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SOME NEW CHEBYSHEV TYPE INEQUALITIES VIA EXTENDED GENERALIZED FRACTIONAL INTEGRAL OPERATOR

BHAGWAT R. YEWALE, DEEPAK B. PACHPATTE

Abstract. In this paper, we establish some new Chebyshev type integral inequalities involving extended generalized fractional integral.

1. Introduction

In 1882, Chebyshev [6] introduced the following functional

$$T(\zeta,\eta) := \frac{1}{b-a}\int_a^b \zeta(\tau)\eta(\tau)d\tau - \left(\frac{1}{b-a}\int_a^b \zeta(\tau))d\tau\right)\left(\frac{1}{b-a}\int_a^b \eta(\tau)d\tau\right), \quad (1.1)$$

where ζ and η are two integrable functions on [a, b]. This functional (1.1) has numerous applications in various field of mathematics, statistics and probability theory. Since then many researchers have provides abundance of results, generalizations and variants and number of integral inequalities related to the above functional (1.1) (see [10, 12, 14, 15, 16]). Fractional calculus plays an important role in mathematical analysis as it extends the order of differentiation and integration from integers to non-integers and also, have various applications in the field of science and engineering. Fractional calculus attracted many researcher's due to large number of applications in distinct fields. For detail description of fractional calculus and applications (see [13, 17, 21]).

In fractional calculus, integral inequalities plays crucial role because they are useful in providing bounds. In recent years, many authors have given generalizations of classical inequalities by employing fractional integral operators. For instance, Aljaaidi and Pachpatte [1], proved Gruss-type inequalities for generalized Katugampola fractional integral operators. Sarikaya et al. [23], derived Hermite-Hadamard's inequalities by considering Riemann-Liouville fractional integral. Recently results on Chebyshev type inequalities for the Saigo fractional integrals were proved by Purohit and Raina [19]. Rahman et al. [20], presented Minkowski inequalities using generalized proportional fractional integral operators. Agarwal et

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al. [2], established Ostrowski type inequalities through fractional integral operator. Moreover, in [4, 5, 7, 8, 9, 22, 24, 25], authors investigated integral inequalities for various fractional integral operators.

Motivated by above work, in this paper we establish some new fractional integral inequalities using extended generalized fractional integral operator introduced in [3]. Using the above defined operator we have obtained the results on Chebyshev type inequalities. The paper is organized as follows: In section 2, we enlist some useful definitions. Section 3 is related to Chebyshev functional in case of synchronous functions and concluding remark highlighting the importance of work, can be seen in last section.

2. Preliminaries

In this section, we present some necessary notations and basic definitions which will be used throughout this paper.

Definition 1 [3] Let $\vartheta,\ \alpha,\ \iota,\ \varpi,\ \varsigma \in \mathbb{C}$, $\Re(\vartheta), \Re(\alpha), \Re(\iota) > 0$, $\Re(\varsigma) > \Re(\varpi) > 0$ with $p \geq 0,\ \lambda > 0$ *and* $0 < k \leq \lambda + \Re(\vartheta)$. Then the extended generalized Mittag-Leffler function is denoted by $E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}$ and is defined as

$$E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\varpi + nk, \varsigma - \varpi)}{\beta(\varpi, \varsigma - \varpi)} \frac{(\varsigma)_{nk}}{\Gamma(\vartheta n + \alpha)} \frac{t^n}{(\iota)_{n\lambda}}, \tag{2.1}$$

where β_p is an extension of the beta function

$$\beta_p(r, s) = \int_0^1 t^{r-1}(1-t)^{s-1} e^{-\frac{p}{t(1-t)}}\, dt \quad (\Re(r), \Re(s), \Re(p) > 0).$$

Definition 2 [3] Let $\omega,\ \vartheta,\ \alpha,\ \iota,\ \varpi,\ \varsigma \in \mathbb{C}$, $\Re(\vartheta), \Re(\alpha), \Re(\iota) > 0$, $\Re(\varsigma) > \Re(\varpi) > 0$ with $p \geq 0,\ \lambda > 0$ *and* $0 < k \leq \lambda + \Re(\vartheta)$. Let $u \in L_1[a,\ b]$ *and* $z \in [a,\ b]$. Then the generalized fractional integral operator is denoted by $\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}$ and is defined as

$$(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u)(z;p) = \int_a^z (z-t)^{\alpha-1} E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(z-t)^{\vartheta}; p) u(t) dt. \tag{2.2}$$

In [11], authors give,

$$(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ 1)(z;p) = (z-a)^{\alpha} E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha+1,\iota}(\omega(z-t)^{\vartheta}; p).$$

The beauty of generalized fractional integral operator (2.2) is that it reduces to the several well known fractional integrals such as Riemann-Liouville fractional integral [13], Prabhakar fractional integral [18] and Srivastava-Tomovski fractional integral [26] (see [3]).

