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EXISTENCE AND UNIQUENESS OF MILD SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR CAPUTO-HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL AND ANTI-PERIODIC CONDITIONS

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Abstract. In this paper, we investigate the existence and uniqueness of mild solutions of a boundary value problem for Caputo-Hadamard fractional differential equations with integral and anti-periodic conditions. Our analysis relies on classical fixed point theorems. Examples are given to illustrate our results.

1. Introduction

The concept of fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non integer order. Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, physics, chemistry, biology, medicine, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[16], [18]–[32] and the references therein. Fractional differential equations involving Riemann-Liouville and Caputo type fractional derivatives have been studied extensively by several researchers. However, the literature on Hadamard type fractional differential equations is not yet as enriched. The fractional derivative due to Hadamard, introduced in 1892, differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains a logarithmic function of arbitrary exponent.

Recently in [32], Xu discussed the existence and uniqueness of solutions of the following fractional differential equation

$$\begin{cases} {}^{c}D^{q}x(t) = f\left(t, x\left(t\right)\right), \ t \in \left[0,1\right], \ 1 < q \leq 2, \\ x(1) = \mu \int_{0}^{1} x\left(s\right) ds, \ x'\left(0\right) + x'\left(1\right) = 0, \end{cases}$$

where ${}^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order q, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

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The boundary value problems for the nonlinear fractional differential equation

$$\begin{cases} \mathfrak{D}_1^{\alpha} x(t) = f(t, x(t)), \ t \in [1, T], \ 0 < \alpha \leq 1, \\ ax(1) + bx(T) = c, \end{cases}$$

has been investigated in [9], where $\mathfrak{D}_1^{\alpha}$ is the Caputo-Hadamard fractional derivative, $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, a, b and c are real constants such that $a + b \neq 0$.

Motivated by these works, we study the existence and uniqueness of mild solutions for the following boundary value problem for the fractional differential equation

$$\mathfrak{D}_1^{\alpha} x(t) = f(t, x(t)), \ t \in [1, T], \tag{1}$$

$$x(1) + x(T) = b \int_1^T x(s) \frac{ds}{s}, \tag{2}$$

where $f : [1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mathfrak{D}_1^{\alpha}$ is the Caputo-Hadamard fractional derivative of order $0 < \alpha < 1$ and $b \in \mathbb{R}$ such that $2 - b \log(T) > 0$.

This paper is organized as follows. In Section 2, we recall briefly some basic definitions and preliminary facts which will be used throughout subsequent sections. In Section 3, we shall provide sufficient conditions ensuring the existence and uniqueness of mild solutions for the problem (1)-(2) via applications of classical fixed point theorems. Finally in Section 4, we give examples to illustrate the theory presented in the previous sections.

2. Preliminaries

In this section we present some basic definitions, notations and results of fractional calculus which are used throughout this paper.

Let $C([1, T], \mathbb{R})$ be the Banach space of all real-valued continuous functions defined on the compact interval $[1, T]$, endowed with the maximum norm.

Definition 1 ([21]). *The Hadamard fractional integral of order $\alpha > 0$ for a continuous function $x : [1, +\infty) \to \mathbb{R}$ is defined as*

$$\mathfrak{I}_1^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha - 1} x(s) \frac{ds}{s}, \ \alpha > 0.$$

Definition 2 ([20]). *The Caputo-Hadamard fractional derivative of order α for a continuous function $x : [1, +\infty) \to \mathbb{R}$ is defined as*

$$\mathfrak{D}_1^{\alpha} x(t) = \frac{1}{\Gamma(n - \alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{n - \alpha - 1} \delta^n(x)(s) \frac{ds}{s}, \ n - 1 < \alpha < n,$$

where $\delta^n = \left(t \frac{d}{dt} \right)^n$, $n = [\alpha] + 1$.

Lemma 1 ([20]). *Let $\alpha > 0$. Suppose $x \in C^{n-1}([0, +\infty), \mathbb{R})$ and $x^{(n)}$ exists almost every-where on any bounded interval of $[1, +\infty)$. Then*

$$\mathfrak{D}_1^{\alpha} [\mathfrak{I}_1^{\alpha} x](t) = x(t),$$

and

$$\mathfrak{I}_1^{\alpha} [\mathfrak{D}_1^{\alpha} x](t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(1)}{\Gamma(k+1)} (\log t)^k.$$

In particular, when $0 < \alpha < 1$, $\mathfrak{I}_1^{\alpha} [\mathfrak{D}_1^{\alpha} x](t) = x(t) - x(1)$.

