Journal of Fractional Calculus and Applications
Vol. 12(2) July 2021, pp. 69-84.
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

A MODIFIED BAZILEVIC FUNCTION ASSOCIATED WITH A SPECIAL CLASS OF ANALYTIC FUNCTIONS $U_{\alpha,n}$ IN THEN OPEN UNIT DISK

J. O. HAMZATN AND A. T. OLADIPO

Abstract. In this work, we investigate some properties of a modified Bazilevic function $F_{\alpha,n}$ as related to a special class of analytic functions $U_{\alpha,n}$ satisfying the condition $|U_{F_{\alpha,n}}(z)| < 1, \quad |z| < 1.$ in the open unit disk E. In particular, some fundamental properties such as, characterization properties, sufficient coefficient condition, radius problems, convolution properties as well as application of fractional calculus, for functions $F_{\alpha,n}$ in the class $U_{\alpha,n}(z)$ associated with modified Bazilevic function are considered.

1. Introduction

As usual we denote by A the class of all functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$, with normalization $f(0) = f'(0) - 1 = 0$. Also we denote the subclass of A consisting of analytic and univalent functions $f(z)$ in the unit disk E by S. Here we shall recall some well-known functions and concepts of analytic functions. Let $f \in A$, then $f \in S^*$ if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \qquad z \in E. \tag{2}$$

This class is called the class of starlike functions of order β. In like manner, let $f \in A$, then, $f \in K$ if and only if

$$\Re\left\{1 + \frac{zf'(z)}{f(z)}\right\} > \beta, \qquad z \in E. \tag{3}$$

This class is called the class of convex functions of order β. The above two classes have been widely studied and investigated by various authors and their results have appeared in prints, see ([9]), ([10]), ([12]), ([29]) and ([30]) just to mention but few.

2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Analytic function, univalent function, starlike function, convex function, Bazilevic function.
Submitted April 13, 2019. Revised Aug. 20, 2020.

Now, research on various families of Bazilevic functions has a long history and will continue to play a crucial role geometric function theory. However, the study of the Bazilevic function commenced around 1955 by a Russian Mathematician Bazilevic ([5]), who defined a function *f(z)* (say) in E as

$$f(z)=\left\{\frac{\alpha}{1+\varepsilon^2}\int_0^z\frac{p(v)-i\varepsilon}{V^{\left(1+\frac{i\alpha\varepsilon}{(1+\varepsilon^2)}\right)}}g(v)^{\frac{\alpha}{1+\varepsilon^2}}dv\right\}^{\frac{1+i\varepsilon}{\alpha}} \tag{4}$$

where $p\in P$, $\alpha>0$ and $g\in\Psi^*$. The family of this functions $f(z)$ defined in (4) became known as Bazilevic functions and is usually, denoted by $B(\alpha,\varepsilon)$. Then, very little is known about the said family in (4), except that, he Bazilevic showed that each function $f\in B(\alpha,\varepsilon)$ is univalent in E. By simplifying (4) it is quite possible to understand and investigate the family better. It should be noted that with special choices of parameters α, ε and the function $g(z)$, the family $B(\alpha,\varepsilon)$ reduces to some well-known subclasses of univalent functions defined and studied by different authors, see ([3]), ([4]), ([19]), ([20]), ([23]) and ([31]) among others.
For instance, if we let $\varepsilon=0$ then equation (4) immediately yields

$$f(z)=\left\{\alpha\int_0^z\frac{p(v)}{V}g(v)^\alpha dv\right\}^{\frac{1}{\alpha}}. \tag{5}$$

By differentiating equation (5) we have

$$\frac{z\,f'(z)\,f(z)^{\alpha-1}}{g(z)^\alpha}=p(z),\qquad z\in E \tag{6}$$

or equivalently

$$\Re e\left\{\frac{z\,f'(z)\,f(z)^{\alpha-1}}{g(z)^\alpha}\right\}>0,\qquad z\in E \tag{7}$$

The subclass of Bazilevic functions satisfying equation (6) are called Bazilevic functions of type α and are denoted by $B(\alpha)$ ([36]). In 1973, Noonan ([22]) gave a plausible description of functions of the class $B(\alpha)$ as those functions in Ψ for which each $r>1$, and the tangent to the curve $U_\alpha(r)=\left\{\varepsilon f(re^{i\theta})^\alpha,\ 0\le\theta<2\pi\right\}$ never turns back on itself as much as π radian. If $\alpha=1$, the class $B(\alpha)$ reduces to the family of close-to-convex functions; that is,

$$\Re e\left\{\frac{zf'(z)}{g(z)}\right\}>0\qquad z\in E. \tag{8}$$

If we decide to choose $g(z)=f(z)$ in inequality (4), we have

$$\Re e\left\{\frac{zf'(z)}{f(z)}\right\}>0\qquad z\in E, \tag{9}$$

which implies that $f(z)$ is starlike. Furthermore, if one replace $f(z)$ by$zf'(z)$, then

$$\Re e\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0\qquad z\in E,$$

which shows that $f(z)$ is convex. Moreover, if $g(z)=z$ in inequality (7), then the family $B_1(\alpha)$ (see [36]) of functions satisfying

$$\Re e\left\{\frac{z\,f'(z)\,f(z)^{\alpha-1}}{z^\alpha}\right\}>0,\qquad z\in E. \tag{10}$$

is obtained. Several subfamilies of Bazilevic functions have been studied repeatedly by different authors and their results authenticated diversely in literatures, see ([6]). In 1992, Abdulhalim ([1]) introduced a generalization of functions satisfying inequality (10) as

$$\Re e\left\{\frac{D^n f(z)^\alpha}{z^\alpha}\right\} > 0, \qquad z \in E \tag{11}$$

where the parameter α and the operator D^n is the famous Salagean derivative operator ([35]) defined below. He denoted this class of functions by $B_n(\alpha)$. It is easily seen that his generalization has extraneously included analytic functions satisfying

$$\Re e\left\{\frac{f(z)^\alpha}{z^\alpha}\right\} > 0, \qquad z \in E \tag{12}$$

which largely non-univalent in the unit disk (cf. ([31])). Abdulhalim ([1]) was able to show that for all $n \in N$, each function of the class $B_n(\alpha)$ is univalent in E. Now in 1983, Sălăgean ([35]) introduced the following differential operator:

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= D(D^0 f(z)) = z f'(z) \\ D^n f(z) &= D(D^{n-1} f(z)) = z(D^{n-1} f(z))'. \end{aligned} \tag{13}$$

