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ON A MILD SOLUTION TO HILFER TIME-FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATION

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Abstract. Consider a Hilfer time-fractional stochastic differential equation

$$D_t^{\alpha,\nu}u(x,t)=\lambda\int_{B(0,t^{\nu/\alpha})}\sigma(u(y,t))\dot{w}(y,t)dy,\ t>0$$

of order $0<\alpha<1$ and type $0\le\nu\le 1$. The initial condition $u(x,0)=u_0(x)$, $x\in B(0,t^{\nu/\alpha})\subset\mathbb{R}^2$ is a non-random function assumed to be non-negative and bounded, $D_t^{\alpha,\nu}$ is a generalized Riemann–Liouville time - fractional derivative operator, σ is Lipschitz continuous, $\dot{w}(y,t)$ a space-time white noise and $\lambda>0$ is the level of the noise. The existence and uniqueness of a solution to the class of equation is given under some precise condition, we give moment growth bounds and long-term behaviours of the mild solution for $\alpha>\frac{1}{2}$. We show that the energy growth (second moment) of the solution to the Hilfer time-fractional stochastic differential equation grows exponentially in time at most a precise rate of $c_2\exp\left(c_5\lambda^{\frac{2}{2\alpha-1}}t\right)$ for all $t>0$ and at least a precise rate of $c_7\exp\left(c_{10}\lambda^{\frac{2}{2\alpha-1}}t\right)$ for some time t, for some positive constants $c_2,c_5,c_7,\ c_{10}$. More so, we show that when the non-linear term σ grows faster than linear, the energy of the solution fails to have global existence at all times t for all $\alpha\in(0,1)$.

1. Introduction

Fractional calculus has become very popular and important due to its application in modeling the anomalous diffusion behaviour of physical processes. They best describe systems which have long-time memory and long-range interaction.

Hilfer proposed a generalized Riemann–Liouville fractional derivative, which combines Riemann–Liouville fractional derivative and Caputo fractional derivative. This family of parameters gives certain degree of freedom on the initial conditions and produces more types of stationary states. While there have been studies on both Riemann-Liouville and Caputo fractional derivatives, see [2, 12, 13, 14] and their references, the Hilfer fractional derivative otherwise known as the generalized

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Riemann–Liouville fractional derivative makes it possible for one to interpolate between the Riemann–Liouville fractional derivative and Caputo fractional derivative. This type of fractional derivative operator finds its application in the theoretical stimulation of dielectric relaxation in glass forming materials and aquifer problems (hydrogeology), see [5, 6, 7, 8] and their references.

In [4], Furati et al. considered an initial value problem for a class of nonlinear fractional differential equation involving Hilfer fractional derivative; Sandev et al. in [16] obtained the solution of a fractional diffusion equation with a Hilfer–generalized Riemann–Liouville time fractional derivative in terms of Mittag–Leffler–type functions and Fox's H–function; Gu and Trujillo in [6] also considered the existence of mild solution for evolution equation with Hilfer fractional derivative. Rihan et al. in [15] furthered the study by considering a class of stochastic differential system with Hilfer fractional derivative and Poisson jumps in Hilbert space where the existence and uniqueness of mild solutions using successive approximation theory were studied, see also [1, 9, 10] for recent papers on the study of mild solutions of classes of Hilfer fractional stochastic differential equations.

In what follows, since modeling of most problems of real situations is best described by stochastic differential equations rather than the deterministic counterpart, we therefore perturb a class of Hilfer time-fractional differential equation with space-time white noise and study the properties of mild solution to a class of Hilfer (generalized Riemann–Liouville) time-fractional stochastic differential equation. To the best of our knowledge, this model has not been studied and none of our results exist in literature for this class of time-fractional stochastic differential equation. Thus, we consider the following Hilfer time-fractional stochastic differential equation of order $0 < \alpha < 1$ and type $0 \leq \nu \leq 1$

