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EXISTENCE AND OSCILLATION FOR COUPLED FRACTIONAL q-DIFFERENCE SYSTEMS

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Abstract. This paper deals first with existence of bounded solutions, then followed by some oscillation results, for a coupled fractional q-difference system. For the first results, some applications are made of the fixed point theory, and the diagonalization process. Finally, we give two examples illustrating the applicability of the imposed conditions.

1. Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics, and other applied sciences; see the monographs [5, 6, 7, 23, 27, 28, 31], the papers [1, 3, 4, 8, 24], and the references therein. Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations and inclusions with the Caputo fractional derivative; [6, 22].

Fractional q-difference equations were initiated in the beginning of the 19th century [9, 15], and received significant attention in recent years; see [12, 13, 16, 17] and references therein. In [1, 2, 3, 8], Abbas *et al.* considered some existence results for some coupled fractional differential systems.

In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of several classes of differential equations and inclusions; [11, 14, 18, 20, 29, 30]. In this article; we discuss the existence of solutions and their oscillation for the following coupled fractional q-difference system

$$\begin{cases} ({}^C D_q^{\alpha_1} u_1)(t) = f_1(t, u_2(t)), \\ ({}^C D_q^{\alpha_2} u_2)(t) = f_2(t, u_1(t)), \\ (u_1(0), u_2(0)) = (u_{01}, u_{02}),\ u_1 \text{ and } u_2 \text{ are bounded on } \mathbb{R}_+, \end{cases} \quad ;\ t \in \mathbb{R}_+, \tag{1}$$

where $q \in (0,1)$, $\alpha_i \in (0,1]$, $\mathbb{R}_+ := [0, +\infty)$, $f_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$; $i = 1,2$, are given functions, and ${}^C D_q^{\alpha_i}$ is the Caputo fractional q-difference derivative of order α_i; $i = 1,2$.

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2. Preliminaries

Let $I := [0, T]$ where $T > 0$. As usual, $L^1(I)$ denotes the space of measurable functions $v : I \to \mathbb{R}$ which are Lebesgue integrable with the norm

$$\|v\|_1 = \int_0^T |v(t)|dt.$$

Consider the Banach space $C(I) := C(I, \mathbb{R})$ of continuous functions from I into $\mathbb{R}$ equipped with the norm

$$\|u\|_\infty := \sup_{t\in I} |u(t)|.$$

Now, we recall some definitions and properties of fractional q-calculus. For $a \in \mathbb{R}$, we set

$$[a]_q = \frac{1-q^a}{1-q}.$$

The q-analogue of the power $(a-b)^n$ is

$$(a-b)^{(0)} = 1,\ (a-b)^{(n)} = \Pi_{k=0}^{n-1}(a-bq^k),\ a, b \in \mathbb{R},\ n \in \mathbb{N}.$$

In general,

$$(a-b)^{(\alpha)} = a^\alpha \Pi_{k=0}^{\infty}\left(\frac{a-bq^k}{a-bq^{k+\alpha}}\right),\ a, b, \alpha \in \mathbb{R}.$$

Note that if $b = 0$, then $a^{(\alpha)} = a^\alpha$.

Definition 2.1 [21] The q-gamma function is defined by

$$\Gamma_q(\xi) = \frac{(1-q)^{(\xi-1)}}{(1-q)^{\xi-1}},\ \xi \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$$

Notice that the q-gamma function satisfies $\Gamma_q(1+\xi) = [\xi]_q\Gamma_q(\xi)$.

Definition 2.2 [21] The q-derivative of order $n \in \mathbb{N}$ of a function $u : I \to \mathbb{R}$ is defined by $(D_q^0 u)(t) = u(t)$,

$$(D_q u)(t) := (D_q^1 u)(t) = \frac{u(t) - u(qt)}{(1-q)t},\ t \neq 0,\ \ (D_q u)(0) := \lim_{t\to 0}(D_q u)(t),$$

and

$$(D_q^n u)(t) := (D_q D_q^{n-1} u)(t),\ t \in I,\ n \in \{1, 2, \ldots\}.$$

Set $I_t := \{tq^n : n \in \mathbb{N}\} \cup \{0\}$.

Definition 2.3 [21] The q-integral of a function $u : I_t \to \mathbb{R}$ is defined by

$$(I_q u)(t) = \int_0^t u(s)d_qs := \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n),$$

provided that the series converges.

We note that $(D_q I_q u)(t) = u(t)$, while if u is continuous at 0, then

$$(I_q D_q u)(t) = u(t) - u(0).$$

Definition 2.4 [10] The Riemann-Liouville fractional q-integral of order $\alpha \in \mathbb{R}_+ := [0, \infty)$ of a function $u : I \to \mathbb{R}$ is defined by $(I_q^0 u)(t) := u(t)$, and

$$(I_q^\alpha u)(t) := \int_0^t \frac{(t - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} u(s) d_q s, \; t \in I.$$

Note that for $\alpha = 1$, we have $(I_q^1 u)(t) = (I_q u)(t)$.

