Journal of Fractional Calculus and Applications
Vol. 12(2) July 2021, pp. 85-93.
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

SOME PROPERTIES FOR CERTAIN MULTIVALENT FUNCTIONS ASSOCIATED WITH DIFFER-INTEGRAL OPERATOR AND EXTENDED MULTIPLIER TRANSFORMATIONS

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Abstract. In this paper, the authors study some properties of multivalent functions

$$\mathfrak{D}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)=(1-\sigma)\mathfrak{J}_{p,n}^{\gamma,\ell}(a,c;\mu)f(z)+\sigma\mathfrak{J}_{p,n}^{\gamma,\ell}(a+1,c;\mu)f(z)$$

and

$$\mathfrak{B}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)=(1-\sigma)\mathfrak{J}_{p,n}^{\gamma,\ell}(a,c;\mu)f(z)+\sigma\mathfrak{J}_{p,n+1}^{\gamma,\ell}(a,c;\mu)f(z)$$

$$(n\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ \mu>0,\ \gamma\geq 0,\ \ell\geq 0,\ a,c\in\mathbb{R},\sigma\in\mathbb{C},\ a>-\mu p,\ p\in\mathbb{N}\,\text{and}\ (c-a)>0)$$

defined by Erdélyi-Kober-type integral operator and an extended multiplier transformations.

1. Introduction

Let $\mathcal{A}_p$ be the class of all functions of the form

$$f(z)=z^p+\sum_{\kappa=p+1}^{\infty}a_\kappa z^\kappa \qquad (p\in\mathbb{N},\ \mathbb{N}=\{1,2,...\}) \tag{1.1}$$

which are analytic and multivalent in the open unit disc $\mathcal{U}=\{z:|z|<1\}$. Catas [8] defined the linear operator $\mathcal{I}_p^n(\gamma,\ell)f(z)$ by the following form (see also [24])

$$\mathcal{I}_p^n(\gamma,\ell)f(z)=z^p+\sum_{\kappa=p+1}^{\infty}\left(\frac{p+\ell+\gamma(\kappa-p)}{p+\ell}\right)^n a_\kappa z^\kappa$$
$$(n\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ \gamma\geq 0,\ \ell\geq 0\text{ and }p\in\mathbb{N}).$$

Note that,

$$\mathcal{I}_p^0(1,0)f(z)=f(z),\qquad\text{and}\qquad \mathcal{I}_p^1(1,0)f(z)=\frac{zf'(z)}{p}.$$

2010 *Mathematics Subject Classification.* Primary 30C45, Secondary 30C80.

Key words and phrases. Analytic functions; linear operators; multivalent function; extended multiplier transformations.

Submitted July 23, 2020. Revised Aug. 11, 2020.

Also, for $\mu > 0$, $a, c \in \mathbb{R}$, $a > -\mu p$, $p \in \mathbb{N}$ and $(c - a) > 0$, modified an Erdélyi-Kober-type integral operator [16], El-Ashwah and Drbuk [13] defined the linear operator $\mathcal{J}_p(a, c; \mu)f(z)$ by the following form

$$\begin{aligned}\mathcal{J}_p(a, c; \mu)f(z) &= \frac{\Gamma(c + \mu p)}{\Gamma(a + \mu p)\Gamma(c - a)} \int_0^1 (1 - t)^{c-a-1} t^{a-1} f(zt^{\mu}) dt \\ &= z^p + \frac{\Gamma(c + \mu p)}{\Gamma(a + \mu p)} \sum_{\kappa=p+1}^{\infty} \frac{\Gamma(a + \kappa\mu)}{\Gamma(c + \kappa\mu)} a_{\kappa} z^{\kappa}.\end{aligned}$$

Note that,

$$\mathcal{J}_p(a, a; \mu)f(z) = f(z), \qquad \text{and} \qquad \mathcal{J}_p(1, 0; 1)f(z) = \frac{zf'(z)}{p}.$$