3. Main Results

In this section, we give some integral inequalities for Chebyshev functional using generalized fractional integral operator (2.2).

Now we give the result on Chebyshev functional which deals with two synchronous functions having some non-negative function:

Theorem 3.1 Let r be non-negative and u, v be two synchronous function on

$[0,\infty[$. Then for all $t>a\geq 0$, $\alpha>0$, $\beta>0$, the following inequalities hold:

$$\begin{aligned}
& (t-a)^{\beta}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta+1,\iota}(\omega(t-a)^{\vartheta};p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\,uvr)(t;p)\\
& +(t-a)^{\alpha}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha+1,\iota}(\omega(t-a)^{\vartheta};p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\,uvr)(t;p)\\
\geq\ & (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\,ur)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\,v)(t;p)+(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\,vr)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\,u)(t;p)\\
& -(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\,r)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\,uv)(t;p)-(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\,uv)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\,r)(t;p)\\
& +(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\,u)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\,vr)(t;p)+(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\,v)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\,ur)(t;p).
\end{aligned} \tag{3.1}$$

Proof. Since $r\geq 0$ implies $r(m)+r(n)\geq 0$ for all $m,\ n\in[0,\infty[$. Also u and v are synchronous on $[0,\infty[$, then

$$(u(m)-u(n))(v(m)-v(n))\geq 0,\quad \text{for all } m,\ n\in\ [0,\infty[.$$

Therefore

$$\begin{aligned}
(u(m)-u(n))(v(m)-v(n))(r(m)+r(n))\geq 0,\\
(u(m)v(m)+u(n)v(n)-u(m)v(n)-u(n)v(m))(r(m)+r(n))\geq 0.
\end{aligned}$$

From above we have

$$\begin{aligned}
& u(m)v(m)r(m)+u(n)v(n)r(n)\\
\geq\ & u(m)v(n)r(m)+u(n)v(m)r(m)-u(n)v(n)r(m)\\
& -u(m)v(m)r(n)+u(m)v(n)r(n)+u(n)v(m)r(n).
\end{aligned} \tag{3.2}$$

Multiplying (3.2) by $(t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)$ and integrating obtained inequality with respect to m from a to t, we get

$$\begin{aligned}
& \int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)u(m)v(m)r(m)dm\\
& +u(n)v(n)r(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)dm\\
\geq\ & v(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)u(m)r(m)dm\\
& +u(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)v(m)r(m)dm\\
& -u(n)v(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)r(m)dm\\
& -r(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)u(m)v(m)dm\\
& +v(n)r(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)u(m)dm\\
& +u(n)r(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)dm.
\end{aligned}$$

It follows that

$$\begin{aligned}
&(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uvr)(t;p) + u(n)v(n)r(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ 1)(t;p) \\
\geq\ & v(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ ur)(t;p) + u(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ vr)(t;p) \\
& - u(n)v(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ r)(t;p) - r(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uv)(t;p) \\
& + v(n)r(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u)(t;p) + u(n)r(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ v)(t;p). \qquad (3.3)
\end{aligned}$$

Multiplying (3.3) by $(t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)$ and integrating obtained inequality with respect to n from a to t, then (3.3) becomes

$$\begin{aligned}
&(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uvr)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)dn \\
&+ (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ 1)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)u(n)v(n)r(n)dn \\
\geq\ & (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ ur)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)v(n)dn \\
&+ (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ vr)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)u(n)dn \\
&- (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ r)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)u(n)v(n)dn \\
&- (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uv)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)r(n)dn \\
&+ (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)v(n)r(n)dn \\
&+ (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ v)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)r(n)dn.
\end{aligned}$$

From above we get (3.1). □

Corollary 3.1 Let r be non-negative function (i.e. $r \geq 0$) and u, v be two synchronous function on $[0,\infty[$. Then for all $t > a \geq 0$, $\alpha > 0$, the following inequalities hold:

$$\begin{aligned}
&(t-a)^{\alpha}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha+1,\iota}(\omega(t-a)^{\vartheta};p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uvr)(t;p) \\
\geq\ & (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ ur)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ v)(t;p) + (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ vr)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u)(t;p) \\
& - (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uv)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ r)(t;p).
\end{aligned}$$