Also in 2017, Darus and Owa ([8]) introduced and studied a fractional analytic function $g_\alpha(z)$

$$g_\alpha(z) = \frac{z}{1 - z^\alpha} = z + \sum_{k=1}^{\infty} z^{\alpha+k} \qquad (z \in E) \tag{14}$$

for some real α $(0 < \alpha \leq 2)$ in the open unit disk. See also ([7]), ([14]-[18]) and ([37]) for more details on fractional analytic functions. However, for the sake of present investigation, we shall consider the fractional analytic function $f(z)^\alpha$ which has the form

$$g(z)^\alpha = \frac{z^\alpha}{1 - z} = z^\alpha + \sum_{k=2}^{\infty} z^{\alpha+k-1} \qquad (z \in E) \tag{15}$$

for some real α $(\alpha > 0)$ in the open unit disk.
The Hadamard product or convolution of two functions $f, g \in A$ is denoted by $f * g$ and is defined as follows:

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z),$$

where $f(z)$ is as defined in (1) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k.$$

In view of (1) and (15), a new class, $W_{\alpha,n}$, of fractional analytic function is derived in E such that

$$f(z)^\alpha = f(z) * g(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k z^{\alpha+k-1} \qquad (z \in E) \tag{16}$$

for some real α ($\alpha > 0$) in the open unit disk.
From (13) and (16), we obtain the following differential operator

$$D^n f(z)^\alpha = \alpha^n z^\alpha + \sum_{k=2}^{\infty}(\alpha + k - 1)^n a_k z^{\alpha+k-1}. \tag{17}$$

From (17), we observe that

$$\Re e \left\{ \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \right\} > \beta, \ \ (0 \leq \beta < 1) \ \ z \in E. \tag{18}$$

Incidentally, (18) coincides with the special class of analytic function (Bazilevic) denoted by $T_n^\alpha(\beta)$ studied by different authors (see ([14]-[15]), ([30]-[31]), ([32]) and ([36]) among others) . Here, we define a modified Bazilevic function $F_{\alpha,\,n}(z) \in T_n^\alpha$ such that

$$F_{\alpha,\,n}(z) = z\left(1 + \sum_{k=2}^{\infty} \alpha_{n,\,k} a_k z^{k-1}\right) \tag{19}$$

where

$$\alpha_{n.\,k} = \left(\frac{\alpha + k - 1}{\alpha}\right)^n$$

Interestingly, (19) coincides with (1) if we set $\alpha = 1$ and $n = 0$. This work concerns mainly with the study of the class $U_{\alpha,n}$ of all functions $F_{\alpha,n} \in T_n^\alpha$ satisfying the inequality

$$|U_{F_{\alpha,n}}(z)| < 1, \qquad z \in E, \tag{20}$$

where

$$U_{F_{\alpha,n}}(z) = \Big(\frac{z}{F_{\alpha,n}(z)}\Big)^2 F'_{\alpha,n}(z) - 1$$

is associated with the class of modified Bazilevic functions T_α^n.
Although, several authors have examined the special class U, of analytic function $f(z)$ defined in (1), satisfying the geometric condition:

$$\big|U_f(z)\big| = \Big|\Big(\frac{z}{f(z)}\Big)^2 f'(z) - 1\Big| < 1, \quad z \in E,$$

(see [26], [34] among others), the main object of the present work is to investigate some basic properties of the new class $U_{F_{\alpha,n}}(z)$ satisfying the inequality (20). It is known that each functions in $U_f(z)$ belongs to S, and each function in

$$S_z = \left\{ z, \frac{z}{1 \pm z}, \ \frac{z}{(1 \pm z)^2}, \ \frac{z}{1 \pm z^2}, \ \frac{z}{1 \pm z + z^2} \right\}$$

belong to U. Also, the functions S_z are only function in S having integral coefficients in the power series expansions of $f \in S$. We remark here that the functions in S_z are extremal for certain geometric subclasses of S, (see [2], [11], [24], [25], [26], [27], [28], [33] and [34] among others).

2. Some properties of class $U_{\alpha,n}$

The first theorem given below is the characterisation property for $U_{\alpha,n}$.

Theorem 2.1. Every $F_{\alpha,n} \in U_{\alpha,n}$ has the representation

$$\frac{z}{F_{\alpha,n}(z)} = 1 - \alpha_{n,2} a_2(\alpha) z - z \int_0^z \frac{\omega(t)}{t^2} dt, a_2(\alpha) = a_2(F_{\alpha,n}) = \frac{F''_{\alpha,n}(0)}{2\alpha_{n,2}},$$

where $\alpha_{n,2} = \left(\frac{\alpha+1}{\alpha}\right)^n$, $\omega \in B_1$, the class of analytic functions in the unit disk E such that $\omega(0) = \omega'(0) = 0$ and $|\omega(z)| < 1$ for $z \in E$.

Proof. Suppose that $F_{\alpha,\,n}(z) = z + \sum_{k=2}^{\infty} \alpha_{n,\,k} a_k z^k$ in $U_{\alpha,n}$. Then we have that $\frac{F(z)}{z} \neq 0$ and $\left(\frac{z}{F(z)}\right)^2 F'(z) = 1 + \left(\alpha_{n,3} a_3 - \alpha_{n,2}^2 a_2^2\right) z^2 + ...,\quad z \in E$ where $\alpha_{n,2}^2 = \left(\frac{\alpha+1}{\alpha}\right)^{2n}$ and $\alpha_{n,3} = \left(\frac{\alpha+2}{\alpha}\right)^n$.

This may be written as

$$\frac{z}{F_{\alpha,n}(z)} - z\left(\frac{z}{F_{\alpha,n}(z)}\right)' = \left(\frac{z}{F_{\alpha,n}(z)}\right)^2 F'_{\alpha,n}(z) = 1 + \omega(z), \qquad z \in E \tag{21}$$

where $\omega(z) = \left(\alpha_{n,3}\, a_3 - \alpha_{n,2}^2\, a_2^2\right) z^2 + ...$ and with $\omega \in B_1$. Also, by Schwarz lemma, $|\omega(z)| \leq |z|^2$, $z \in E$. Obviously,

$$\left(\frac{1}{F_{\alpha,n}(z)} - \frac{1}{z}\right)' = -\frac{\omega(z)}{z^2}.$$

Since

$$\left(\frac{1}{F(z)} - \frac{1}{z}\right|_{z=0} = -\alpha_{n,2} a_2 \ ,$$

then by simple integration

$$\frac{1}{F(z)} - \frac{1}{z} = -\alpha_{n,2} a_2 - \int_0^z \frac{\omega(t)}{t^2} dt$$

and thus the desired representation follows.

This representation together with many others that follow from it led to a number of recent investigations (see ([24]-([27])) and ([33]) for more details).