Now, for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mu > 0$, $\gamma \geq 0$, $\ell \geq 0$, $a, c \in \mathbb{R}$, $a > -\mu p$, $p \in \mathbb{N}$ and $(c - a) > 0$, we define the linear operator $\mathfrak{J}_{p,n}^{\gamma,\ell}(a, c; \mu)f(z)$ by the following form

$$\mathfrak{J}_{p,n}^{\gamma,\ell}(a, c; \mu)f(z) = z^p + \frac{\Gamma(c + \mu p)}{\Gamma(a + \mu p)} \sum_{\kappa=p+1}^{\infty} \left(\frac{p + \ell + \gamma(\kappa - p)}{p + \ell} \right)^n \frac{\Gamma(a + \kappa\mu)}{\Gamma(c + \kappa\mu)} a_{\kappa} z^{\kappa}. \tag{1.2}$$

The above-defined operator includes several simpler operators. We point out here some of these special cases as follows:

(i): Putting $\gamma = 1$ and $a = c$, we obtain $I_p(n, \ell)f(z)$, which was studied by Kumar et al. [17] (see also [28]);
(ii): Putting $\gamma = 1$, $\ell = 0$ and $a = c$, we obtain $D_p^n f(z)$, which was studied by Kamali and Orhan [15] (see also [2, 22]);
(iii): Putting $a = c$, we obtain $D_{\gamma,p}^n f(z)$, which was studied by Aouf et al. [4];
(iv): Putting $n = -m$ and $a = c$, we obtain $J_p^m(\gamma, \ell)f(z)$, which was studied by El-Ashwah and Aouf [12] (see also [5, 27]);
(v): Putting $n = -m$ $(m \in \mathbb{Z})$, $\gamma = 1$, $\ell = 1$ and $a = c$, we obtain $D_p^m f(z)$, which was studied by Patel and Sahoo [23];
(vii): Putting $\gamma = 1$, $p = 1$ and $a = c$, we obtain $I_\ell^n f(z)$, which was studied by Cho and Srivastava [10] (see also [9]);
(viii): Putting $\ell = 0$, $p = 1$ and $a = c$, we obtain $I_\gamma^n f(z)$, which was studied by Al-Oboudi [1];
(ix): Putting $\gamma = 1$, $\ell = 0$, $p = 1$ and $a = c$, we obtain $D^n f(z)$, which was studied by Salagean [26];
(x): Putting $a = \beta$, $c = \alpha + \beta - \delta + 1$, $\mu = 1$ and $n = 0$, we obtain $\mathfrak{R}_{\beta,p}^{\alpha,\delta} f(z)$ $(\delta > 0;\ \alpha \geq \delta - 1;\ \beta > -p)$ which was studied by Aouf et al. [3];
(xi): Putting $a = \beta$, $c = \alpha + \beta$, $\mu = 1$ and $n = 0$, we obtain $Q_{\beta,p}^{\alpha} f(z)$ $(\alpha \geq 0;\ \beta > -p)$ which was studied by Liu and Owa [19];
(xii): Putting $p = 1$, $a = \beta$, $c = \alpha + \beta$, $\mu = 1$ and $n = 0$, we obtain $Q_\beta^\alpha f(z)$ $(\alpha \geq 0,\ \beta > -1)$ which was studied by Jung et al. [14];
(xiii): Putting $p = 1$, $a = \alpha - 1$, $c = \beta - 1$, $\mu = 1$ and $n = 0$, we obtain $L(\alpha, \beta)f(z)$ $(\alpha,\ \beta \in \mathbb{C} \setminus \mathbb{Z}_0,\ \mathbb{Z}_0 = \{0, -1, -2, ...\})$ which was studied by Carlson and Shaffer [7];
(xiv): Putting $p = 1$, $a = \nu - 1$, $c = v$, $\mu = 1$, and $n = 0$, we obtain $I_{\nu,v} f(z)$ $(\nu > 0;\ v > -1)$ which was studied by Choi et al. [11];