$$\begin{cases} D_t^{\alpha,\nu} u(x,t) = \lambda \displaystyle\int_{B(0,t^{\nu/\alpha})} \sigma(u(y,t))\dot{w}(y,t)dy, \ 0 < t < \infty, \\ \\ I_t^{1-\mu} u(x,0^+) = u_0(x), \ x \in B(0,t^{\nu/\alpha}), \ \mu = \alpha + \nu - \alpha\nu, \end{cases} \tag{1}$$

where $\dot{w}(y,t)$ is a space-time white noise, $\lambda > 0$ the noise level, $\sigma : B(0,t^{\nu/\alpha}) \to \mathbb{R}$ a Lipschitz continuous function, and $D_t^{\alpha,\nu}$ the generalized R–L tempered time-fractional derivative of order $0 < \alpha \leq 1$ and type $0 \leq \nu \leq 1$ given by Hilfer as follows [7]

$$D_t^{\alpha,\nu} u(x,t) = \left(I_t^{\nu(1-\alpha)} D\big(I_t^{(1-\nu)(1-\alpha)}\big)\right) u(x,t).$$

The two parameter family of fractional derivatives allows one to interpolate between Riemann-Liouville and the Caputo derivatives.
That is, for $\nu = 0$, it reduces to the classical Riemann–Liouville fractional derivative

$$D_t^{\alpha,0} u(x,t) = D^{\alpha}\big(I_t^{(1-\alpha)} u(x,t)\big) = D_t^{\alpha} u(x,t),$$

with

$$D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t-s)^{-\alpha} u(x,s)ds, \ \alpha \in (0,1)$$

and

$$I_t^{\alpha} u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(x,s)ds, \ \alpha \in (0,1).$$

For $\nu = 1$, it reduces to the Caputo fractional derivative

$$D_t^{\alpha,1}u(x,t) = \big(I_t^{(1-\alpha)}D^\alpha\big)u(x,t) = {}_*D_t^\alpha u(x,t),$$

with

$${}_*D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\partial_s u(x,s)ds,\ \alpha \in (0,1).$$

For $0 < \nu < 1$, it interpolates continuously between the two derivatives.

Now, define the mild solution to equation (1) in $L^2(\mathbf{P})$ as follows:

Definition 1.1. We say that $\big(u(x,t), x \in B(0,\, t^\mu), 0 \le t \le T^*\big)$ is a mild solution of Equation (1) if $a.s$, the following is satisfied

$$\begin{aligned} u(x,t) &= \frac{u_0(x)}{\Gamma(\mu)}t^{(\alpha-1)(1-\nu)} \\ &+ \frac{\lambda}{\Gamma(\alpha)}\int_0^t\int_{B(0,t^{\nu/\alpha})}(t-s)^{\alpha-1}\sigma(u(y,s))\dot{w}(y,s)dyds \\ &= \frac{u_0(x)}{\Gamma(\mu)}t^{(\mu-1)} + \frac{\lambda}{\Gamma(\alpha)}\int_0^t\int_{B(0,t^{\nu/\alpha})}(t-s)^{\alpha-1}\sigma(u(y,s))w(dy,ds),\ \mu > 1. \end{aligned}$$

Also, if $\big(u(x,t), x \in B(0,\, t^\mu), 0 \le t \le T^*\big)$ satisfies the following additional condition

$$\sup_{t\in[0,T^*]}\ \sup_{x\in B(0,\, t^\mu)} \mathbb{E}|u(x,t)|^2 < \infty,$$

then we say that $\big(u(x,t), x \in B(0,\, t^\mu), 0 \le t \le T^*\big)$ is a random field solution to Equation (1).

The above problem is motivated by the following proposition

Proposition 1.2 ([6])**.** *If $0 < \alpha < 1$ and $0 \le \nu \le 1$, then the solution of the generalized fractional differential equation*

$$\begin{cases} D_{a^+}^{\alpha,\nu}x(t) = f(t,x(t)),\ t > a \\ I_{a^+}^{1-\mu}x(a^+) = x_a,\ \mu = \alpha + \nu - \alpha\nu, \end{cases}$$

is given by

$$x(t) = \frac{x_a}{\Gamma(\mu)}(t-a)^{(\alpha-1)(1-\nu)} + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s,x(s))ds.$$

Next, we give an estimates (bounds) on an incomplete gamma function.

