_1) = {}^{\Psi}F_{\rho_1}^{1}\left[\begin{array}{cc} (\xi_i,\zeta_i)_{1,\gamma} & \\ & | \; p_1,p_2;p_3;x_1 \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right],$$

And

$${}^{\Psi}\Phi_{\rho_1}(p_2;p_3;x_1) = {}^{\Psi}\Phi_{\rho_1}^{1}\left[\begin{array}{cc} (\xi_i,\zeta_i)_{1,\gamma} & \\ & | \; p_2;p_3;x_1 \\ (\ell_j,\varepsilon_j)_{1,\lambda} & \end{array}\right].$$

Here B, F and Φ denote classical beta, Gauss and confluent hypergeometric functions (see for details [1], [2] and [3]); B_{ρ_1}, F_{ρ_1} and Φ_{ρ_1} are beta, Gauss and confluent hypergeometric functions defined in (refer to [7], [8] and [9]); $B_{\rho_1}^{\aleph}$, $F_{\rho_1}^{\aleph}$ and $\Phi_{\rho_1}^{\aleph}$ are beta, Gauss and confluent hypergeometric functions introduced in [11]; $B_{\rho_1}^{(\rho_2,\rho_3)}$,

$F_{\rho_1}^{(\rho_2,\rho_3)}$ and $\Phi_{\rho_1}^{(\rho_2,\rho_3)}$ are beta, Gauss and confluent hypergeometric functions established in [12]; $B_{\rho_1}^{(\rho_2,\rho_3;\aleph)}$, $F_{\rho_1}^{(\rho_2,\rho_3;\aleph)}$ and $\Phi_{\rho_1}^{(\rho_2,\rho_3;\aleph)}$, are beta, Gauss and confluent hypergeometric functions proposed in [14]; ${}^{\Psi}B_{\rho_1}$, ${}^{\Psi}F_{\rho_1}$ and ${}^{\Psi}\Phi_{\rho_1}$ are beta, Gauss and confluent hypergeometric functions presented in [17]. The new generalized beta, Gauss and confluent hypergeometric functions as generalization of many known extension hence they become of paramount important from application point of view in the field of mathematical physics, statistic, engineering and other applied mathematics related areas . This generalization of beta, Gauss and confluent hypergeometric function can be use to study two and three variable hypergeometric function (see for example [10], [13], [15], [49] and [50]) and theory of fractional calculus (see, [51], [52] and [53]).

References

[1] H.M. Srivastava and J. Choi, Zeta and q-zeta functions and associated series and integrals, Elsevier Amsterdam, 2012.

[2] M.A. Chaudhry and S.M. Zubair, On a class of incomplete gamma functions with applications, Chapman & Hall/ CRS, 2002.

[3] M.A. Chaudhry, A.K. Rathie, R.K. Parmar and Y.S. Kim, Generalized extended beta function and confluent hypergeometric functions, Applied Mathematics and Computation, Vol. 159, 589-602, 2002.

[4] R. Sahin and O. Yagci, A new generalization of pochammer symbol and its application, Applied Mathematics and Nonlinear Science, Vol. 5, No. 1, 255-266, 2020.

[5] U.M. Abubakar, S.R. Kabara, M.A. Lawan and F.A. Idris, A new extension of modified gamma and beta functions, Cankaya University Journal of Science and Engineering, In production, 2020.

[6] A. Goswani, S. Jain, P. Agarwal and S. Araci, A note on the new extended beta and Gauss hypergeometric functions, Applied Mathematics & Information Science: An International Journal, Vol. 19, No. 1, 139-144, 2018.

[7] M.A. Chaudhry and S.M. Zubair, Generalzed incomplete gamma functions with applications, Journal of Computational and Applied Mathematics, Vol. 55, 99-124, 1994.

[8] M.A. Chaudhry, A. Qadir, M. Rafiq and S.M. Zubair, Extension of Euler's beta function, Journal of Computational and Applied Mathematics, Vol. 78, 19-32, 1997.

[9] M.A. Chaudhry, A. Qadir, H.M. Srivastava and R.B. Paris, Extended hypergeometric and confluent hypergeometric functions, Journal of Computational and Applied Mathematics, Vol. 159, 589-602, 2004.

[10] M.A. Ozarslan and E. Ozergin, Some generating relations for extended hypergeometric functions via generalized fractional operator, Mathematical and Computer Modeling, Vol. 52, 1525-1833, 2010.

[11] D.M. Lee, A.K. Rathie, R.K. Parmar and S.P. Kim, Generalization of extended beta function, hypergeometric and confluent hypergeometric functions, Honan Mathematical Journal, Vol. 33, No. 2, 187-206, 2011.