(xv): Putting $p = 1$, $a = \alpha$, $c = 0$, $\mu = 1$ and $n = 0$, we obtain $D^{\alpha} f(z)$ $(\alpha > -1)$ which was studied by Ruscheweyh [25];

(xvi): Putting $p = 1$, $a = 1$, $c = m$, $\mu = 1$ and $n = 0$, we obtain the operator $I_m f(z)$ $(m \in \mathbb{N}_0)$ which was studied by Noor [21];

(xvii): Putting $p = 1$, $a = \beta$, $c = \beta + 1$, $\mu = 1$ and $n = 0$, we obtain $J_{\beta} f(z)$ which was studied by Bernardi [6];

(xviii): Putting $p = 1$, $a = 1$, $c = 2$, $\mu = 1$ and $n = 0$, we obtain $Jf(z)$ which was studied by Libera [18].

It is readily verified from (1.2) that

$$\mathfrak{I}_{p,n}^{\gamma,\ell}(a+1,c;\mu)f(z) = \frac{a}{a+\mu p}\mathfrak{I}_{p,n}^{\gamma,\ell}(a,c;\mu)f(z) + \frac{\mu}{a+\mu p} z(\mathfrak{I}_{p,n}^{\gamma,\ell}(a,c;\mu)f(z))' \quad (1.3)$$

and

$$\mathfrak{I}_{p,n+1}^{\gamma,\ell}(a,c;\mu)f(z) = \frac{p+\ell-p\gamma}{p+\ell}\mathfrak{I}_{p,n}^{\gamma,\ell}(a,c;\mu)f(z) + \frac{\gamma}{p+\ell} z(\mathfrak{I}_{p,n}^{\gamma,\ell}(a,c;\mu)f(z))'. \quad (1.4)$$

Now, we define the two functions $\mathfrak{D}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)$ and $\mathfrak{B}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)$ as follows

$$\mathfrak{D}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)(z) = (1-\sigma)\mathfrak{I}_{p,n}^{\gamma,\ell}(a,c;\mu)f(z) + \sigma\mathfrak{I}_{p,n}^{\gamma,\ell}(a+1,c;\mu)f(z)$$
$$(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \mu > 0,\ \gamma \geq 0,\ \ell \geq 0,\ a,c \in \mathbb{R}, \sigma \in \mathbb{C},\ a > -\mu p,\ p \in \mathbb{N},\ (c-a) > 0) \quad (1.5)$$

and

$$\mathfrak{B}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)(z) = (1-\sigma)\mathfrak{I}_{p,n}^{\gamma,\ell}(a,c;\mu)f(z) + \sigma\mathfrak{I}_{p,n+1}^{\gamma,\ell}(a,c;\mu)f(z)$$
$$(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \mu > 0,\ \gamma \geq 0,\ \ell \geq 0,\ a,c \in \mathbb{R}, \sigma \in \mathbb{C},\ a > -\mu p,\ p \in \mathbb{N},\ (c-a) > 0). \quad (1.6)$$

We note that:

(i): If $n = 0$ in (1.5), then the function $\mathfrak{D}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)(z)$ reduces to

$$\mathfrak{D}_{p}(a,c;\mu,\sigma)(z) = (1-\sigma)\mathfrak{I}_{p}(a,c;\mu,\sigma)f(z) + \sigma\mathfrak{I}_{p}(a+1,c;\mu,\sigma)f(z)$$
$$(\mu > 0,\ a,c \in \mathbb{R}, \sigma \in \mathbb{C},\ a > -\mu p,\ p \in \mathbb{N},\ (c-a) > 0); \quad (1.7)$$