Theorem 1.3 ([11])**.** *The following inequalities*

$$\exp\left(\frac{-ax}{a+1}\right) \le \frac{a}{x^a}\gamma(a,x) \le {}_1F_1(a;a+1;-x) \le \frac{1}{a+1}\Big(1 + ae^{-x}\Big)\ \textit{for all}\ x > 0,$$

hold, where ${}_1F_1(a;a+1;-x)$ is a confluent hypergeometric (Kummer) function.

Also, for $0 < a \le 1$,

$$\frac{1-e^{-x}}{x} \le \frac{a}{x^a}\gamma(a,x).$$

The function $\gamma(z,x)$ is an incomplete gamma function given by

$$\gamma(z,x)=\int_0^x e^{-t}t^{z-1}dt,\ x>0$$

with the relation

$$\Gamma(z)=\gamma(z,x)+\Gamma(z,x),$$

where $\Gamma(z,x)$ is the complement of the incomplete gamma function given by

$$\Gamma(z,x)=\int_x^\infty e^{-t}t^{z-1}dt,\ x>0.$$

The paper is outlined as follows. In section 2, we gave the main results and their proofs and section 3 contains a concise summary of the paper.

2. Main Results

Here, we make the following assumption on σ; that is, σ is globally Lipschitz:

Condition 2.1. There exist a finite positive constant, Lip_σ such that for all x, $y\in B(0,t^{\nu/\alpha})\subset\mathbb{R}^2$, we have

$$|\sigma(x)-\sigma(y)|\leq \mathrm{Lip}_\sigma|x-y|,$$

with $\sigma(0)=0$ for convenience.

We define the $L^2(\mathbf{P})$ norm as follows:

$$\|u\|_{2,\alpha,\beta,\nu}:=\left\{\sup_{0\leq t\leq T^*}\sup_{x\in B(0,t^{\nu/\alpha})}e^{-\beta t}\mathbb{E}|u(x,t)|^2\right\}^{1/2},$$

where T^* is a fixed finite number and obtain the following result

Theorem 2.2. *Suppose* $c_3<\frac{1}{(\lambda\mathrm{Lip}_\sigma)^2}$ *for positive constant* Lip_σ *together with Condition* (2.1)*, then there exists solution* u *that is unique up to modification, with* $c_3:=\frac{\pi}{\Gamma^2(\alpha)}\frac{T^{*2\frac{\nu}{\alpha}+2\alpha-1}}{2\alpha-1}$, $\alpha>\frac{1}{2}$.

The proof of the above result is by Banach's fixed point theorem and is based on the following lemma(s):

For $\mu>1$, we define the operator

$$\mathcal{A}u(x,t)=\frac{u_0(x)}{\Gamma(\mu)}t^{(\mu-1)}+\frac{\lambda}{\Gamma(\alpha)}\int_0^t\int_{B(0,t^{\nu/\alpha})}(t-s)^{\alpha-1}\sigma(u(y,s))w(dy,ds),$$

and the fixed point of the operator $\mathcal{A}$ gives the solution of Equation (1).

Lemma 2.3. *Given a random solution* u *such that* $\|u\|_{2,\alpha,\beta,\nu}<\infty$ *and Condition* (2.1) *holds. Then there exist positive constants* c_2 *and* c_3 *such that*

$$\|\mathcal{A}u\|^2_{2,\alpha,\beta,\nu}\leq c_2+c_3\lambda^2\mathrm{Lip}^2_\sigma\|u\|^2_{2,\alpha,\beta,\nu},$$

where $c_2:=\frac{c_1}{\Gamma^2(\mu)}T^{*2(\mu-1)}$, $c_3:=\frac{\pi}{\Gamma^2(\alpha)}\frac{T^{*2\frac{\nu}{\alpha}+2\alpha-1}}{2\alpha-1}$, $\alpha>\frac{1}{2}$.