Theorem 1 (Banach's fixed point theorem [17]). *Let Ω be a non-empty closed subset of a Banach space $(S, \|.\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Theorem 2 (Schaefer's fixed point theorem [17]). *Let S be a Banach space, and $\Phi : S \to S$ completely continuous operator. If the set $E = \{x \in S : x = \lambda \Phi x$, for some $\lambda \in (0,1)\}$ is bounded, then N has fixed points.*

3. Main results

Let us start by defining what we mean by a solution of the problem (1)-(2).

Definition 3. *A function $x \in C([1,T], \mathbb{R})$ is said to be a mild solution of the problem (1)-(2) if x satisfies the corresponding integral equation of (1)-(2).*

For the existence of solutions for the problem (1)-(2), we need the following auxiliary lemma.

Lemma 2. *Let $\Delta = 2 - b\log(T)$, $x \in C([1,T], \mathbb{R})$ and x' exists. If x is a solution of the boundary value problem (1)-(2), then x is a solution of the integral equation*

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \\ &\quad + \frac{b}{\Delta\Gamma(\alpha)} \int_1^T \left(\int_1^s \left(\log \frac{s}{\sigma}\right)^{\alpha-1} f(\sigma, x(\sigma)) \frac{d\sigma}{\sigma} \right) \frac{ds}{s} \\ &\quad - \frac{1}{\Delta\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s}, \end{aligned} \tag{3}$$

for $t \in [1,T]$.

Proof. Suppose x satisfies the problem (1)-(2). Then, by applying $\mathfrak{I}_1^\alpha$ to both sides of (1), we have

$$\mathfrak{I}_1^\alpha \left(\mathfrak{D}_1^\alpha x(t)\right) = \mathfrak{I}_1^\alpha \left(f(t, u(t))\right).$$

In view of Lemma 1, we get

$$x(t) = x(1) + \mathfrak{I}_1^\alpha \left(f(t, u(t))\right). \tag{4}$$

The condition (2) implies that

$$\begin{aligned} &2x(1) + \frac{1}{\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s} \\ &= b\log(T) x(1) + \frac{b}{\Gamma(\alpha)} \int_1^T \left(\int_1^s \left(\log \frac{s}{\sigma}\right)^{\alpha-1} f(\sigma, x(\sigma)) \frac{d\sigma}{\sigma} \right) \frac{ds}{s}, \end{aligned}$$

so

$$\begin{aligned} x(1) &= \frac{b}{\Delta\Gamma(\alpha)} \int_1^T \left(\int_1^s \left(\log \frac{s}{\sigma}\right)^{\alpha-1} f(\sigma, x(\sigma)) \frac{d\sigma}{\sigma} \right) \frac{ds}{s} \\ &\quad - \frac{1}{\Delta\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} f(s, x(s)) \frac{ds}{s}. \end{aligned} \tag{5}$$

Substituting (5) in (4) we get the integral equation (3). The proof is completed. $\square$

Now, we transform the integral equation (3) to be applicable to fixed point theorems, we define the operator $\Phi : C([1,T],\mathbb{R}) \to C([1,T],\mathbb{R})$ by

$$\begin{aligned}(\Phi x)(t) &= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}\\ &+ \frac{b}{\Delta\Gamma(\alpha)}\int_1^T \left(\int_1^s \left(\log\frac{s}{\sigma}\right)^{\alpha-1} f(\sigma,x(\sigma))\frac{d\sigma}{\sigma}\right)\frac{ds}{s}\\ &- \frac{1}{\Delta\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}.\end{aligned}$$

Where figured fixed point must satisfy the identity operator equation $\Phi u = u$.

In the following subsections we prove existence, as well as existence and uniqueness results, for the boundary value problem (1)-(2) by using a variety of fixed point theorems.