(ii): If $a = c$ in (1.6), then the function $\mathfrak{B}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)(z)$ reduces to (see [4])

$$\mathfrak{B}_{p,n}^{\gamma,\ell}(\sigma)(z) = (1-\sigma)\mathfrak{I}_{p,n}(\gamma,\ell)f(z) + \sigma\mathfrak{I}_{p,n+1}(\gamma,\ell)f(z)\ (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \gamma \geq 0,\ \ell \geq 0,\ \sigma \in \mathbb{C},\ p \in \mathbb{N}); \quad (1.8)$$

(iii): If $a = c = 0, \mu = 1$ and $n = 0$ in (1.5) or $a = c$, $n = 0$, $\gamma = 1$ and $\ell = 0$ in (1.6), then the two functions $\mathfrak{D}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)(z)$ and $\mathfrak{B}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)(z)$ reduce to

$$\mathfrak{F}_{p}(\sigma)(z) = (1-\sigma)f(z) + \sigma\frac{zf'(z)}{p} \qquad (f(z) \in \mathcal{A}_p,\ \sigma \in \mathbb{C},\ p \in \mathbb{N}); \quad (1.9)$$

To prove our main works, we need that lemma:

Lemma 1.1. [20] *Let $\varphi(x,y)$, $\varphi : D \to \mathbb{C}$, $D \subset \mathbb{C} \times \mathbb{C}$ be complex valued function, $x = x_1 + ix_2$ and $y = y_1 + y_2$. Suppose that $\varphi(x,y)$ satisfies the following conditions:*

- *$\varphi(x,y)$ is continuous in D,*
- *$(1,0) \in D$ and $\Re(\varphi(1,0)) > 0$,*
- *for all $(ix_2, y_1) \in D$ and $y_1 \leq -\frac{1+x_2^2}{2}$, $\Re(\varphi(ix_2,y_1)) \leq 0$.*

Let $q(z) = 1 + q_1 z + q_2 z^2 + ...$ *be regular in the unit disc* $\mathcal{U}$ *such that* $(q(z), zq'(z)) \in D$, $(z \in \mathcal{U})$. *If*

$$\Re\{\varphi(q(z), zq'(z))\} > 0 \qquad (z \in \mathcal{U}),$$

then $\Re(q(z)) > 0$.

In this paper, the authors study some properties of multivalent functions $\mathfrak{D}_{p,n}^{\gamma,\ell}(a, c; \mu, \sigma)f(z)$ and $\mathfrak{B}_{p,n}^{\gamma,\ell}(a, c; \mu, \sigma)f(z)$ defined by Erdélyi-Kober-type integral operator and an extended multiplier transformations.

2. The main results

Unless otherwise mentioned, we suppose that $f(z) \in \mathcal{A}_p$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mu > 0$, $\gamma \geq 0$, $\ell \geq 0$, $a, c \in \mathbb{R}, \sigma \in \mathbb{C}$, $a > -\mu p$, $p \in \mathbb{N}$ and $(c - a) > 0$.

Theorem 2.1. *Let* $\mathfrak{D}_{p,n}^{\gamma,\ell}(a, c; \mu, \sigma)f(z)$ *be defined by (1.5). If*

$$\Re\left(\frac{\mathfrak{D}_{p,n}^{\gamma,\ell}(a, c; \mu, \sigma)f(z)}{z^p}\right) > \tau, \qquad (0 \leq \tau < 1;\ \Re(\sigma) \geq 0),$$

then

$$\Re\left(\frac{\mathfrak{J}_{p,n}^{\gamma,\ell}(a, c; \mu)f(z)}{z^p}\right) > \frac{2(a + \mu p)\tau + \mu\Re(\sigma)}{2(a + \mu p) + \mu\Re(\sigma)}.$$