However, because $\omega \in B_1$, Schwarz lemma give $|\omega(z)| \leq |z|^2$. Consequently,

$$\left|\frac{z}{F(z)} + \alpha_{n,2} a_2 z - 1\right| \leq |w(z)| = |z|^2, z \in E. \tag{22}$$

It was observed that if z is fixed $(0 \leq |z| < 1)$, then this inequality determines the range of the functional

$$\frac{z}{F_{\alpha,n}(z)} + (\alpha_{n,2} a_2 - 1) z$$

in the class $U_{\alpha,n}$. Particularly, if $a_2 = 0$ then by a simple computation, (22) yields

$$\left|\frac{F_{\alpha,n}(z)}{z} - \frac{1}{1 - |z|^4}\right| \leq \frac{|z|^2}{1 - |z|^4}, z \in E. \tag{23}$$

So that for every $F_{\alpha,n} \in U_{\alpha,n}$ with $F''_{\alpha,n}(0) = 0$,

$$\frac{|z|}{1 + |z|^2} \leq |F_{\alpha,n}(z)| \leq \frac{|z|}{1 - |z|^2}$$

and

$$\Re\left(\frac{F_{\alpha,n}(z)}{z}\right) \geq \frac{1}{1+|z|^2} > \frac{1}{2}, z \in D. \tag{24}$$

Corollary 2.2. Let$F_{\alpha,n} \in U_{\alpha,n}$. Then

(1) $\left|\frac{z}{F_{\alpha,n}(z)} - 1\right| \leq |z|\left(\alpha_{\alpha,2}|a_2| + |z|\right)$, $z \in D$.

(2) $\Re\left(\frac{F_{\alpha,n}(z)}{z}\right) > \frac{1}{2}$ *in* D *if* $F''_{\alpha,n}(0) = 0$.

Remark 2.1. It can easily be shown that if $F(z) = \frac{f(z)}{1+z} \in U$, then

(i) $\left|\frac{z}{F(z)} - 1\right| \leq |z|\bigg(|a_2 - 1| + |z|\bigg), \qquad z \in E.$

(ii) $\Re\bigg(\frac{F(z)}{z}\bigg) > 1/3$ in E if $F''(0) = 0$.

Here, we note that one of the sufficient conditions for function $F_{\alpha,n}$ of the form (19) to be in S^* is that $\sum_{k=2}^{\infty} \alpha_{n,k} k |a_k(\alpha)| \leq 1$. However, the coefficient condition is also sufficient for $F_{\alpha,n}$ to belong to H, where H denote the class of normalized analytic function $F_{\alpha,n}$ satisfying the condition

$$\left|F'_{\alpha,n}(z) - 1\right| < 1 \; in \; E.$$

Theorem 2.3. Suppose that $F_{\alpha,\,n}(z) = z + \sum_{k=2}^{\infty} \alpha_{n,\,k} a_k z^k$ such that $\sum_{k=2}^{\infty} \alpha_{n,k}\; k\; |a_k(\alpha)| \leq 1$, then, $F_{\alpha,n} \in U_{\alpha,n}$, where $\alpha_{n,k} = \left(\frac{\alpha+k-1}{\alpha}\right)^n$. The result is sharp.

Proof. Following the assumption that $\sum_{k=2}^{\infty} \alpha_{n,k} k |a_k| \leq 1$, then

$$\begin{aligned}
\left|F'_{\alpha,n}(z) - \left(\frac{F_{\alpha,n}(z)}{z}\right)^2\right| &= \left|1 + \sum_{k=2}^{\infty} k\alpha_{n,k} a_k z^{k-1} - \left(1 + \sum_{k=2}^{\infty} \alpha_{n,k} a_k z^{k-1}\right)^2\right| \\
&= \left|\sum_{k=2}^{\infty} \alpha_{n,k}(k-2)\, a_k z^{k-1} - \left(\sum_{k=2}^{\infty} \alpha_{n,k} a_k z^{k-1}\right)^2\right| \\
&= |z|^2 \left|\sum_{k=2}^{\infty} \alpha_{n,k}(k-2)\, a_k z^{k-3} - \left(\sum_{k=2}^{\infty} \alpha_{n,k} a_k(\alpha) z^{k-2}\right)^2\right| .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|F'_{\alpha,n}(z) - \left(\frac{F_{\alpha,n}(z)}{z}\right)^2\right| &< \textstyle\sum_{k=2}^{\infty} \alpha_{n,k}\,(k-2)\; |a_k| - \left(\sum_{k=2}^{\infty} \alpha_{n,k}\,|a_k|\right)^2 \\
&\leq\; 1 - 2\textstyle\sum_{k=2}^{\infty} \alpha_{n,k}\;|a_k| + \left(\sum_{k=2}^{\infty} \alpha_{n,k}\,|a_k|\right)^2 \\
&\leq\; \left(1 - \textstyle\sum_{k=2}^{\infty} \alpha_{n,k}\,|a_k|\right)^2 \\
&\leq \left|\frac{F_{\alpha,n}(z)}{z}\right|^2 .
\end{aligned}$$

That is

$$\left|F'_{\alpha,n}(z) - \left(\frac{F_{\alpha,n}(z)}{z}\right)^2\right| \leq \left|\frac{F_{\alpha,n}(z)}{z}\right|^2$$

from which it is obvious that $F_{\alpha,n} \in U_{\alpha,n}$. The result is sharp.

To show that the constant 1 in the coefficient estimate cannot be replaced by a larger number, for instance, $1 + \delta$ $(\delta > 0)$, we consider the function

$$F_{\alpha,n}(z) = z + \frac{1+\delta}{k} z^k, \quad (k \geq 2).$$

It is observed that $F'_{\alpha,n}(z) = 1+(1+\delta)z^{k-1}$ has a Zero in E since $\delta > 0$. Therefore, the result is the best possible.

3. Special Form of Functions in Class $U_{\alpha,n}$

Our prime focus in this section is to investigate the analytic function $F_{\alpha,n}(z)$ in E having the form

$$F_{\alpha,n} = \frac{z}{1+\sum_{k=1}^{\infty} \alpha_{n,k} c_k z^k} \tag{25}$$

where

$$\alpha_{n,k} = \Big(\frac{\alpha+k-1}{\alpha}\Big)^n.$$

We shall remark here that if $F_{\alpha,n} \in S$ then $\frac{z}{F_{\alpha,n}(z)}$ is non-vanishing in the unit disk E and hence, can be represented as Taylor's series of the form (25) which is convenient for our investigation. Now, we recall that if $F_{\alpha,n} \in S$ and has the above form, then from the well-known Area Theorem (see ([12]) and ([28])) we have that

$$\sum_{k=2}^{\infty} (k-1)\alpha_{n,k}^2 \big|c_k\big|^2 \leq 1. \tag{26}$$

But that condition is not sufficient for the univalence of the analytic function $F_{\alpha,n}$ of the form (25) (see Theorem 3.3 below). In the next theorem, we present a sufficient condition for the univalence in terms of the coefficients a_k of analytic function $F_{\alpha,n}$ of the form (25).