Proof. By Itô isometry and assuming that $|u_0(x)|^2 \leq c_1$, we obtain

$$\begin{aligned}
\mathbb{E}|\mathcal{A}u(x,t)|^2 &= \frac{|u_0(x)|^2}{\Gamma^2(\mu)} t^{2(\mu-1)} \\
&+ \frac{\lambda^2}{\Gamma^2(\alpha)} \int_0^t \int_{B(0,t^{\nu/\alpha})} (t-s)^{2\alpha-2} \mathbb{E}|\sigma(u(y,s))|^2 dyds \\
&\leq \frac{c_1}{\Gamma^2(\mu)} t^{2(\mu-1)} \\
&+ \frac{\lambda^2 \mathrm{Lip}_\sigma^2}{\Gamma^2(\alpha)} \int_0^t \int_{B(0,t^{\nu/\alpha})} (t-s)^{2\alpha-2} \mathbb{E}|u(y,s)|^2 dyds \\
&\leq \frac{c_1}{\Gamma^2(\mu)} t^{2(\mu-1)} \\
&+ \frac{\lambda^2 \mathrm{Lip}_\sigma^2}{\Gamma^2(\alpha)} \pi t^{2\nu/\alpha} \int_0^t \sup_{y\in B(0,t^{\nu/\alpha})} \mathbb{E}|u(y,s)|^2 (t-s)^{2\alpha-2} ds.
\end{aligned}$$

Multiply through by $e^{-\beta t}$ we have

$$\begin{aligned}
e^{-\beta t}\mathbb{E}|\mathcal{A}u(x,t)|^2 &\leq \frac{c_1}{\Gamma^2(\mu)} t^{2(\mu-1)} e^{-\beta t} \\
&+ \frac{\lambda^2 \mathrm{Lip}_\sigma^2}{\Gamma^2(\alpha)} \pi t^{2\nu/\alpha} \|u\|^2_{2,\alpha,\beta,\nu} \int_0^t (t-s)^{2\alpha-2} e^{-\beta(t-s)} ds \\
&= \frac{c_1}{\Gamma^2(\mu)} t^{2(\mu-1)} e^{-\beta t} \\
&+ \frac{\lambda^2 \mathrm{Lip}_\sigma^2}{\Gamma^2(\alpha)} \pi t^{2\nu/\alpha} \|u\|^2_{2,\alpha,\beta,\nu} \beta^{1-2\alpha} \big[\Gamma(2\alpha-1) - \Gamma(2\alpha-1,\beta t)\big] \\
&= \frac{c_1}{\Gamma^2(\mu)} t^{2(\mu-1)} e^{-\beta t} \\
&+ \frac{\lambda^2 \mathrm{Lip}_\sigma^2}{\Gamma^2(\alpha)} \pi t^{2\nu/\alpha} \|u\|^2_{2,\alpha,\beta,\nu} \beta^{1-2\alpha} \gamma(2\alpha-1,\beta t),\ \mathcal{R}(\alpha) > \frac{1}{2}.
\end{aligned}$$

Now take sup over $t \in [0,T^*]$, $T^* < \infty$, and sup over $x \in B(0,t^{\nu/\alpha})$ and upper bound estimate of Theorem 1.3 to obtain