3.1. **Existence and uniqueness results via Banach's fixed point theorem.**

Theorem 3. *Assume the following hypothesis*

(H1) There exists a constant $k > 0$ such that

$$|f(t,x) - f(t,y)| \le k|x-y|,$$

for $t \in [1,T]$ and $x, y \in \mathbb{R}$.

If

$$\frac{k(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{k|b|(\log T)^{\alpha+1}}{\Delta\Gamma(\alpha+2)} + \frac{k(\log T)^\alpha}{\Delta\Gamma(\alpha+1)} < 1, \tag{6}$$

then the boundary value problem (1)-(2) has a unique mild solution in $[1,T]$.

Proof. Let Φ defined by (3). Clearly, the fixed points of operator Φ are mild solutions of the problem (1)-(2). Let $x, y \in C([1,T],\mathbb{R})$. Then for $t \in [1,T]$, we have

$$\begin{aligned}&|(\Phi x)(t) - (\Phi y)(t)|\\ &\le \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} |f(s,x(s)) - f(t,y(s))|\frac{ds}{s}\\ &+ \frac{|b|}{\Delta\Gamma(\alpha)}\int_1^T \left(\int_1^s \left(\log\frac{s}{\sigma}\right)^{\alpha-1} |f(\sigma,x(\sigma)) - f(t,y(s))|\frac{d\sigma}{\sigma}\right)\frac{ds}{s}\\ &+ \frac{1}{\Delta\Gamma(\alpha)}\int_1^T \left(\log\frac{T}{s}\right)^{\alpha-1} |f(s,x(s)) - f(t,y(s))|\frac{ds}{s}\\ &\le \frac{k(\log t)^\alpha}{\Gamma(\alpha+1)}\|x-y\| + \frac{k|b|(\log T)^{\alpha+1}}{\Delta\Gamma(\alpha+2)}\|x-y\| + \frac{k(\log T)^\alpha}{\Delta\Gamma(\alpha+1)}\|x-y\|\\ &\le \left(\frac{k(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{k|b|(\log T)^{\alpha+1}}{\Delta\Gamma(\alpha+2)} + \frac{k(\log T)^\alpha}{\Delta\Gamma(\alpha+1)}\right)\|x-y\|.\end{aligned}$$

Therefore

$$\|\Phi x - \Phi y\| \le \left(\frac{k(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{k|b|(\log T)^{\alpha+1}}{\Delta\Gamma(\alpha+2)} + \frac{k(\log T)^\alpha}{\Delta\Gamma(\alpha+1)}\right)\|x-y\|.$$

From (6), Φ is a contraction. As a consequence of Banach's fixed point theorem, we get that Φ has a unique fixed point which is the unique mild solution of the problem (1)-(2). $\square$

3.2. **Existence results via Schaefer's fixed point theorem.**

Theorem 4. *Assume the following hypothesis*
(H2) There exists a constant $M > 0$ such that

$$|f(t, x)| \leq M,$$

for $t \in [1, T]$ and each $x \in \mathbb{R}$.
Then the boundary value problem (1)-(2) has at least one mild solution in $[1, T]$.

Proof. We shall use Schaefer's fixed point theorem to prove that Φ defined by (3) has a fixed point. The proof will be given in several steps.

Step 1. The continuity of f implies the continuity of the operator Φ defined by (3).

Step 2. Φ maps bounded sets into bounded sets in $C([1, T], \mathbb{R})$.

Indeed, it is enough to show that for any $\eta > 0$, there exists a positive constant l such that for each $x \in B_\eta = \{x \in C([1, T], \mathbb{R}) : \|x\| \leq \eta\}$, we have $\|\Phi x\| \leq l$. In fact, we have

$$\begin{aligned}
|(\Phi x)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s, x(s))| \frac{ds}{s} \\
&\quad + \frac{|b|}{\Delta\Gamma(\alpha)} \int_1^T \left(\int_1^s \left(\log \frac{s}{\sigma}\right)^{\alpha-1} |f(\sigma, x(\sigma))| \frac{d\sigma}{\sigma}\right) \frac{ds}{s} \\
&\quad + \frac{1}{\Delta\Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} |f(s, x(s))| \frac{ds}{s} \\
&\leq \frac{M}{\Gamma(\alpha+1)} (\log t)^\alpha + \frac{M|b|}{\Delta\Gamma(\alpha+2)} (\log T)^{\alpha+1} + \frac{M}{\Delta\Gamma(\alpha+1)} (\log T)^\alpha \\
&\leq \left(\Delta + \frac{|b| \log T}{\alpha+1} + 1\right) \frac{M}{\Delta\Gamma(\alpha+1)} (\log T)^\alpha .
\end{aligned}$$