Theorem 3.1. Let $F_{\alpha,n} \in T_n^{\alpha}$ have the form (25), if

$$\sum_{k=2}^{\infty} (k-1)\alpha_{n,k}\big|c_k\big| \leq 1$$

$$\alpha_{n,k} = \Big(\frac{\alpha+k-1}{\alpha}\Big)^n$$

then $F_{\alpha,n} \in U_{\alpha,n}$ and the constant 1 is the best possible in a sense: if

$$\sum_{k=2}^{\infty} (k-1)\alpha_{n,k}\big|c_k\big| = \Big(\frac{1+\alpha}{\alpha}\Big)^n\big(1+\sqrt{\delta}\big)$$

for some $\delta > 0$, $\alpha > 0$ and $n \in \mathbb{N}_0$, then there exists an $F_{\alpha,n}$ such that $F_{\alpha,n}$ is not univalent in E.

Proof. For the first part of the statements, we have

$$\Big|U_{F_{\alpha,n}}(z)\Big| = \Big| - z\Big(\frac{z}{F_{\alpha,n}(z)}\Big)' + \frac{z}{F_{\alpha,n}(z)} - 1\Big| = \Big| - \sum_{k=2}^{\infty} (k-1)\alpha_{n,k} a_k z^{k-1}\Big|$$

$$\leq \sum_{k=2}^{\infty} (k-1)\alpha_{n,k}\big|a_k\big| \leq 1.$$

To show that the theorem is sharp, we consider the function $F_{\alpha,n}(z) = z - mz^2$ where $m = \frac{\sqrt{1+\sqrt{\delta}}}{1+\sqrt{1+\sqrt{\delta}}}$, $\delta > 0$, so that $1/2 < m < 1$.

Then, we have

$$\frac{z}{F_{\alpha,n}(z)} = \frac{1}{1-mz} = 1 + \sum_{k=1}^{\infty} m^k z^k.$$

Also, we can say that

$$\sum_{k=2}^{\infty}\big(k-1\big)\alpha_{n,k}|c_k| = \sum_{k=2}^{\infty}\big(k-1\big)\alpha_{n,k}m^k = \alpha_{n,k}\Big(\frac{m}{m-1}\Big)^2 = \alpha_{n,2}\big(1+\sqrt{\delta}\big).$$

Now, it is observed that $F'_{\alpha,n}(z) = 1-2mz$, therefore, $F'_{\alpha,n}(1/2m) = 0$ proving that $F_{\alpha,n}$ is not univalent in the unit disk E. The coefficient condition of Theorem 3.1 is only a sufficient condition for $F_{\alpha,n}$ to be in the class $U_{\alpha,n}$. In fact, it is not too difficult to see that the condition of Theorem 3.1 is not a necessary condition for the corresponding function to be in that class.

Theorem 3.2. Let $F_{\alpha,n} \in U_{\alpha,n}$ have the form (25). Then

$$\sum_{k=2}^{\infty}\big(k-1\big)^2\alpha_{n,k}^2|c_k|^2 \le 1 \tag{27}$$

In particular, we have $|c_1| \le 2$ and $|c_k| \le \frac{1}{(k-1)\alpha_{n,k}}$ for $k \ge 2$ and $\alpha_{n,k}$ is as earlier defined. The result is sharp.

Proof. Recall that $F_{\alpha,n} \in U_{\alpha,n}$ if and only if

$$\Big|U_{F_{\alpha,n}}(z)\Big| = \Big|\frac{z}{F_{\alpha,n}(z)} - z\Big(\frac{z}{F_{\alpha,n}}\Big)' - 1\Big| = \Big|\sum_{k=2}^{\infty}\big(k-1\big)\alpha_{n,k}c_kz^k\Big|.$$

We note that $g_{\alpha,n}(z) = \sum_{k=3}^{\infty}\big(k-2\big)\alpha_{n,k}a_kz^{k-1}$ is analytic in E and therefore, with $z = re^{i\theta}$, we have

$$\sum_{k=2}^{\infty}\big(k-1\big)^2\alpha_{n,k}^2|c_k|^2r^{2(k)} = \frac{1}{2\pi}\int_0^{2\pi}\big|g(re^{i\theta})\big|^2d\theta < 1$$

so that, as $r \to 1^-$, we obtain the desired inequality. Because $c_1 = -\frac{F''_{\alpha,n}(0)}{2\alpha_{n,2}}$ and the Bieberbach inequality gives $|c_1| \le 2$ and the fact that the Koebe function $k(z) = \frac{z}{(1-z)^2}$, $(\alpha > 0)$ belong to $U_{\alpha,n}$ shows that the result is best possible. Further, the inequality (27) implies that for $k \ge 2$ we have $|c_k| \le \frac{1}{(k-1)\alpha_{n,k}}$.

It is observed that the necessary coefficient condition of Theorem 3.2 for the class $U_{\alpha,n}$ is stronger than that for the class S, namely the inequality (26).

Theorem 3.3. Let $F_{\alpha,n} \in T_n^{\alpha}$ and have the form (25) satisfying the condition

$$\sum_{k=2}^{\infty}\big(k-1\big)\alpha_{n,k}^2|c_k|^2 \le 1.$$

Then, $F_{\alpha,n}$ is univalent in the disk $|z| < \frac{1}{\sqrt{2}}$ and the result is the best possible.

Proof. Consider the function $g_{\alpha,n}(z) = \frac{1}{r}F_{\alpha,n}(rz)$ where $0 < r \le 1$. Then

$$\frac{z}{g_{\alpha,n}(z)} = 1 + \sum_{k=1}^{\infty}\alpha_{n,k}c_kr^k.$$

Because

$$\begin{aligned}\sum_{k=2}^{\infty}\big(k-1\big)\alpha_{n,k}|c_k|r^k &= \sum_{k=2}^{\infty}\sqrt{\big(k-1\big)}\alpha_{n,k}|c_k|\sqrt{\big(k-1\big)}r^k\\ &\le \Big(\sum_{k=2}^{\infty}\big(k-1\big)\alpha_{n,k}^2|c_k|^2\Big)^{1/2}\Big(\sum_{k=2}^{\infty}\big(k-1\big)r^{2(k)}\Big)^{1/2}\end{aligned}$$

$$= \frac{r^2}{1-r^2} \leq 1$$

for $0 < r \leq 1/\sqrt{2}$, it follows easily that g_α is in the class $U_{\alpha,n}$. In particular $F_{\alpha,n}$ is univalent in the disk $|z| < 1/\sqrt{2}$.
For the function $F_{\alpha,n,0}(z) = z - \frac{1}{\sqrt{2}}z^2$, we have

$$\frac{z}{F_{\alpha,n,0}(z)} = \frac{1}{1-\frac{1}{\sqrt{2}}z^2} = 1 + \sum_{k=1}^{\infty}\big(\frac{1}{\sqrt{2}}\big)^k z^k$$

and

$$\sum_{k=2}^{\infty}\big(k-1\big)\alpha_{n,k}^2|c_k|^2 = \sum_{k=2}^{\infty}\big(k-1\big)\alpha_{n,k}^2(1/2)^k = 1$$