$$\begin{aligned}
\|\mathcal{A}u\|^2_{2,\alpha,\beta,\nu} &\leq \frac{c_1}{\Gamma^2(\mu)} \sup_{0<t\leq T^*} t^{2(\mu-1)} e^{-\beta t} + \frac{\lambda^2 \mathrm{Lip}_\sigma^2}{\Gamma^2(\alpha)} \frac{\pi T^{*2\nu/\alpha}}{\beta^{2\alpha-1}} \frac{1}{(2\alpha-1)(2\alpha)} \\
&\times \sup_{0<t\leq T^*} (\beta t)^{2\alpha-1} \left(1 + (2\alpha-1)e^{-\beta t}\right) \|u\|^2_{2,\alpha,\beta,\nu} \\
&\leq \frac{c_1}{\Gamma^2(\mu)} \sup_{0<t\leq T^*} t^{2(\mu-1)} + \frac{\lambda^2 \mathrm{Lip}_\sigma^2}{\Gamma^2(\alpha)} \frac{\pi T^{*2\frac{\nu}{\alpha}+2\alpha-1}}{2\alpha-1} \|u\|^2_{2,\alpha,\beta,\nu} \\
&= c_2 + c_3 \lambda^2 \mathrm{Lip}_\sigma^2 \|u\|^2_{2,\alpha,\beta,\nu},
\end{aligned}$$

with positive constants and the second line follows since $e^{-\beta t} \leq 1$ for all $t \in [0,T^*]$. □

Similarly, we obtain the following result

Lemma 2.4. *Suppose* u *and* v *are random solutions such that* $\|u\|_{2,\alpha,\beta,\nu} + \|v\|_{2,\alpha,\beta,\nu} < \infty$ *and Condition* (2.1) *holds. Then there exists a positive constant* c_3 *such that*

$$\|\mathcal{A}u - \mathcal{A}v\|^2_{2,\alpha,\beta,\nu} \leq c_3 \lambda^2 \mathrm{Lip}_\sigma^2 \|u-v\|^2_{2,\alpha,\beta,\nu}.$$

Proof of Theorem 2.2. By fixed point theorem we have $u(x,t) = \mathcal{A}u(x,t)$ and from Lemma 2.3,

$$\|u\|^2_{2,\alpha,\beta,\nu} = \|\mathcal{A}u\|^2_{2,\alpha,\beta,\nu} \leq c_2 + c_3\lambda^2 \mathrm{Lip}^2_\sigma \|u\|^2_{2,\alpha,\beta,\nu}$$

which follows that

$$\|u\|^2_{2,\alpha,\beta,\nu}\big[1 - c_3\lambda^2\mathrm{Lip}^2_\sigma\big] \leq c_2 \Rightarrow \|u\|_{2,\alpha,\beta,\nu} < \infty \Leftrightarrow c_3 < \frac{1}{(\lambda \mathrm{Lip}_\sigma)^2}.$$

Similarly, from Lemma 2.4,

$$\|u - v\|^2_{2,\alpha,\beta,\nu} = \|\mathcal{A}u - \mathcal{A}v\|^2_{2,\alpha,\beta,\nu} \leq c_3\lambda^2\mathrm{Lip}^2_\sigma\|u - v\|^2_{2,\alpha,\beta,\nu},$$

thus

$$\|u - v\|^2_{2,\alpha,\beta,\nu}\big[1 - c_3\lambda^2\mathrm{Lip}^2_\sigma\big] \leq 0$$

and therefore

$$\|u - v\|_{2,\alpha,\beta,\nu} < 0 \;\; (\Rightarrow \|u - v\|_{2,\alpha,\beta,\nu} = 0)$$

if and only if

$$c_3 < \frac{1}{(\lambda\mathrm{Lip}_\sigma)^2}.$$

The existence and uniqueness result follows by Banach's contraction principle. □

2.1. **Growth moment estimates.** For the growth moment results, we present the following renewable inequalities, which give bounds on the functions involved:

Proposition 2.5 ([3])**.** *Let $\rho > 0$ and suppose that $f(t)$ is a locally integrable function satisfying*

$$f(t) \leq c_1 + \kappa\int_0^t (t-s)^{\rho-1} f(s)ds, \; for \; all\, t > 0,$$

where c_1 is some positive number. Then, we have

$$f(t) \leq c_2 \exp\big(c_3(\Gamma(\rho))^{1/\rho}\kappa^{1/\rho}t\big), \; for \; all\, t > 0,$$

for some positive constants c_2 and c_3.