Thus

$$\|\Phi x\| \leq \left(\frac{(\Delta+1)(\alpha+1) + |b| \log T}{\alpha+1}\right) \frac{M}{\Delta\Gamma(\alpha+1)} (\log T)^\alpha = l.$$

Step 3. Φ maps bounded sets into equicontinuous sets of $C([1, T], \mathbb{R})$.

Let $t_1, t_2 \in [1,T]$ with $t_1 < t_2$, B_η be a bounded set of $C([1,T],\mathbb{R})$ as in Step 2, and let $x \in B_\eta$. Then

$$\begin{aligned}
&\left|(\Phi x)(t_2) - (\Phi x)(t_1)\right| \\
&\leq \left|\frac{1}{\Gamma(\alpha)}\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s} - \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}\right| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left|\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right| \left|f(s,x(s))\right|\frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\left|f(s,x(s))\right|\frac{ds}{s} \\
&\leq \frac{M}{\Gamma(\alpha)}\int_1^{t_1}\left(\left(\log\frac{t_1}{s}\right)^{\alpha-1} - \left(\log\frac{t_2}{s}\right)^{\alpha-1}\right)\frac{ds}{s} + \frac{M}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha-1}\frac{ds}{s} \\
&\leq \frac{M}{\Gamma(\alpha+1)}\left((\log t_1)^\alpha + \left(\log\frac{t_2}{t_1}\right)^\alpha - (\log t_2)^\alpha + \left(\log\frac{t_2}{t_1}\right)^\alpha\right) \\
&\leq \frac{2M}{\Gamma(\alpha+1)}\left(\log\frac{t_2}{t_1}\right)^\alpha.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero and the convergence is independent of x in B_η. As consequence of Step 1 to Step 3, together with the Arzela-Ascoli theorem, we can conclude that Φ is completely continuous.

Step 4. Apriori bounds.

Now it remains to show that the set

$$E = \{x \in C([1,T],\mathbb{R}) : x = \lambda\Phi x \text{ for some } 0 < \lambda < 1\},$$

is bounded. Let $x \in E$, then $x = \lambda\Phi x$ for some $0 < \lambda < 1$. Thus, for each $t \in [1,T]$ we have

$$\begin{aligned}
x(t) = \lambda\Bigg[&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s} \\
&+\frac{b}{\Delta\Gamma(\alpha)}\int_1^T\left(\int_1^s\left(\log\frac{s}{\sigma}\right)^{\alpha-1} f(\sigma,x(\sigma))\frac{d\sigma}{\sigma}\right)\frac{ds}{s} \\
&-\frac{1}{\Delta\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-1} f(s,x(s))\frac{ds}{s}\Bigg].
\end{aligned}$$

For $\lambda \in (0,1)$, we have

$$\begin{aligned}
|(\Phi x)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s,x(s))| \frac{ds}{s} \\
&+ \frac{|b|}{\Delta \Gamma(\alpha)} \int_1^T \left(\int_1^s \left(\log \frac{s}{\sigma}\right)^{\alpha-1} |f(\sigma, x(\sigma))| \frac{d\sigma}{\sigma}\right) \frac{ds}{s} \\
&+ \frac{1}{\Delta \Gamma(\alpha)} \int_1^T \left(\log \frac{T}{s}\right)^{\alpha-1} |f(s,x(s))| \frac{ds}{s} \\
&\leq \frac{M}{\Gamma(\alpha+1)} (\log t)^{\alpha} + \frac{M|b|}{\Delta\Gamma(\alpha+2)} (\log T)^{\alpha+1} + \frac{M}{\Delta\Gamma(\alpha+1)} (\log T)^{\alpha} \\
&\leq \left(\Delta + \frac{|b| \log T}{\alpha+1} + 1\right) \frac{M}{\Delta\Gamma(\alpha+1)} (\log T)^{\alpha} = R.
\end{aligned}$$

Thus

$$\|\Phi x\| \leq \left(\frac{(\Delta+1)(\alpha+1) + |b| \log T}{\alpha+1}\right) \frac{M}{\Delta\Gamma(\alpha+1)} (\log T)^{\alpha} = R.$$

This implies that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that Φ has a fixed point which is a mild solution of the problem (1)-(2). □

4. Examples

In this section, we present some examples to illustrate our results of the previous section.