Otherwise, $\Re F'_{\alpha,n,0}(z) = \Re\big(1-\sqrt{2}z\big) > 0$ for $|z| < \frac{1}{\sqrt{2}}$ and $F'_{\alpha,n,0}(1/\sqrt{2})$.
Theorem 3.4. Let $F_{\alpha,n} \in T_n^\alpha$ and have the form (25) satisfying the condition

$$\sum_{k=2}^{\infty}\big(k-1\big)^2\alpha_{n,k}^2|c_k|^2 \leq 1.$$

Then $F_{\alpha,n}$ is univalent in the disk $|z| < \sqrt{\frac{\sqrt{5}-1}{2}}$ and the result is best possible.
Proof. As in the proof of the theorem just concluded. It suffices to see that

$$\sum_{k=2}^{\infty}\big(k-1\big)\alpha_{n,k}|c_k|r^k \leq \Big(\sum_{k=2}^{\infty}\big(k-1\big)^2\alpha_{n,k}^2|c_k|^2\Big)^{1/2}\Big(\sum_{k=2}^{\infty}r^{2k}\Big)^{1/2}$$

$$= \frac{r^2}{\sqrt{1-r^2}} \leq 1,$$

where $r^4 + r^2 - 1 \leq 0$, that is if $0 < r \leq r_0 = \sqrt{\frac{\sqrt{5}-1}{2}} \approx 0.78615$. It means that the function $g_{\alpha,n}$ defined as $g_{\alpha,n}(z) = \frac{1}{r}F_{\alpha,n}(rz)$ is in the class $U_{\alpha,n}$ and hence $F_{\alpha,n}(z)$ is univalent in the disk $|z| < r_0 = \sqrt{\frac{\sqrt{5}-1}{2}} \approx 0.78615$. Now, for function $F_{\alpha,n,0}(z)$ defined as

$$\frac{z}{F_{\alpha,n,0}(z)} = 1 + \sum_{k=2}^{\infty}\frac{r^k}{(k-1)\alpha_{n,k}}z^k = 1 - \frac{r_0 z}{(\alpha_{n,k})^2}log\Big(1-\frac{r_0 z}{\alpha_{n,k}}\Big)$$

where $\alpha_n^k = \alpha_{n,k}$, i.e. $\alpha_n^2 = \alpha_{n,2}$, $\alpha_n^3 = \alpha_{n,3}$ etc., then we have that $\Re(F_{\alpha,n,0}(z)) > 0$ in E. so that $F_{\alpha,n} \in A$ and

$$\sum_{k=3}^{\infty}(k-2)^2(\alpha_{n,k})^2|a_k|^2 = \sum_{k=3}^{\infty}(k-2)^2(\alpha_{n,k})^2\frac{r^{2(k-1)}}{(k-2)^2(\alpha_{n,k})^2} = 1.$$

On the other hand side for $|z| < r_0$ we find that

$$\Big|(\frac{z}{F_{\alpha,n,0}(z)})^2F'_{\alpha,n,0}(z) - 1\Big| = \Big| - \frac{r_0^2z^2}{\alpha_n^4 - \alpha_n^3 r_0 z}\Big| < \frac{r_0^4}{\alpha_n^4 - \alpha_n^3 r_0^2} = 1,$$

while for $r_0 \leq z = r < 1$:

$$\Big|(\frac{z}{F_{\alpha,n,0}(z)})^2F'_{\alpha,n,0}(z) - 1\Big|_{z=r} = \frac{r^4}{\alpha_n^4 - \alpha_n^3 r^2} \geq 1.$$

It means that $g_{\alpha,n,0}(z) = \frac{1}{r}F_{\alpha,n,0}(rz)$ is in the class $U_{\alpha,n}$ for $r \leq r_0$, but not in a larger value of r, and hence, $F_{\alpha,n}$ is univalent in the disk $|z| < r_0$, but not in a larger disk. Furthermore, a simple computation yields

$$F'_{\alpha,n,0}(z) = \frac{1 - \frac{r_0 z}{\alpha_n} - \frac{r_0^2 z^2}{\alpha_n^3}}{\Big(1 - \frac{r_0 z}{\alpha_n}\Big)\Big[1 - \frac{r_0 z}{\alpha_n^2} log\Big(1 - \frac{r_0 z}{\alpha_n}\Big)\Big]^2}$$

and therefore, $F'_{\alpha,n,0}(r_0) = 0$. Thus, $F_{\alpha,n}$ cannot be univalent in any disk larger than the disk $|z| < r_0$.

4. Further Properties of Functions in $U_{\alpha,n}$

Theorem 4.1. Let $F_{\alpha,n} \in T_n^\alpha$ of the form (25) with $c_k \geq 0$ and for all $k \geq 2$. Then we have the following equivalence:
$(a) F_{\alpha,n} \in S$
$(b) \frac{F_{\alpha,n}(z)F'_{\alpha,n}(z)}{z} \neq 0$ for $z \in E$
$(c) \sum_{k=2}^{\infty} \alpha_{n,k} c_k \leq 1$
$(d) F_{\alpha,n} \in U_{\alpha,n}$.
where $\alpha_{n,k} = (\frac{\alpha+k-1}{\alpha})^n$ and $z \in E$.
Proof. $(a) \Rightarrow (b)$: Let $F_{\alpha,n} \in U_{\alpha,n}$ be of the form (25) with $a_k \geq 0$ for all $k \geq 2$. Then,

$$F'_{\alpha,n}(z) \neq 0 \quad and \quad \frac{F_{\alpha,n}(z)}{z} \neq 0 \; in \; E.$$

$(b) \Rightarrow (c)$: From the representation of $F_{\alpha,n}$ and (21) we see that for $z \in E$,

$$\Big(\frac{rz}{F_{\alpha,n}(rz)}\Big)^2 F'_{\alpha,n}(rz) = 1 - \sum_{k=2}^{\infty}(k-1)\alpha_{n,k}c_k r^k z^k, \qquad \alpha_{n,k} = \Big(\frac{\alpha+k-1}{\alpha}\Big)^n$$

from which as $\frac{z}{F_{\alpha,n}(z)} \neq 0$, it follows that $F'_{\alpha,n}(rz) \neq 0$ is equivalence to

$$1 - \sum_{k=2}^{\infty}(k-1)\alpha_{n,k}c_k r^k z^k \neq 0.$$

We claim that

$$\sum_{k=2}^{\infty}(k-1)\alpha_{n,k}c_k \leq 1.$$

Suppose on the contrary that

$$\sum_{k=2}^{\infty}(k-1)\alpha_{n,k}c_k > 1.$$

Then, on the other hand, their exists a positive integer m such that

$$\sum_{k=2}^{\infty}(k-1)\alpha_{n,k}c_k > 1$$

and so there exists an r_0 with $0 < r_0 < 1$ and

$$\sum_{k=2}^{m}(k-1)\alpha_{n,k}c_k r_0^k > 1.$$

On the other hand, as $a_k \geq 0$ for $k \geq 2$, we have that

$$\Big(\frac{r_0}{F_{\alpha,n}(r_0)}\Big)^2 F'_{\alpha,n}(r_0) = 1 - \sum_{k=2}^{\infty}(k-1)\alpha_{n,k}a_k r_0^k \leq 1 - \sum_{k=2}^{m}(k-1)\alpha_{n,k}a_k r_0^k < 0$$