We also give the converse of the above proposition:

Proposition 2.6 ([3])**.** *Let $\rho > 0$ and suppose that $f(t)$ is a nonnegative, locally integrable function satisfying*

$$f(t) \geq c_1 + \kappa\int_0^t (t-s)^{\rho-1} f(s)ds, \; for \; all\, t > 0,$$

where c_1 is some positive number. Then, we have

$$f(t) \geq c_2 \exp\big(c_3(\Gamma(\rho))^{1/\rho}\kappa^{1/\rho}t\big), \; for \; all\, t > \frac{e}{\rho}\big(\Gamma(\rho)\kappa\big)^{-1/\rho},$$

for some positive constants c_2 and c_3.

Theorem 2.7. *Given that Condition 2.1 holds, then for all $t > 0$ and $\alpha > \frac{1}{2}$ we have*

$$\sup_{x\in B(0,t^{\nu/\alpha})} \mathbb{E}|u(x,t)|^2 \leq c_2 \exp\big(c_5\lambda^{\frac{2}{2\alpha-1}}t\big),$$

for some positive constants c_2 and c_5.

Proof. Following similar steps as in Lemma 2.3 and the assumption that the initial condition is bounded above,

$$\begin{aligned}\sup_{x\in B(0,t^{\nu/\alpha})} \mathbb{E}|u(x,t)|^2 &\leq \frac{c_1}{\Gamma^2(\mu)} t^{2(\mu-1)} \\ &+ \frac{\lambda^2 \mathrm{Lip}_\sigma^2}{\Gamma^2(\alpha)} \pi t^{2\nu/\alpha} \int_0^t \sup_{y\in B(0,t^{\nu/\alpha})} \mathbb{E}|u(y,s)|^2 (t-s)^{2\alpha-2} ds \\ &\leq \frac{c_1}{\Gamma^2(\mu)} \sup_{0<t\leq T^*} t^{2(\mu-1)} \\ &+ \frac{\lambda^2 \mathrm{Lip}_\sigma^2}{\Gamma^2(\alpha)} \pi T^{*2\nu/\alpha} \int_0^t \sup_{y\in B(0,t^{\nu/\alpha})} \mathbb{E}|u(y,s)|^2 (t-s)^{2\alpha-2} ds.\end{aligned}$$

Let $f(t) := \sup_{x\in B(0,t^{\nu/\alpha})} \mathbb{E}|u(x,t)|^2$ then we have

$$\begin{aligned} f(t) &\leq c_2 + c_3\lambda^2 \int_0^t (t-s)^{2\alpha-2} f(s) ds \\ &= c_2 + c_3\lambda^2 \int_0^t (t-s)^{(2\alpha-1)-1} f(s) ds.\end{aligned}$$

Thus by applying Proposition 2.5 for $\alpha > \frac{1}{2}$, we obtain

$$f(t) \leq c_2 \exp\left[c_4 \lambda^{\frac{2}{2\alpha-1}} (\Gamma(2\alpha-1))^{\frac{1}{2\alpha-1}} t\right] = c_2 \exp\left[c_5 \lambda^{\frac{2}{2\alpha-1}} t\right],$$

and the result follows. □

Similarly, we have the lower bound estimate by assuming that the initial condition $u_0(x) > c_6$ for some positive constant c_3 and

Condition 2.8. There exist a finite positive constant, L_σ such that for all $x \in B(0,t^{\nu/\alpha})$, we have

$$|\sigma(x)| \geq L_\sigma |x|.$$

Thus

Theorem 2.9. *Given that Condition 2.8 holds, then there exists a time t such that for $\alpha > \frac{1}{2}$ we have*

$$\inf_{x\in B(0,t^{\nu/\alpha})} \mathbb{E}|u(x,t)|^2 \geq c_7 \exp\left(c_{10} \lambda^{\frac{2}{2\alpha-1}} t\right),$$

for some positive constants c_7 and c_{10}.