In next theorem we obtain new inequalities associated with extended generalized fractional integral operator in the case where functions are monotonic:

Theorem 3.2 If u, v and r are monotonic functions on $[0,\infty[$ such that for all m, n $\in$ [a, t],

$$(u(m)-u(n))(v(m)-v(n))(r(m)-r(n)) \geq 0.$$

Then

$$\begin{aligned}
&(t-a)^{\beta}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta+1,\iota}(\omega(t-a)^{\vartheta};p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uvr)(t;p)\\
&-(t-a)^{\alpha}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha+1,\iota}(\omega(t-a)^{\vartheta};p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ uvr)(t;p)\\
\geq\ &(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ ur)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ v)(t;p)+(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ vr)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ u)(t;p)\\
&-(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ r)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ uv)(t;p)+(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uv)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ r)(t;p)\\
&-(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ vr)(t;p)-(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ v)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ ur)(t;p).
\end{aligned}\tag{3.4}$$

Proof. Since

$$(u(m)-u(n))(v(m)-v(n))(r(m)-r(n))\geq 0,$$

$$(u(m)v(m)-u(m)v(n)-u(n)v(m)+u(n)v(n))(r(m)-r(n))\geq 0.$$

We have

$$\begin{aligned}
u(m)v(m)r(m)-u(n)v(n)r(n)\geq\ &u(m)v(m)r(n)+u(m)v(n)r(m)\\
&+u(n)v(m)r(m)-u(m)v(n)r(n)\\
&-u(n)v(m)r(n)-u(n)v(n)r(m).
\end{aligned}\tag{3.5}$$

Multiplying (3.5) by $(t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)$ on both sides and integrating resulting inequality with respect to m from a to t, following inequality obtained

$$\begin{aligned}
&\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)u(m)v(m)r(m)dm\\
&-u(n)v(n)r(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)dm\\
\geq\ &r(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\alpha-1};p)u(m)v(m)dm\\
&+v(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)u(m)r(m)dm\\
&+u(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)v(m)r(m)dm\\
&-v(n)r(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)u(m)dm\\
&-u(n)r(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)v(m)dm\\
&-u(n)v(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)r(m)dm.
\end{aligned}$$

That is

$$\begin{aligned}
&(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uvr)(t;p)-u(n)v(n)r(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ 1)(t;p)\\
\geq\ &r(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uv)(t;p)+v(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ ur)(t;p)\\
&+u(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ vr)(t;p)-v(n)r(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u)(t;p)\\
&-u(n)r(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ v)(t;p)-u(n)v(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ r)(t;p).
\end{aligned}\tag{3.6}$$

Multiplying (3.6) by $(t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)$ and integrating obtained inequality with respect to n from a to t, we get

$$\begin{aligned}
& (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uvr)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)dn \\
& -(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ 1)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)u(n)v(n)r(n)dn \\
\geq\ & (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uv)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)r(n)dn \\
& +(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ ur)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)v(n)dn \\
& +(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ vr)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)u(n)dn \\
& -(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)v(n)r(n)dn \\
& -(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ v)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)u(n)r(n)dn \\
& -(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ r)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)u(n)v(n)dn.
\end{aligned}$$

We can write

$$\begin{aligned}
& (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uvr)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ 1)(t;p)-(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ 1)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ uvr)(t;p) \\
\geq\ & (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uv)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ r)(t;p)+(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ ur)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ v)(t;p) \\
& +(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ vr)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ u)(t;p)-(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ vr)(t;p) \\
& -(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ v)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ ur)(t;p)-(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ r)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ uv)(t;p).
\end{aligned}$$

From above we get required inequality (3.4). □

Now we give our next result in which we prove fractional integral inequalities for any two functions not necessary synchronous or monotonic defined on $[0,\infty[$:

Theorem 3.3 Let u, v:$[0,\infty[\ \to\ [0,\infty[$ be two functions, then following fractional integral inequalities are valid:

For all $t>a\geq 0$, $\alpha>0$, $\beta>0$,

$$\begin{aligned}
& (t-a)^{\beta}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta+1,\iota}(\omega(t-a)^{\vartheta};p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u^2)(t;p) \\
& +(t-a)^{\alpha}E^{(\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha+1,\iota}(\omega(t-a)^{\vartheta};p)(\varepsilon^{\gamma,\delta,k,c}_{\mu,\beta,\iota,\omega,a}\ v^2)(t;p) \\
\geq\ & 2(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ v)(t;p)
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
& (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u^2)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ v^2)(t;p) \\
& +(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ v^2)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ u^2)(t;p) \\
\geq\ & 2(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uv)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ uv)(t;p).
\end{aligned} \tag{3.8}$$