and since $F'_{\alpha,n}(r)$ is a continuous function of r with $F'_{\alpha,n}(0) = 1$ and $F'_{\alpha,n}(r) < 0$, there exists an $r_1 (0 < r_1 < r_0 < 1)$ such that $F'_{\alpha,n}(r) = 0$. This is a contradiction. Consequently, we must have

$$\sum_{k=2}^{\infty}(k-1)\alpha_{n,k}c_k \leq 1.$$

$(c) \Rightarrow (d)$: Suppose that $\sum_{k=2}^{\infty}(k-1)\alpha_{n,k}c_k \leq 1$. Then, by Theorem 3.1, it follows that $F_{\alpha,n} \in U_{\alpha,n}$.

$(d) \Rightarrow (a)$: $U_{\alpha,n} \in S$.

Finally, we consider the radius property of univalent functions as well as the convolution property with $U_{\alpha,n}$. We noted that if for every $F_{\alpha,n} \in S$ the function $\frac{1}{r}F_{\alpha,n}(rz)$ for $0 < r \leq r_0$, and r_0 is the largest number for which this holds, then we say that r_0 is the $U_{\alpha,n}$ radius (or the radius of $U_{\alpha,n}$-property) in the class S. In this case, we may conveniently write $r_0 = r_{u_{\alpha,n}}(S)$.

Theorem 4.2.

$$r_{u_{\alpha,n}}(S) = \frac{1}{\sqrt{2}}.$$

Proof. Let $F_{\alpha,n} \in S$. Then every such an $F_{\alpha,n}$ has the form

$$\frac{z}{F_{\alpha,n}(z)} = 1 + \sum_{k=1}^{\infty}\alpha_{n,k}c_k z^k.$$

Then by (26) we obtain

$$\sum_{k=2}^{\infty}(k-1)\alpha_{n,k}^2|c_k|^2 \leq 1.$$

The desired conclusion clearly follows from theorem 3.3. Moreover, to see that the number $\frac{1}{\sqrt{2}}$ is the best possible, we consider the function

$$F_{\alpha,n}(z) = \frac{z(1-\frac{1}{\sqrt{2}}z)}{1-z^2}.$$

If we put $z = \rho e^{i\theta} \in E$, then

$$\Re\big((1-z^2)F'_{\alpha,n}(z)\big) = \frac{(1-\rho^2)\big(1+\rho^2-\sqrt{2}\rho cos\theta\big)}{\big|1-\rho^2 e^{i2\theta}\big|} > 0$$

for $0 \leq \rho < 1$. Thus, $F_{\alpha,n}$ is close-to-convex in E and therefore, $F_{\alpha,n} \in S$.

Next, we note that

$$\bigg|\Big(\frac{z}{F_{\alpha,n}(z)}\Big)^2 F'_{\alpha,n}(z) - 1\bigg| = \Big|\frac{z}{\sqrt{2}-z}\Big|^2$$

is less than 1 for $|z| < \frac{1}{\sqrt{2}}$, equal to 1 for $|z| = \frac{1}{\sqrt{2}}$ and bigger than 1 for $\frac{1}{\sqrt{2}} < z = r < 1$. The sharpness part follows.

Theorem 4.3. Let $F_{\alpha,n}, G_{\alpha,n} \in S$ with the representations

$$\frac{z}{F_{\alpha,n}(z)} = 1 + \sum_{k=1}^{\infty}\alpha_{n,k}a_k z^k, \quad \frac{z}{G_{\alpha,n}(z)} = 1 + \sum_{k=1}^{\infty}\alpha_{n,k}b_k z^k.$$

If

$$\Phi(z) = \frac{z}{F_{\alpha,n}(z)} * \frac{z}{G_{\alpha,n}(z)} = 1 + \sum_{k=1}^{\infty} \alpha_{n,k} a_k b_k z^k \neq 0$$

for every $z \in E$, then

$$F_{\alpha,n} = \frac{z}{\Phi(z)} \in U_{\alpha,n}$$

and, in particular, $F_{\alpha,n}$ is univalent in E.

Proof. For $F_{\alpha,n}, G_{\alpha,n} \in S$ with their representations we have that

$$\sum_{k=2}^{\infty}(k-1)\alpha_{n,k}|a_k|^2 \leq 1 \quad and \quad \sum_{k=2}^{\infty}(k-1)\alpha_{n,k}|b_k|^2 \leq 1.$$

By assumption

$$\Phi(z) = \frac{z}{F_{\alpha,n}(z)} * \frac{z}{G_{\alpha,n}(z)} = 1 + \sum_{k=1}^{\infty} \alpha_{n,k} a_k b_k z^k \neq 0,$$

and therefore, the function $F_{\alpha,n}$ is analytic in E. By the classical Cauchy-Schwarz inequality, we conclude that

$$\sum_{k=2}^{\infty}(k-1)\alpha_{n,k}|a_k b_k| \leq \Big(\sum_{k=2}^{\infty}(k-1)\alpha_{n,k}|a_k|^2\Big)^{\frac{1}{2}} \Big(\sum_{k=2}^{\infty}(k-1)\alpha_{n,k}|b_k|^2\Big)^{\frac{1}{2}} \leq 1,$$

which by theorem (4.1), $F_{\alpha,n} \in U_{\alpha,n}$.

Remark 4.1. If we let $\alpha = 1$ and $n = 0$ in all the results obtained above, we obtain the results due to Obradovic and Ponnusamy ([28]).

5. Application of Fractional Calculus

Before proceeding to the result in this section, the following useful definitions shall be necessary .