[12] E. Ozergin and M.A. Ozarslan, Extension of gamma, beta and hypergeometric functions, Journal of Computational and Applied Mathematics, Vol. 235, 4601-4610, 2011.

[13] H. Liu, Some generating relations for extended Appell's and Lauricella's hypergeometric functions, Rocky Mountain Journal of Mathematics, Vol. 44, No. 6, 1987-2007, 2014.

[14] R.K. Paemar, A new generalization of gamma, beta, hypergeometric and confluent hypergeometric functions, Le Matematiche, Vol., LXVIII, Fasc. II, 33-42, 2013.

[15] A. Agarwa, J. Choi and S. Jain, extended hypergeometric functions of two and three variable, Communication of Korean Mathematical Society, Vol. 30, No. 4, 403-414, 2015.

[16] E. Ata, Generalized beta function defined by Wright function, available at https://arXiv.org/abs/1803.031212.

[17] E. Ata and I.O. Kiymaz, A study on certain properties of generalized special functions definded by Fox-Wright function, Applied Mathematics and Nonlinear Sciences, Vol. 5, No. 1, 147-162, 2020.

[18] M.A.H. Kulip, F.F. Mohsen, and S.S. Barahmah, Futher extended gamma and beta functions in term of generealized Wright function, Electronic Journal of University of Aden for Basic and Appled Sciences, Vol. 1, No. 2, 78-83, 2020.

[19] F. He, A. Bakhet, M. Abdallah and M. Hidam, On the extended hypergeometric function and their application for the derivatives of extended Jacobi matrix polynomials, Mathematical Problems in Engineering, Article ID: 4268361, 8 pages, 2020.

[20] R. Salim and O. Yagci, Fractional calculus of extended hypergeometric function, Applied Mathematics and Nonlinear Sciences, Vol. 5, No. 1, 369-384, 2020.

[21] M. Gyassuddin, M. Ali and R.B. Paris, Certain new extension of beta and related functions, https://arXiv.org/abs/2006.15352v1.

[22] E.C. Emmanual, On analytical review of the gamma functions, Asian Research Journal of Current Sciences, Article no ARJoCS.162, Vol. 12, No. 1, 28-33, 2020.

[23] R. Sahim and O. Yagci, A new generalized pochammer symbol and its applications, Applied Mathematics and Nonlinear Sciences, Vol. 5, No. 1, 255-266, 2020.

[24] U.M. Abubakar, S.R. Kabara, A note on a new extension of extended gamma and beta functions and their properties, IOSR Journal of Mathematics, Vol.15, No. 5, 1-6, 2019.

[25] U.M. Abubakar, S.R. Kabara, Some result on the extension of the extended beta function, IOSR Journal of Mathematics, Vol.15, No. 5, 7-12, 2019.

[26] K.S. Gehlot and K.S. Nisar, Extension of two parameter gamma, beta functions and its properties, Applications and Applied Mathematics: An International Journal, Special Issue 6, 39-55, 2020.

[27] N. Khan, T. Usman and M. Aman, Extended beta, hypergeometric and confluent hypergeometric function, Transactions of National Academy of Science of Azerbaijan. Series of Physical-Technical and Mathematical Sciences, Issue Mathematics, Vol. 39, No. 1, 83-97, 2019.

[28] K.S. Nisar, G. Rahman and Tomovski, On a certain extension of the Riemann - Liouville fractional derivative operator, Communication of Korean Mathematical Society, Vol. 34, No. 2, 507-522, 2019.

[29] M. Gayasuddin, N. Khan, M. Ali, A study of the extended beta, Gauss and confluent hypergeometric functions, International Journal of Applied mathematics, Vol. 33, No. 10, 1-13, 2020.

[30] K. Oraby, A. Rizq, M. Ahmed, M. Khaled, E. Ahmed and M. Magdy, Generalization of beta function in term of Mittag-Leffler function, Frontiers in Science Research and Technology, Vol. 1, 81-88, 2020.

[31] L. Debnath and D. Bhatta, Integral transforms and their applications, Chapman & Hall/ CRC, 2007.

[32] M. Ali, M. Ghayasuddin, W.A. Khan and K.S. Nisar, A novel kind of the multi-index beta, Gauss and confluent hypergeometric functions, Journal of Mathematics and Computer Science, Vol. 23, 145-154, 2021.