Lemma 2.5 [25] For $\alpha \in \mathbb{R}_+ := [0, \infty)$ and $\lambda \in (-1, \infty)$ we have

$$(I_q^\alpha (t-a)^{(\lambda)})(t) = \frac{\Gamma_q(1+\lambda)}{\Gamma(1+\lambda+\alpha)} (t-a)^{(\lambda+\alpha)}, \; 0 < a < t < T.$$

In particular,

$$(I_q^\alpha 1)(t) = \frac{1}{\Gamma_q(1+\alpha)} t^{(\alpha)}.$$

Definition 2.6 [26] The Riemann-Liouville fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to \mathbb{R}$ is defined by $(D_q^0 u)(t) := u(t)$, and

$$(D_q^\alpha u)(t) := (D_q^{[\alpha]} I_q^{[\alpha]-\alpha} u)(t), \; t \in I,$$

where $[\alpha]$ is the integer part of α.

Definition 2.7 [26] The Caputo fractional q-derivative of order $\alpha \in \mathbb{R}_+$ of a function $u : I \to \mathbb{R}$ is defined by $({}^C D_q^0 u)(t) := u(t)$, and

$$({}^C D_q^\alpha u)(t) := (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} u)(t), \; t \in I.$$

Lemma 2.8 [26] Let $\alpha \in \mathbb{R}_+$. Then the following equality holds:

$$(I_q^\alpha \; {}^C D_q^\alpha u)(t) = u(t) - \sum_{k=0}^{[\alpha]-1} \frac{t^k}{\Gamma_q(1+k)} (D_q^k u)(0).$$

In particular, if $\alpha \in (0, 1)$, then

$$(I_q^\alpha \; {}^C D_q^\alpha u)(t) = u(t) - u(0).$$

From the above lemma, we conclude with the following result.

Lemma 2.9 Let $h \in C(I)$. Then the problem

$$\begin{cases} ({}^C D_q^{\alpha_1} u_1)(t) = h(t) \\ u_0 = u_0, \end{cases} \qquad (2)$$

has a unique solution given by

$$u(t) = u_0 + (I_q^\alpha h)(t).$$

In the sequel we will make use of the following fixed point theorem.

Theorem 2.10 (Schauder fixed point theorem, [19]). Let E be a Banach space and Q be a nonempty bounded convex and closed subset of E, and let $N : Q \to Q$ be a compact and continuous map. Then N has at least one fixed point in Q.

3. Existence of bounded solutions

In this section, we are concerned with the existence of solutions of the coupled system (1).

Definition 3.1 By a solution of the coupled system (1) we mean a pair of bounded coupled functions $(u_1, u_2) \in C(I) \times C(I)$ that satisfies the system

$$\begin{cases} ({}^C D_q^{\alpha_1} u_1)(t) = f_1(t, u_2(t)), \\ ({}^C D_q^{\alpha_2} u_2)(t) = f_2(t, u_1(t)), \end{cases}$$

on $\mathbb{R}_+ \times \mathbb{R}_+$ and the initial conditions $(u_1(0), u_2(0)) = (u_{01}, u_{02})$.

For $n \in \mathbb{N}$, let $I_n := [0, n]$. We denote by $X_n := C(I_n) \times C(I_n)$ the Banach space with the norm

$$\|(u, v)\|_{X_n} = \|u\|_\infty + \|v\|_\infty.$$

The following hypotheses will be used in the sequel.

(H_1) The functions $t \mapsto f_1(t, v)$ and $t \mapsto f_2(t, u)$ are measurable on $I_n := [0, n]$, $n \in \mathbb{N}$, for each $u, v \in \mathbb{R}$, and the functions $u \mapsto f_1(t, v)$ and $v \mapsto f_2(t, u)$ is continuous for a.e. $t \in I_n$.

(H_2) There exist continuous functions $p_{in} : I_n \to \mathbb{R}_+,\ n = 1, 2$, such that

$$|f_i(t, u_i)| \le p_{in}(t), \text{ for a.e. } t \in I_n, \text{ and each } u_i \in \mathbb{R}.$$

Set

$$p_{in}^* = \sup_{t \in I_n} p_{in}(t),\ i = 1, 2.$$

Theorem 3.2 Assume that hypotheses (H_1) and (H_2) hold. Then the problem (1) has at least one bounded solution defined on $\mathbb{R}_+$.

Proof. The proof will be given in two parts. Fix $n \in \mathbb{N}$ and consider the problem

$$\begin{cases} ({}^C D_q^{\alpha_1} u_1)(t) = f_1(t, u_2(t)) \\ ({}^C D_q^{\alpha_2} u_2)(t) = f_2(t, u_1(t)) \quad ; \ t \in I_n := [0, n]. \\ (u_1(0), u_2(0)) = (u_{01}, u_{02}) \end{cases} \tag{3}$$

Part 1. We begin by showing that (3) has a solution $(u_{1n}, u_{2n}) \in X_n$ with

$$\|(u_{1n}, u_{2n})\|_{X_n} \le R_n := \|u_{01}\| + \|u_{02}\| + \sum_{i=1}^{2} \frac{n^\alpha p_{in}^*}{\Gamma_q(1 + \alpha_i)}.$$

Consider the operators $N_i : C(I_n) \to C(I_n),\ i = 1, 2$, and $N : X_n \to X_n$ defined by

$$(N_1 u_1)(t) = u_{01} + (I_q^{\alpha_1} f_1(\cdot, u_2(\cdot)))(t),\ t \in I, \tag{4}$$

$$(N_2) u_2(t) = u_{02} + (I_q^{\alpha_2} f_2(\cdot, u_1(\cdot)))(t),\ t \in I, \tag{5}$$

and

$$(N(u_1,u_2))(t) = ((N_1u_1)(t),(N_2u_2)(t)). \tag{6}$$

Clearly, the fixed points of the operator N are solutions of our coupled system (3).