Example 1. *We consider the fractional boundary value problem*

$$\begin{aligned}
\mathfrak{D}^{\frac{1}{2}} x(t) &= \frac{\sin(x(t))}{5t}, \ t \in [1,e], && (7) \\
x(1) + x(e) &= \int_1^e x(s) \frac{ds}{s}, && (8)
\end{aligned}$$

where $\alpha = \frac{1}{2}$, $T = e$, $b = 1$ *and* $f(t,x) = \frac{\sin(x)}{5t}$. *For any* $x, y \in \mathbb{R}$ *and* $t \in [1,e]$, *we have*

$$|f(t,x) - f(t,y)| \leq \frac{1}{5} |x-y|.$$

Therefore, the condition $\frac{k(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{k|b|(\log T)^{\alpha+1}}{\Delta\Gamma(\alpha+2)} + \frac{k(\log T)^{\alpha}}{\Delta\Gamma(\alpha+1)} < 1$ *holds with* $k = \frac{1}{5}$ *and* $\Delta = 1$. *Indeed,* $\frac{2}{5\Gamma\left(\frac{1}{2}+1\right)} + \frac{1}{5\Gamma\left(\frac{1}{2}+2\right)} \simeq 0.60 < 1$. *By Theorem 3, the problem (7)-(8) has a unique mild solution in* $[1,e]$.

Example 2. *We consider the fractional boundary value problem*

$$\begin{aligned}
\mathfrak{D}^{\frac{1}{2}} x(t) &= \frac{\cos(x(t))}{2\exp(-t)}, \ t \in [1,e], && (9) \\
x(1) + x(e) &= \frac{1}{2} \int_1^e x(s) \frac{ds}{s}, && (10)
\end{aligned}$$

where $\alpha = \frac{1}{2}$, $T = e$, $b = \frac{1}{2}$, $\Delta = \frac{3}{2}$ *and* $f(t,x) = \frac{\cos(x)}{2\exp(-t)}$. *We have*

$$|f(t,x)| \leq \frac{|\cos(x)|}{2\exp(-t)} \leq \frac{1}{2e^{-e}}.$$

Choosing $M = \frac{1}{2e^{-e}}$*, then by Theorem 4, the problem (9)-(10) has a mild solution in* $[1, e]$.

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References

[1] Y. Adjabi, F. Jarad, D. Baleanu, T. Abdeljawad, On Cauchy problems with Caputo Hadamard fractional derivatives, Journal of Computational Analysis and Applications, 21(1), 661-681, 2016.

[2] B. Ahmad, S. K. Ntouyas, Existence and uniqueness of solutions for Caputo-Hadamard sequential fractional order neutral functional differential equations, Electron. J. Differential Equations, 2017(36), 1-11, 2017.

[3] B. Ahmad, S. K. Ntouyas, Initial value problems for functional and neutral functional Hadamard type fractional differential inclusions, Miskolc Math. Notes, 17, 15-27, 2016.

[4] B. Ahmad, S. K. Ntouyas, Initial value problems of fractional order Hadamard-type functional differential equations, Electron. J. Differential Equations, 2015(77), 1-9, 2015.

[5] B. Ahmad, S. K. Ntouyas, Initial-value problems for fractional differential equations, Electronic Journal of Differential Equations, 2014(161), 1-8, 2014.

[6] B. Ahmad, S K Ntouyas, J. Tariboon, A study of mixed Hadamard and Riemann-Liouville fractional integro-differential inclusions via endpoint theory, Applied Mathematics Letters, 52, 9-14, 2016.

[7] M. Benchohra, J. R. Graef and S. Hamani, Existence results for boundary value problems of nonlinear fractional differential equations with integral conditions, Appl. Anal., 87(7), 851-863, 2008.