[33] S.Mubeen, K.S. Nisar, G. Rahman and M. Arshad, Inequalities of extended (p, q)-beta and confluent hypergeometric functions, https://arXiv.org/abs/1703.08852

[34] G. Rahman, K.S. Nisar, J. Choi, S. Mubeen and M. Arshad, Extension of real matrix - variable gamma and beta functions and their applications, Far East Journal of Mathematical Sciences, Vol. 101, Issue 10, 2333-2347, 2017.

[35] H.M. Sriastava, G. Rahman and K.S. Nisar, Some extension of the pochammer symbol and associated hypergeometric functions, Iranian Journal of Science and Technoloy, Transactions A: Science Vol. 43, 2601-2606, 2019.

[36] F. Qi, K.S. Nisar and G. Rahman, Convexity and inequalities related to extended beta and confluent hypergeometric functions, AIM Mathematics, Vol. 4, Issue 5, 149-1507, 2019.

[37] G. Rahman, S. Mubeen and K.S. Nisar, Further extension of extended fractional derivative operator of Riemann-Liouville, https://dx.doi.org/10.20944/preprints201712.0013.v1

[38] G. Rahman, S. Mubeen and K.S. Nisar, J. Choi, Certain extended specia function and fractiona integral and derivative operators via an extended beta function, Nonlinear Functional Analysis and Applications, Vol. 24, Issue 1, 1-13, 2019.

[39] G. Rahman, S. Mubeen and K.S. Nisar, On generalized k-fractional operator, AIM Mathematics, Vol. 5, Issue 3, 2020.

[40] G. Rahman, G. Kanwal, K.S. Nisar and Ghaffar, A new extension of beta and hypergeometric functions, https://dx.doi.org/10.20944/preprints201801.0074.v1

[41] K. Nantomah, K.S. Nisar and K.S. Ghelot, On a k - extension of the Nielsen's β-functions, International Journal of Nonlinear Analysis and Application, Vol. 9, Issue 2, 191-201, 2018.
[42] K.S. Ghelot and K.S. Nisar, Extension of two parameter gamma, beta functions and its properties, Applications and Applied Mathematics, An International Journal, Special Issue 6, 39-55, 2020.
[43] G. Rahman, K.S. Nisar and M. Arshad, A new extension of Hurwitz - Lerch zeta function, https://arXiv.org/abs/1802.07823
[44] K.S. Nisar, F. Qi, G. Rahman, S. Mubeen and M. Arshad, Some inequalities involving the extended gamma function and the Kummer confluent hypergeometric k-function, Journal of Inequalities and Applications, Vol. 135, 12 pages, 2018.
[45] K.S. Nisar, F. Qi, G. Rahman, S. Mubeen and M. Arshad, Some inequalities involving the extended gamma and beta functions, https://hal.archieves-ouvertes.fr/hal-01701647
[46] M. Bohner, G. Rahman, S. Mubeen and K.S. Nisar, A further extension o the extended Riemann-Liouille fractional derivative operator, Turkish Jornal of Mathematics, Vol. 42, 2631-2642, 2018.
[47] G. Rahman, K.S. Nisar and S. Mubeen, A new generalization of extended beta and hypergeometric functions, https://dx.doi.org/10.20944/preprints201802.0036.v1
[48] E. Erdogan, S. Kocabas and A.N. Dernek, Some results on the generalized Mellin transforms and applications, Konuralp Journal of Mathematics, Vol. 7, No. 1, 175-181, 2019.
[49] S.A. Dar, and R.B. Paris, A (p, v)-extension of the Appell function $F_1(.)$ and its properties, https://arXiv.org/abs/1711.077801.
[50] M. Shadab and J. Choi, Extension of Appell, Lauricella hypergeometric functions, Far East Journal of Mathematical science, Vol, 102, No. 6, 1301-1317, 2017.
[51] D. Baleanu, P. Agarwal, R.K. Parmar, A.AL. Qurashi and S. Salashour, Extension of the fractional derivative operator of the Riemann-Liouville, Journal of Nonlinear Science and Applications, Vol. 10 Issue 6, 2914-2924, 2017.
[52] P. Agarwal, J. Choi and R.B. Paris, Extended Riemann-Liouville fractional dervatie operator and its applications, Journal of Nonlinear Science and Applications, Vol. 8, 451-466, 2015.
[53] M. Shadab, M.F. Khan and J.L. Lopez-Bonilla, A new Rieman-Liouville type fractional derivative operator and its application in generating function, Advance in Difference Equation, Vol. 167, 16 pages, 2018.

U. M. Abubakar
Faculty of Computing and Mathematical Sciences, Kano University of Science and Technology, Wudil, P.M.B. 3244 Kano State, Nigeria
E-mail address: umabubakar347@gmail.com, umabubakar347@yahoo.com