Proof. Since we know

$$(u(m)-v(n))^2\geq 0,\quad \text{for all } m,\ n\in[0,\infty[.$$

Therefore

$$u^2(m) + v^2(n) \geq 2u(m)v(n). \tag{3.9}$$

Now multiplying both sides of (3.9) by $(t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)$, and then integrating resulting inequality with respect to m from a to t, we get

$$\begin{aligned} &\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)u^2(m)dm \\ &+ v^2(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)dm \\ \geq\ & 2v(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)u(m)dm. \end{aligned}$$

We can be write

$$\begin{aligned} &(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u^2)(t;p) + v^2(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ 1)(t;p) \\ \geq\ & 2v(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u)(t;p). \end{aligned} \tag{3.10}$$

Multiplying (3.10) by $(t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)$ and integrating with respect to n from a to t, we obtain

$$\begin{aligned} &(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u^2)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)dn \\ &+ (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ 1)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)v^2(n)dn \\ \geq\ & 2(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)v(n)dn. \end{aligned}$$

Consequently

$$\begin{aligned} &(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u^2)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ 1)(t;p) \\ &+ (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ 1)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ v^2)(t;p) \\ \geq\ & 2(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ v)(t;p). \end{aligned}$$

From above we get (3.7).
Next, since

$$(u(m)v(n) - u(n)v(m))^2 \geq 0, \quad \text{for all } m,\ n\ \in\ [0,\infty[.$$

$$u^2(m)v^2(n) + u^2(n)v^2(m) \geq 2u(m)v(m)u(n)v(n). \tag{3.11}$$

Multiplying (3.11) by $(t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)$ and integrating resulting inequality with respect to m from a to t, we obtain

$$\begin{aligned} &v^2(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)u^2(m)dm) \\ &+ u^2(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)v^2(m)dm) \\ \geq\ & 2u(n)v(n)\int_a^t (t-m)^{\alpha-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota}(\omega(t-m)^{\vartheta};p)u(m)v(m)dm. \end{aligned}$$

We can write

$$\begin{aligned} & v^2(n)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u^2)(t;p)+u^2(n)(\varepsilon^{\varpi,\delta,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ v^2)(t;p) \\ \geq\ & 2u(n)v(n)(\varepsilon^{\varpi,\delta,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uv)(t;p). \end{aligned} \tag{3.12}$$

Multiplying both sides of (3.12) by $(t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)$ and integrating resulting inequality with respect to n from a to t, following inequality obtained

$$\begin{aligned} & (\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u^2)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)v^2(n)dn \\ & +(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ v^2)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)u^2(n)dn \\ \geq\ & 2(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uv)(t;p)\int_a^t (t-n)^{\beta-1}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota}(\omega(t-n)^{\vartheta};p)u(n)v(n)dn. \end{aligned}$$

From above we get (3.8). □

Corollary 3.3 Let u, v:$[0,\infty[\ \rightarrow\ [0,\infty[$ be two functions, then for all $t>a\geq 0$, $\alpha>0$, we have

$$\begin{aligned} & (t-a)^{\alpha}E^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha+1,\iota}(\omega(t-a)^{\mu};p)\left[(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u^2)(t;p)+(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ v^2)(t;p)\right] \\ \geq\ & 2(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\beta,\iota,\omega,a}\ v)(t;p) \end{aligned}$$

and

$$(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ u^2)(t;p)(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ v^2)(t;p)\geq\left[(\varepsilon^{\varpi,\lambda,k,\varsigma}_{\vartheta,\alpha,\iota,\omega,a}\ uv)(t;p)\right]^2. \quad \square$$

4. Conclusion

In this paper, we derived certain new integral inequalities, related to the Chebyshev functional via extended generalized fractional integral operator. The inequalities obtained in this paper are reduces to various fractional integral inequalities by choosing particular values of parameters involved in extended generalized fractional integral operator such as: choosing $p=\omega=0$, fractional integral inequalities for Riemann-Liouville integral operator defined in [13], choosing $p=0$ *and* $\iota=\lambda=k=1$, fractional integral inequalities for fractional integral operator defined by Prabhakar in [18], choosing $p=0$ *and* $\iota=\lambda=1$, fractional integral inequalities for fractional integral operator defined in [26].

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Bhagwat R. Yewale
Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, (M.S), 431001, India
E-mail address: yewale.bhagwat@gmail.com

Deepak B. Pachpatte
Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, (M.S), 431001, India
E-mail address: pachpatte@gmail.com