Proof. Let $q(z)$ be a function defined by

$$\frac{\mathfrak{J}_{p,n}^{\gamma,\ell}(a, c; \mu)f(z)}{z^p} = \zeta + (1 - \zeta)q(z) \tag{2.1}$$

such that

$$\zeta = \frac{2(a + \mu p)\tau + \mu\Re(\sigma)}{2(a + \mu p) + \mu\Re(\sigma)}$$

and $q(z) = 1 + q_1 z + q_2 z^2 + ...$ is regular in $\mathcal{U}$. By using (1.3), we obtain

$$\begin{aligned}\frac{\mathfrak{D}_{p,n}^{\gamma,\ell}(a, c; \mu, \sigma)f(z)}{z^p} &= (1 - \sigma)\frac{\mathfrak{J}_{p,n}^{\gamma,\ell}(a, c; \mu)f(z)}{z^p} + \sigma\frac{\mathfrak{J}_{p,n}^{\gamma,\ell}(a + 1, c; \mu)f(z)}{z^p} \\ &= \zeta + (1 - \zeta)q(z) + \frac{\mu\sigma}{a + \mu p}(1 - \zeta)zq'(z).\end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we have

$$\Re\left(\frac{\mathfrak{D}_{p,n}^{\gamma,\ell}(a, c; \mu, \sigma)f(z)}{z^p} - \tau\right) = \Re\left(\zeta - \tau + (1 - \zeta)q(z) + \frac{\mu\sigma}{a + \mu p}(1 - \zeta)zq'(z)\right) > 0. \tag{2.3}$$

If

$$\varphi(x, y) = \zeta - \tau + (1 - \zeta)x + \frac{\mu\sigma}{a + \mu p}(1 - \zeta)y$$

with

$$q(z) = x = x_1 + ix_2 \qquad \text{and} \qquad zq'(z) = y = y_1 + iy_2,$$

and using Lemma 1.1, then

- $\varphi(x, y)$ is continuous in D,
- $(1, 0) \in D$ and $\Re(\varphi(1, 0)) = 1 - \tau > 0$,

- for all $(ix_2, y_1) \in D$ and $y_1 \leq -\frac{1+x_2^2}{2}$,

$$\begin{aligned}\Re\left(\varphi(x_2 i, y_1)\right) &= \zeta - \tau + (1-\zeta) y_1 \frac{\mu \Re(\sigma)}{a+\mu p} \\ &\leq \zeta - \tau - (1-\zeta)\frac{\mu(1+x_2^2)\Re(\sigma)}{2(a+\mu p)} \leq 0.\end{aligned}$$

We have $\Re(q(z)) > 0$, that is

$$\Re\left(\frac{\mathfrak{I}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)}{z^p}\right) > \zeta = \frac{2(a+\mu p)\tau + \mu\Re(\sigma)}{2(a+\mu p)+\mu\Re(\sigma)}.$$

The proof of Theorem 2.1 is completed. □

Putting $n = 0$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.2. *Let $\mathfrak{D}_p(a,c;\mu,\sigma)(z)$ be defined by (1.7). If*

$$\Re\left(\frac{\mathfrak{D}_p(a,c;\mu,\sigma)f(z)}{z^p}\right) > \tau, \qquad (0 \leq \tau < 1;\ \Re(\sigma) \geq 0),$$

then

$$\Re\left(\frac{\mathfrak{I}_p(a,c,\mu)f(z)}{z^p}\right) > \frac{2(a+\mu p)\tau + \mu\Re(\sigma)}{2(a+\mu p)+\mu\Re(\sigma)}.$$

Theorem 2.3. *Let $\mathfrak{B}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)$ be defined by (1.6). If*

$$\Re\left(\frac{\mathfrak{B}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)}{z^p}\right) > \tau, \qquad (0 \leq \tau < 1;\ \Re(\sigma) \geq 0),$$

then

$$\Re\left(\frac{\mathfrak{I}_{p,n}^{\gamma,\ell}(a,c;\mu)f(z)}{z^p}\right) > \frac{2(p+\ell)\tau + \gamma\Re(\sigma)}{2(p+\ell)+\gamma\Re(\sigma)}.$$

Proof. Using the same technique as in the proof of Theorem 2.1 with Equation (1.4), we obtain the proof of Theorem 2.3 □

Remark 2.4. *Putting $a = c$ in Theorem 2.3, we obtain the result which was studied by Aouf et al.* [4, Theorem 1].