Definition 5.1 Given function $f(z)$ of the form (1). The fractional integral of order ϵ $(0 < \epsilon \leq 1)$ is defined such that

$$D_z^{-\epsilon} f(z) = \frac{1}{\Gamma(\epsilon)} \int_0^z \frac{f(t)}{(z-t)^{1-\epsilon}} dt \tag{28}$$

where $f(z)$ is analytic function in a simply connected region of $z-$plane containing the origin and the multiplicity of $(z-t)^{\epsilon-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Definition 5.2. Similarly, the fractional derivative of order ϵ $(0 \leq \epsilon < 1)$ denoted by $D_z^{\epsilon} f(z)$ is given such that

$$D_z^{\epsilon} f(z) = \frac{1}{\Gamma(1-\epsilon)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\epsilon}} dt \tag{29}$$

where the multiplicity of $(z-t)^{-\epsilon}$ is as removed in Definition 5.1. It can be verified from (30) that the fractional derivative of order m is given by

$$D_z^{\epsilon} f(z) = \frac{d^m}{dz^m}\big(D_z^{\epsilon-m} f(z)\big), \quad m \leq \epsilon < m+1,\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$$

and that of order $m+\epsilon$ is given by

$$D_z^{m+\epsilon} f(z) = \frac{d^m}{dz^m}\big(D_z^{\epsilon} f(z)\big), \quad m \leq \epsilon < m+1,\ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\,.$$

Interestingly both (28) and (29) have the series representations

$$D_z^{-\epsilon} f(z) = \frac{1}{\Gamma(2+\epsilon)} z^{\epsilon+1} + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1+\epsilon)} c_k z^{k+\epsilon} \tag{30}$$

and

$$D_z^{\epsilon} f(z) = \frac{1}{\Gamma(2-\epsilon)} z^{1-\epsilon} + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1-\epsilon)} c_k z^{k-\epsilon} \tag{31}$$

respectively. (see [13], [21] and [38] among others).

Theorem 5.1. Let $F_{\alpha,n}(z) \in T_n^{\alpha}$ of the form (25) belongs to $U_{\alpha,n}$, then

$$\frac{|z|^{1+\epsilon}}{\Gamma(2+\epsilon)} \left\{1 - \frac{2}{2+\epsilon}\left(\frac{\alpha}{\alpha+1}\right)^n |z|\right\} \le \left|D_z^{-\epsilon} f(z)\right| \le \frac{|z|^{1+\epsilon}}{\Gamma(2+\epsilon)} \left\{1 + \frac{2}{2+\epsilon}\left(\frac{\alpha}{\alpha+1}\right)^n |z|\right\} \tag{32}$$

where all the parameters involved are as earlier defined.
The inequality (32) is attained for function $F(z)$ given as

$$F(z) = \frac{z}{1+z}.$$

Proof. With reference to Theorem 3.1, we have

$$\sum_{k=2}^{\infty} c_k \le \left(\frac{\alpha}{\alpha+1}\right)^k. \tag{33}$$

Also, from definition (28), we have

$$D_z^{-\epsilon} f(z) = \frac{1}{\Gamma(2+\epsilon)} z^{\epsilon+1} + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k+1+\epsilon)} c_k z^{k+\epsilon}.$$

It follows that

$$\Gamma(2+\delta) z^{-\epsilon} D_z^{-\epsilon} f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\epsilon)}{\Gamma(k+1+\epsilon)} c_k z^k = z + \sum_{k=2}^{\infty} \mu(k) c_k z^k \tag{34}$$

where $\mu(k) = \frac{\Gamma(k+1)}{\Gamma(k+1+\epsilon)}$. It is noteworthy to say that $\mu(k)$ is a decreasing function of k and

$$0 < \mu(k) \le \mu(2) = \frac{2}{2+\epsilon}.$$

Now, appealing to (33) and (34), we obtain

$$\left|\Gamma(2+\delta) z^{-\epsilon} D_z^{-\epsilon} f(z)\right| \le |z| + \mu(2)|z|^2 \sum_{k=2}^{\infty} c_k \le |z| + \left(\frac{2}{2+\epsilon}\right)\left(\frac{\alpha}{\alpha+1}\right)^n |z|^2.$$

Similarly,

$$\left|\Gamma(2+\delta) z^{-\epsilon} D_z^{-\epsilon} f(z)\right| \ge |z| - \mu(2)|z|^2 \sum_{k=2}^{\infty} c_k \ge |z| - \left(\frac{2}{2+\epsilon}\right)\left(\frac{\alpha}{\alpha+1}\right)^n |z|^2.$$

This completes the proof of Theorem 5.1.

Theorem 5.2. Let $F_{\alpha,n}(z) \in T_n^{\alpha}$ of the form (25) belongs to $U_{\alpha,n}$, then

$$\frac{|z|^{1-\epsilon}}{\Gamma(2-\epsilon)} \left\{1 - \frac{2}{2-\epsilon}\left(\frac{\alpha}{\alpha+1}\right)^n |z|\right\} \le \left|D_z^{-\epsilon} f(z)\right| \le \frac{|z|^{1-\epsilon}}{\Gamma(2-\epsilon)} \left\{1 + \frac{2}{2-\epsilon}\left(\frac{\alpha}{\alpha+1}\right)^n |z|\right\} \tag{35}$$

where all the parameters involved are as earlier defined.
The inequality (35) is attained for function $F(z)$ given as

$$F(z) = \frac{z}{1+z}.$$

Proof. The proof is similar to that of Theorem 5.1.
However, for various choices of the parameters, n, α, δ in the Theorem 5.1 and Theorem 5.2, several corollaries follow as simple consequences. Few of them are listed below:

Illustration 5.1. Let $F_{1,n}(z) \in T_n^1$ be in the class $U_{1,n}$, then

$$\frac{|z|^{1+\epsilon}}{\Gamma(2+\epsilon)}\left\{1-\frac{2}{2+\epsilon}\left(\frac{1}{2}\right)^n|z|\right\} \leq \left|D_z^{-\epsilon}f(z)\right| \leq \frac{|z|^{1+\epsilon}}{\Gamma(2+\epsilon)}\left\{1+\frac{2}{2+\epsilon}\left(\frac{1}{2}\right)^n|z|\right\}$$

Illustration 5.2. Let $F_{1,n}(z) \in T_n^1$ be in the class $U_{1,n}$, then for $\epsilon = 1$

$$\frac{|z|^2}{2}\left\{1-\frac{2}{3}\left(\frac{1}{2}\right)^n|z|\right\} \leq \left|D_z^{-\epsilon}f(z)\right| \leq \frac{|z|^2}{2}\left\{1+\frac{2}{3}\left(\frac{1}{2}\right)^n|z|\right\}$$

Illustration 5.3. Let $F_{1,n}(z) \in T_n^1$ of the form (25) belongs to $U_{1,n}$, then

$$\frac{|z|^{1-\epsilon}}{\Gamma(2-\epsilon)}\left\{1-\frac{2}{2-\epsilon}\left(\frac{1}{2}\right)^n|z|\right\} \leq \left|D_z^{-\epsilon}f(z)\right| \leq \frac{|z|^{1-\epsilon}}{\Gamma(2-\epsilon)}\left\{1+\frac{2}{2-\epsilon}\left(\frac{1}{2}\right)^n|z|\right\} \tag{36}$$

Illustration 5.4. Let $F_{1,n}(z) \in T_n^1$ of the form (25) belongs to $U_{1,n}$, then for $\alpha = 1$ and $\epsilon = 0$

$$|z|\left\{1-\left(\frac{1}{2}\right)^n|z|\right\} \leq \left|D_z^{-\epsilon}f(z)\right| \leq |z|\left\{1+\left(\frac{1}{2}\right)^n|z|\right\} \tag{37}$$

References

[1] S. Abdulhalim, On a class of analytic function involving the Sălăgean differential Operator.Tamkang Journal of Mathematics, vol. 23, no.1, 51-58, 1992.