Proof. With the assumption that the initial condition $u_0(x) > c_6$, we have

$$\begin{aligned}\inf_{x\in B(0,t^{\nu/\alpha})} \mathbb{E}|u(x,t)|^2 &\geq \frac{c_6}{\Gamma^2(\mu)} t^{2(\mu-1)} \\ &+ \frac{\lambda^2 L_\sigma^2}{\Gamma^2(\alpha)} \pi t^{2\nu/\alpha} \int_0^t \inf_{y\in B(0,t^{\nu/\alpha})} \mathbb{E}|u(y,s)|^2 (t-s)^{2\alpha-2} ds \\ &\geq \frac{c_6}{\Gamma^2(\mu)} \inf_{0<t\leq T^*} t^{2(\mu-1)} \\ &+ \frac{\lambda^2 L_\sigma^2}{\Gamma^2(\alpha)} \pi \inf_{0<t\leq T^*} t^{2\nu/\alpha} \int_0^t \inf_{y\in B(0,t^{\nu/\alpha})} \mathbb{E}|u(y,s)|^2 (t-s)^{2\alpha-2} ds.\end{aligned}$$

Let $g(t) := \inf_{x\in B(0,t^{\nu/\alpha})} \mathbb{E}|u(x,t)|^2$ then we have

$$\begin{aligned} g(t) &\geq c_7 + c_8\lambda^2 \int_0^t (t-s)^{2\alpha-2} g(s)ds \\ &= c_7 + c_8\lambda^2 \int_0^t (t-s)^{(2\alpha-1)-1} g(s)ds. \end{aligned}$$

Applying Proposition 2.6 for $\alpha > \frac{1}{2}$, then for $t > \frac{e}{2\alpha-1}\big(c_8\lambda^2\Gamma(2\alpha-1)\big)^{-\frac{1}{2\alpha-1}}$ we obtain

$$g(t) \geq c_7 \exp\left[c_9 \lambda^{\frac{2}{2\alpha-1}} (\Gamma(2\alpha-1))^{\frac{1}{2\alpha-1}} t\right] = c_7 \exp\left[c_{10}\lambda^{\frac{2}{2\alpha-1}} t\right],$$

and the result follows. □

We now give immediate consequences of the above Theorem 2.7 and Theorem 2.9. First is the long time behaviour of the energy solution, which says that the rate of growth depends on the operator and the noise term (the noise level) as time t grows to T^*:

Corollary 2.10. *Suppose that $\alpha > \frac{1}{2}$ and that positive constants c_5, c_{10} are as in Theorem 2.7 and Theorem 2.9. Then for all $x \in B(0, t^{\nu/\alpha})$ and $\lambda > 0$,*

$$c_{10}\lambda^{\frac{2}{2\alpha-1}} \leq \liminf_{t\to T^*} \frac{\log \mathbb{E}|u(x,t)|^2}{t}.$$

Remark 2.11. From Theorem 2.9, we obtain the following estimate: $\log \mathbb{E}|u(x,t)|^2 \geq \log c_7 + c_{10}\lambda^{\frac{2}{2\alpha-1}} t \geq c_{10}\lambda^{\frac{2}{2\alpha-1}} t$.

The next result gives the rate of growth of the second moment with respect to the noise parameter λ:

Corollary 2.12. *Suppose that $\alpha > \frac{1}{2}$ and conditions of Theorem 2.7 and Theorem 2.9 hold. Then for all $x \in B(0, t^{\nu/\alpha})$ and for some time t,*

$$\frac{2}{2\alpha-1} \leq \liminf_{\lambda\to\infty} \frac{\log\log \mathbb{E}|u(x,t)|^2}{\log \lambda} \leq \limsup_{\lambda\to\infty} \frac{\log\log \mathbb{E}|u(x,t)|^2}{\log\lambda} \leq \frac{2}{2\alpha-1}.$$

2.2. **Global non-existence of solution.** We show that if the function σ grows faster than linear, then the second moment $\mathbb{E}|u(x,t)|^2$ of the solution to (1) ceases to exist for all time t.