Putting $a = c = 0, \mu = 1$ and $n = 0$ in Theorem 2.1 or $a = c$, $n = 0$, $\gamma = 1$ and $\ell = 0$ in Theorem 2.3, we obtain the following corollary:

Corollary 2.5. *Let $\mathfrak{F}_p(\sigma)(z)$ be defined by (1.9). If*

$$\Re\left(\frac{\mathfrak{F}_p(\sigma)(z)}{z^p}\right) > \tau, \qquad (f(z) \in \mathcal{A}_p,\ 0 \leq \tau < 1;\ \Re(\sigma) \geq 0),$$

then

$$\Re\left(\frac{f(z)}{z^p}\right) > \frac{2p\tau + \Re(\sigma)}{2p+\Re(\sigma)}.$$

Theorem 2.6. *Let $\mathfrak{D}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)$ be defined by (1.5). If*

$$\Re\left(\frac{\mathfrak{D}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)}{z^p}\right) < \tau, \qquad (\tau > 1;\ \Re(\sigma) \geq 0),$$

then

$$\Re\left(\frac{\mathfrak{I}_{p,n}^{\gamma,\ell}(a,c;\mu)f(z)}{z^p}\right) < \frac{2(a+\mu p)\tau+\mu\Re(\sigma)}{2(a+\mu p)+\mu\Re(\sigma)}.$$

Proof. Let $q(z)$ be a function defined by

$$\frac{\mathfrak{I}_{p,n}^{\gamma,\ell}(a,c;\mu)f(z)}{z^p} = \zeta+(1-\zeta)q(z) \tag{2.4}$$

such that

$$\zeta = \frac{2(a+\mu p)\tau+\mu\Re(\sigma)}{2(a+\mu p)+\mu\Re(\sigma)} > 1$$

and $q(z) = 1+q_1z+q_2z^2+...$ is regular in $\mathcal{U}$. By using (1.3) and (2.4), we obtain

$$\Re\left(\tau-\frac{\mathfrak{D}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)}{z^p}\right) = \Re\left(\tau-\zeta-(1-\zeta)q(z)-\frac{\mu\sigma}{a+\mu p}(1-\zeta)zq'(z)\right) > 0. \tag{2.5}$$

If

$$\varphi(x,y) = \tau-\zeta-(1-\zeta)x-\frac{\mu\sigma}{a+\mu p}(1-\zeta)y$$

with

$$q(z) = x = x_1+ix_2 \qquad \text{and} \qquad zq'(z) = y = y_1+iy_2,$$

and using Lemma 1.1, then

- $\varphi(x,y)$ is continuous in D,
- $(1,0)\in D$ and $\Re\left(\varphi(1,0)\right) = \tau-1 > 0$,
- for all $(ix_2,y_1)\in D$ and $y_1 \le -\frac{1+x_2^2}{2}$,

$$\begin{aligned}\Re\left(\varphi(x_2i,y_1)\right) &= \tau-\zeta-(1-\zeta)y_1\frac{\mu\Re(\sigma)}{a+\mu p}\\ &\le \tau-\zeta+(1-\zeta)\frac{\mu(1+x_2^2)\Re(\sigma)}{2(a+\mu p)} \le 0.\end{aligned}$$

We have $\Re(q(z)) > 0$, that is

$$\Re\left(\frac{\mathfrak{I}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)}{z^p}\right) < \zeta = \frac{2(a+\mu p)\tau+\mu\Re(\sigma)}{2(a+\mu p)+\mu\Re(\sigma)}.$$