[8] W. Benhamida, J. R. Graef, and S. Hamani, Boundary value problems for fractional differential equations with integral and anti-periodic conditions in a Banach space, Prog. Frac. Differ. Appl., 4(2), 1-7, 2018.

[9] W. Benhamida, S. Hamani, J. Henderson, Boundary value problems for Caputo-Hadamard fractional differential equations, Advances in the Theory of Nonlinear Analysis and its Application, 2(3), 138-145, 2018.

[10] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Composition of Hadamard-type fractional integration operators and the semigroup property, J. Math. Anal. Appl., 269, 387-400, 2002.

[11] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, J. Math. Anal. Appl., 269, 1-27, 2002.

[12] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, J. Math. Anal. Appl., 270, 1-15, 2002.

[13] M. Cichon and H. A. H. Salem, On the solutions of Caputo-Hadamard Pettis-type fractional differential equations, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM, 113(4), 3031-3053, 2019.

[14] M. Cichon and H. A. H. Salem, On the lack of equivalence between differential and integral forms of the Caputo-type fractional problems, J. Pseudo-Differ. Oper. Appl., 2020. DOI: 10.1007/s11868-020-00345-z.

[15] K. Diethelm, The Analysis of Fractional Differential Equations, Lecture Notes in Mathematics, Springer-verlag, Berlin, Heidelberg, 2010.

[16] Y. Gambo, F. Jarad, D. Baleanu, F. Jarad, On Caputo modification of the Hadamard fractional derivatives, Advances in Difference Equations, 2014(10), 1-12, 2014.

[17] A. Granas and J. Dugundji, Fixed Point Theory, Springer, New York, 2003.

[18] F. Jarad, T. Abdeljawad, Generalized fractional derivatives, and Laplace transform, Discrete and Continuous dynamic systems, 2020. DOI:10.3934/dcdss.2020039.

[19] F. Jarad, T. Abdeljawad, D. Baleanu, On the generalized fractional derivatives and their Caputo modification, Journal of Nonlinear Sciences and Applications, 10(5), 2607-2619, 2017.

[20] F. Jarad, T. Abdeljawad, D. Baleanu, Caputo-type modification of the Hadamard fractional derivatives, Advances in Difference Equations, 2012(142), 1-8, 2012.

[21] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies 204, Editor: Jan Van Mill, Elsevier, Amsterdam, The Netherlands, 2006.

[22] K. D. Kucche, J. J. Nieto and V. Venktesh, Theory of nonlinear implicit fractional differential equations, Differ, Equ. Dyn. Syst., 28, 1?17, 2020.
[23] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.
[24] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1993.
[25] B. Ross, S. G. Samko, E. R. Love, Functions that have no first order derivative might have fractional derivatives of any order less than one, Real Anal. Exchange, 20, 140-157, 1994/5.
[26] H. A. H. Salem, Weakly absolutely continuous functions without weak, but fractional weak derivatives, J. Pseudo-Differ. Oper. Appl., 2019, https://doi.org/10.1007/s11868-019-00274-6.
[27] H. A. H. Salem, Hadamard-type fractional calculus in Banach spaces, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A. Math., 2019, 113:987 1006, https://doi.org/10.1007/s13398-018-0531-y.
[28] H. A. H. Salem, On functions without pseudo derivatives having fractional pseudo derivatives, Quaestiones Mathematicae, 2018, http://dx.doi.org/10.2989/16073606.2018.1523247.
[29] S. Samko, A. Kilbas, O. L. Marichev, Fractional Integrals and Derivatives, Gordon and Breach Science Publisher, 1993.
[30] S. T. Sutar, K. D. Kucche, Global existence and uniqueness for implicit differential equations of arbitrary order, Fractional Differential Calculus, 5(2), 199-208, 2015.
[31] P. Thiramanus, S. K. Ntouyas and J. Tariboon, Existence and uniqueness results for Hadamard- type fractional differential equations with nonlocal fractional integral boundary conditions, Abstr. Appl. Anal., 2014, 1-9, 2014.
[32] Y. Xu, Fractional boundary value problems with integral and anti-periodic boundary conditions, Bulletin of the Malaysian Mathematical Sciences Society, 39(2), 571-587, 2016.

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