Now, suppose that instead of Condition 2.8, we have the following condition:

Condition 2.13. There exist a finite positive constant, L_σ such that for all $x \in B(0, t^{\nu/\alpha}) \subset \mathbb{R}^2$, we have

$$|\sigma(x)| \geq L_\sigma |x|^\beta, \;\; \beta > 1.$$

Theorem 2.14. *Suppose that Condition 2.13 is in force. Then there does not exist a solution to Equation* (1) *for all* $0 < \alpha < 1$.

Proof. Assuming the lower bound condition on the initial condition $u_0(x)$, then by Condition 2.13, we have

$$\begin{aligned} \mathbb{E}|u(x,t)|^2 &\geq c_7 + c_8\lambda^2 \int_a^t (t-s)^{2(\alpha-1)} \mathbb{E}|u(x,s)|^{2(\beta)} ds \\ &\geq c_7 + c_8\lambda^2 \int_0^t (t-s)^{2(\alpha-1)} \left(\inf_{x\in B(0,t^{\nu/\alpha})} \mathbb{E}|u(x,s)|^2\right)^\beta ds. \end{aligned}$$

Let $g(t) := \inf_{x\in B(0,t^{\nu/\alpha})} \mathbb{E}|u(x,t)|^2$. Then it follows that

$$\begin{aligned} g(t) &\geq c_7 + c_8\lambda^2 \int_0^t (t-s)^{2(\alpha-1)} g^\beta(s)ds \\ &\geq c_7 + c_8\lambda^2 \int_0^t t^{2(\alpha-1)} g^\beta(s)ds, \end{aligned}$$

since $t \geq t-s$ and $(t-s)^{2(\alpha-1)} \geq t^{2(\alpha-1)}$ for all $2(\alpha-1) < 0$. Multiply through by $t^{2(\alpha-1)}$, then

$$\begin{aligned} g(t)t^{2(\alpha-1)} &\geq c_7 t^{2(\alpha-1)} + c_8\lambda^2 \int_0^t g^\beta(s)ds \\ &= c_7 t^{2(\alpha-1)} + c_8\lambda^2 \int_0^t \frac{\left(s^{2(\alpha-1)}g(s)\right)^\beta}{s^{2\beta(\alpha-1)}} ds. \end{aligned}$$

Let $y(t) = t^{2(\alpha-1)}g(t),\ 0 < t \leq T^*$, then we have

$$y(t) \geq c_7 T^{*2(\alpha-1)} + c_8\lambda^2 \int_0^t \frac{y^\beta(s)}{s^{2\beta(\alpha-1)}} ds.$$

Applying the fundamental theorem of calculus, thus

$$\dot{y}(t) \geq c_8\lambda^2 \frac{y^\beta(t)}{t^{2\beta(\alpha-1)}},$$

and solving the differential equation $\dot{y}(t) = c_8\lambda^2 \frac{y^\beta(t)}{t^{2\beta(\alpha-1)}}$, that is, $\frac{\dot{y}(t)}{y^\beta(t)} = c_8\lambda^2 t^{2\beta(1-\alpha)}$ with $y(0) = c_7 T^{*2(\alpha-1)}$ we have

$$y(t) = \left\{ \frac{(1-\beta)c_8\lambda^2}{1+2\beta(1-\alpha)} t^{1+2\beta(1-\alpha)} + y(0)^{1-\beta} \right\}^{\frac{1}{1-\beta}}$$

and the solution fails to exist for all $1 + 2\beta(1-\alpha) > 0$, that is, for all $\beta > 1$ and $0 < \alpha < 1$. □

3. Conclusion

Long term behaviours (with respect to time t and with respect to the noise level λ) of the mild solution to a Hilfer time-fractional stochastic differential equation were studied. The existence and uniqueness result was obtained under some precise conditions and we proved the second moment energy bounds (upper and lower estimates) of the solution. The result showed that our solution grows exponentially in time and the solution fails to exist for all time t when the nonlinear term grows more than linear for $\alpha \in (0, 1)$.

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