The proof of Theorem 2.6 is completed. □

Putting $n=0$ in Theorem 2.6, we obtain the following corollary:

Corollary 2.7. *Let $\mathfrak{D}_p(a,c;\mu,\sigma)(z)$ be defined by (1.7). If*

$$\Re\left(\frac{\mathfrak{D}_p(a,c;\mu,\sigma)f(z)}{z^p}\right) < \tau, \qquad (\tau > 1;\ \Re(\sigma)\ge 0),$$

then

$$\Re\left(\frac{\mathfrak{I}_p(a,c,\mu)f(z)}{z^p}\right) < \frac{2(a+\mu p)\tau+\mu\Re(\sigma)}{2(a+\mu p)+\mu\Re(\sigma)}.$$

Theorem 2.8. *Let $\mathfrak{B}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)$ be defined by (1.6). If*

$$\Re\left(\frac{\mathfrak{B}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)}{z^p}\right)<\tau, \qquad (\tau>1;\ \Re(\sigma)\geq 0),$$

then

$$\Re\left(\frac{\mathfrak{I}_{p,n}^{\gamma,\ell}(a,c;\mu)f(z)}{z^p}\right)<\frac{2(p+\ell)\tau+\gamma\Re(\sigma)}{2(p+\ell)+\gamma\Re(\sigma)}.$$

Proof. Using the same technique as in the proof of Theorem 2.6 with Equation (1.4), we obtain the proof of Theorem 2.8 □

Remark 2.9. *Putting $a=c$ in Theorem 2.8, we obtain the result which was studied by Aouf et al.* [4, Theorem 2].

Putting $a=c=0,\mu=1$ and $n=0$ in Theorem 2.6 or $a=c$, $n=0$, $\gamma=1$ and $\ell=0$ in Theorem 2.8, we obtain the following corollary:

Corollary 2.10. *Let $\mathfrak{F}_p(\sigma)(z)$ be defined by (1.9). If*

$$\Re\left(\frac{\mathfrak{F}_p(\sigma)(z)}{z^p}\right)<\tau, \qquad (f(z)\in\mathcal{A}_p,\ \tau>1;\ \Re(\sigma)\geq 0),$$

then

$$\Re\left(\frac{f(z)}{z^p}\right)<\frac{2p\tau+\Re(\sigma)}{2p+\Re(\sigma)}.$$

Using the same technique as in the proof of the above theorems and putting $f(z)=\frac{zf'(z)}{p}$, we obtain the following theorems:

Theorem 2.11. *Let $\mathfrak{D}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)$ be defined by (1.5). If*

$$\Re\left(\frac{(\mathfrak{D}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z))'}{pz^{p-1}}\right)>\tau, \qquad (0\leq\tau<1;\ \Re(\sigma)\geq 0),$$

then

$$\Re\left(\frac{(\mathfrak{I}_{p,n}^{\gamma,\ell}(a,c;\mu)f(z))'}{pz^{p-1}}\right)>\frac{2(a+\mu p)\tau+\mu\Re(\sigma)}{2(a+\mu p)+\mu\Re(\sigma)}.$$

Putting $n=0$ in Theorem 2.11, we obtain the following corollary:

Corollary 2.12. *Let $\mathfrak{D}_p(a,c;\mu,\sigma)(z)$ be defined by (1.7). If*

$$\Re\left(\frac{(\mathfrak{D}_p(a,c;\mu,\sigma)f(z))'}{pz^{p-1}}\right)>\tau, \qquad (0\leq\tau<1;\ \Re(\sigma)\geq 0),$$

then

$$\Re\left(\frac{(\mathfrak{I}_p(a,c,\mu)f(z))'}{pz^{p-1}}\right)>\frac{2(a+\mu p)\tau+\mu\Re(\sigma)}{2(a+\mu p)+\mu\Re(\sigma)}.$$