[2] L. A. Aksentev, Sufficient conditions for Univalence of regular functions,(Russian), Izu Vyss. Ucebn. zaved. Mathematika 1958(4), 3-7, 1958.

[3] F. M. Al-Aboudi, n-Bazilevic functions, Abstr. Appl. Anal., Article ID383592, 1-10, 2012.

[4] A. A. Amer and M. Darus, Distortion theorem for certain class of Bazilevic functions. Internat. J. Math. Anal. 6, 591-597, 2012.

[5] I. E. Bazilevic, On a class of integrability in quadratures of the Loewner-Kufarev Equation.Mathematicheskii sbornik, Russian, vol.37, no.79, 471-476, 1955.

[6] S. D. Bernardi, Bibliography of Schlicht Functions. Reprinted by Mariner Publishing, Tampa, Fla, USA, Courant Institute of Mathematical Sciences, New York University, 1983.

[7] M. Darus and R.W. Ibrahim, Partial sums of analytic functions of bounded turning with applications, Computational and Applied Mathematics, 29(1), 81-88, 2010.

[8] M. Darus and S. Owa, New subclasses concerning some analytic and univalent functions. Chinese J. Math. Article ID4674782, 2017, 4 pages. http//doi.org/10.1155/2017/4674782.

[9] P. L. Duren, Univalent Functions, Grundlehren der mathematischen Wissenschaften 259, New York, Berlin, Heidelberg, Tokyo, Springer-Verlag, 1983.

[10] R. Founier and S. Ponnusamy, A class of locally univalent functions defined by a differential inequality, Complex var. Elliptic Equ. 52(1), 1-8, 2007.

[11] B. Friedman, Two theorems on schlicht functions, Duke Math. J. 13, 171-177, 1946.

[12] A. W. Goodman, Univalent functions, Vol. 1-2, Mariner, Tampa, Florida, 1983.

[13] J. O. Hamzat and M. O. Olayiwola, Application of fractional calculus on certain new subclasses of analytic function, Int. J. Sci. Tech. vol. 3, Issue 10, (2015), 235-245.

[14] J.O. Hamzat, Subordination Results Associated with Generalized Bessel Functions, J. Nepal Math. Soc. vol. 2, Issue 1, 2019, 57-64.

[15] J.O. Hamzat and O. Fagbemiro, Some Properties of a New Subclass of Bazilevic Functions Defined by Catas et al Differential Operator, Trends in Science and Tech. J. vol.3, no. 2B, 909-917, 2018.
[16] J.O. Hamzat and D. O. Makinde, Coefficient Bounds for Bazilevic Functions Involving Logistic Sigmoid Function Associated with Conic Domains, Int. J. Math. Anal. Opt.: Theory and Applications, vol. 2018, no. 2, 392-400, 2018.
[17] R. W. Ibrahim, Fractional complex transforms for fractional differential equations, Advances in Difference Equations 2012.1(2012): 192.
[18] R. W. Ibrahim and M. Darus, On subordination theorems for new classes of normalize analytic functions, Appl. Math. Sci. 2.56, 2785-2794, 2008.
[19] Y. C. Kim and H. M. Srivastava, The hardy space of a certain subclass of Bazilevic Functions. Appl. Math. Comput. 183, 1201-1207, 2006.
[20] Y. C. Kim and T. Sugawa, A note on Bazilevic functions. Taiwanese J. Math. 13, 1489- 1495, 2009.
[21] Y. Komatu, On analytic prolongation family of integral operators, Mathematics (cluj). 32(55), (1990), 141-145.
[22] J. W. Noonan, On close-to-convex functions of order β, Pacific journal of Mathematics, vol.44, no.1, 263-280, 1973.
[23] K. I. Noor and K. Ahmad, On higher order Bazilevic functions, Internat. J. Modern Phys. B27(4), Article ID1250203, 1-14, 2013.
[24] M. Nunokawa, M. Obradovic, and S. Owa, One criterion for univalency, Proc. Amer. Math. Soc. 106, 1035-1037, 1989.
[25] M. Obradovic and S. Ponnusamy, New criteria and distortion theorems for unvalent functions, Complex Var. Theory Appl., 44, 173-191, 2001.
[26] M. Obradovic and S. Ponnusamy, Radius properties for subclasses of univalent functions , Analysis (Munich) 25, 183-188, 2005.
[27] M. Obradovic, S. ponnusamy, V. Singh and P. Vasundhra, Univalency, starlikeness and convexity applied to certain classes of rational functions, Analysis (Munich) 22(3)(2002), 225-242.
[28] M. Obradovic and S. Ponnusamy, On the class U, Proc. 21st Annual conference of the Jammu Math. soc. and a National seminar on Analysis Application Feb 25-27, 2011.
[29] A. T. Oladipo, On subclasses of analytic and univalent functions, Advances in Applied Mathematical Analysis, 4(1), 87-93, 2009.
[30] A. T. Oladipo and D. Breaz, On the family of Bazilevic functions, Acta Universitatis Apulensis, no.24, 319-330, 2010.
[31] A. T. Oladipo and D. Breaz, A brief study of certain class of Harmonic Functions of Bazilevic Type. ISRN Math. Anal.Article ID 179856, 11 pages, 2013.
[32] T. O. Opoola, On a new subclass of univalent functions. Mathematica, vol.36, no.2, tome 36, 195-200, 1994.
[33] S. Ponnusamy and P. Vasundhra, Criteria for univalence, starlikeness and convexity, Ann. polon. Math. 85, 121-133, 2005.
[34] S. Ponnusamy and S. K. Sahoo, Study of some subclasses of univalent functions and their Radius properties, KODAI MATH. J. 29, 391-405, 2009.
[35] G. S. Salagean, Subclasses of univalent functions, Lecture Notes in math. (Springer-Verlag), 362-372, 1983.
[36] R. Singh, On Bazilevic functions. Proceedings of the American Mathematical Society, vol.38, no.1, 263-280, 1963.
[37] H. M. Srivastava, M. Darus and R. W. Ibrahim, Classes of analytic functions with fractional powers defined by means of a certain linear operator, Integral Transforms and Special Functions, 22.1, 17-28, 2011.
[38] G. A. Waggas, Fractional calculus on a subclass of Spiralike functions defined by Komatu operator, Int. Math. Forum, 3, 32, (2008), 1587-1594.

Hamzat J. O.
Department of Mathematics, University of Lagos, Akoka, Lagos State, Nigeria.
E-mail address: jhamzat@unilag.edu.ng, emmanther2012@gmail.com

Oladipo A. T.
Department of Pure and Applied Mathematics,
Ladoke Akintola University of Technology, Ogbomoso, Oyo State, Nigeria.
E-mail address: atoladipo@lautch.edu.ng