Theorem 2.13. *Let $\mathfrak{B}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z)$ be defined by (1.6). If*

$$\Re\left(\frac{(\mathfrak{B}_{p,n}^{\gamma,\ell}(a,c;\mu,\sigma)f(z))'}{pz^{p-1}}\right)>\tau, \qquad (0\leq\tau<1;\ \Re(\sigma)\geq 0),$$

then

$$\Re\left(\frac{(\mathfrak{I}_{p,n}^{\gamma,\ell}(a,c;\mu)f(z))'}{pz^{p-1}}\right)>\frac{2(p+\ell)\tau+\gamma\Re(\sigma)}{2(p+\ell)+\gamma\Re(\sigma)}.$$

Remark 2.14. *(i) Putting $a = c$ in Theorem 2.13, we obtain the result which was studied by Aouf et al.* [4, Theorem 3].
(ii) Putting $a = c = 0, \mu = 1$ and $n = 0$ in Theorem 2.11 or $a = c$, $n = 0$, $\gamma = 1$ and $\ell = 0$ in Theorem 2.13, we obtain the result which was studied by Aouf et al. [4, Corollary 1].

Theorem 2.15. *Let $\mathfrak{D}_{p,n}^{\gamma,\ell}(a, c; \mu, \sigma)f(z)$ be defined by (1.5). If*

$$\Re\left(\frac{(\mathfrak{D}_{p,n}^{\gamma,\ell}(a, c; \mu, \sigma)f(z))'}{pz^{p-1}}\right) < \tau, \qquad (\tau > 1;\ \Re(\sigma) \geq 0),$$

then

$$\Re\left(\frac{(\mathfrak{J}_{p,n}^{\gamma,\ell}(a, c; \mu)f(z))'}{pz^{p-1}}\right) < \frac{2(a + \mu p)\tau + \mu\Re(\sigma)}{2(a + \mu p) + \mu\Re(\sigma)}.$$

Putting $n = 0$ in Theorem 2.15, we obtain the following corollary:

Corollary 2.16. *Let $\mathfrak{D}_p(a, c; \mu, \sigma)(z)$ be defined by (1.7). If*

$$\Re\left(\frac{(\mathfrak{D}_p(a, c; \mu, \sigma)f(z))'}{pz^{p-1}}\right) < \tau, \qquad (\tau > 1;\ \Re(\sigma) \geq 0),$$

then

$$\Re\left(\frac{(\mathfrak{J}_p(a, c, \mu)f(z))'}{pz^{p-1}}\right) < \frac{2(a + \mu p)\tau + \mu\Re(\sigma)}{2(a + \mu p) + \mu\Re(\sigma)}.$$

Theorem 2.17. *Let $\mathfrak{B}_{p,n}^{\gamma,\ell}(a, c; \mu, \sigma)f(z)$ be defined by (1.6). If*

$$\Re\left(\frac{(\mathfrak{B}_{p,n}^{\gamma,\ell}(a, c; \mu, \sigma)f(z))'}{pz^{p-1}}\right) < \tau, \qquad (\tau > 1;\ \Re(\sigma) \geq 0),$$

then

$$\Re\left(\frac{(\mathfrak{J}_{p,n}^{\gamma,\ell}(a, c; \mu)f(z))'}{pz^{p-1}}\right) < \frac{2(p + \ell)\tau + \gamma\Re(\sigma)}{2(p + \ell) + \gamma\Re(\sigma)}.$$

Remark 2.18. *(i) Putting $a = c$ in Theorem 2.17, we obtain the result which was studied by Aouf et al.* [4, Theorem 4].
(ii) Putting $a = c = 0, \mu = 1$ and $n = 0$ in Theorem 2.15 or $a = c$, $n = 0$, $\gamma = 1$ and $\ell = 0$ in Theorem 2.17, we obtain the result which was studied by Aouf et al. [4, Corollary 2].

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