For any $n \in \mathbb{N}^\star := \mathbb{N}\backslash\{0\}$, we consider the ball

$$B_{R_n} := B(0,R_n) = \{w = (w_1,w_2) \in X_n : \|w\|_{X_n} \le R_n\}.$$

We shall show that the operator $N : B_{R_n} \to B_{R_n}$ satisfies all the assumptions of Theorem 2.10. The proof will be given in several steps.

Step 1. $N : B_{R_n} \to B_{R_n}$ *is continuous.*
Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence such that $u_k := (u_{1k},u_{2k}) \to u := (u_1,u_2)$ in B_{R_n}. Then, for each $t \in I_n$, we have

$$\|(N_1u_{1k})(t) - (N_1u_1)(t)\| \le \int_0^t \frac{(tq-s)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}\|f_1(s,u_{2k}(s)) - f_1(s,u_2(s))\|d_qs,$$

and

$$\|(N_2u_{2k})(t) - (N_2u_2)(t)\| \le \int_0^t \frac{(tq-s)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}\|f_2(s,u_{1k}(s)) - f_1(s,u_1(s))\|d_qs.$$

Since $u_{ik} \to u_i$ as $k \to \infty$, the Lebesgue dominated convergence theorem implies that

$$\|N_i(u_{ik}) - N_i(u_i)\|_\infty \to 0 \quad \text{as } k \to \infty.$$

Hence

$$\|N(u_k) - N(u)\|_{X_n} \to 0 \quad \text{as } k \to \infty.$$

Step 2. $N(B_{R_n})$ *is uniformly bounded.*
For any $u := (u_1,u_2) \in X_n$, and each $t \in I_n$ we have

$$\begin{aligned}
|(N_1(u_1)(t)| &\le |u_{01}| + \int_0^t \frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}|f_1(s,u_2(s))|d_qs \\
&\le |u_{01}| + \int_0^t \frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha)}p_{1n}(s)d_qs \\
&\le |u_{01}| + p_{1n}^* \int_0^t \frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)}d_qs \\
&\le |u_{01}| + \frac{n^\alpha p_{1n}^*}{\Gamma_q(1+\alpha_1)}.
\end{aligned}$$

Also,

$$\begin{aligned}
|(N_2(u_2)(t)| &\le |u_{02}| + \int_0^t \frac{(t-qs)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)}|f_2(s,u_1(s))|d_qs \\
&\le |u_{02}| + \frac{n^\alpha p_{2n}^*}{\Gamma_q(1+\alpha_2)}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\|(Nu)(t)\| &= \|(N_1u_1)(t)\| + \|(N_2u_2)(t)\| \\
&\le \|u_{01}\| + \frac{n^{\alpha_1} p_{1n}^*}{\Gamma_q(1+\alpha_1)} + \|u_{02}\| + \frac{n^{\alpha_2} p_{2n}^*}{\Gamma_q(1+\alpha_2)} \\
&= R_n.
\end{aligned}$$

Hence

$$\|N(u)\|_{X_n} \leq R_n. \tag{7}$$

This proves that $N(B_{R_n}) \subset B_{R_n}$.

Step 3. *$N(B_{R_n})$ is equicontinuous.*
Let $t_1, t_2 \in I_n$, $t_1 < t_2$ and let $u := (u_1, u_2) \in B_{R_n}$. Thus we have

$$\begin{aligned}
&|(N_1u_1)(t_2) - (N_1u_1)(t_1)| \\
&\leq \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha_1-1)}|}{\Gamma_q(\alpha_1)} |f+1(s,u_2(s))| d_qs \\
&+ \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha_1-1)}|}{\Gamma_q(\alpha_1)} |f_1(s,u_2(s))| d_qs \\
&\leq p_{1n}^* \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha_1-1)} - (t_1-qs)^{(\alpha_1-1)}|}{\Gamma_q(\alpha_1)} d_qs \\
&+ p_{1n}^* \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha_1-1)}|}{\Gamma_q(\alpha_1)} d_qs.
\end{aligned}$$

Also, we get

$$\begin{aligned}
&|(N_2u_2)(t_2) - (N_2u_2)(t_1)| \\
&\leq p_{2n}^* \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha_2-1)} - (t_1-qs)^{(\alpha_2-1)}|}{\Gamma_q(\alpha_2)} d_qs \\
&+ p_{2n}^* \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha_1-1)}|}{\Gamma_q(\alpha_2)} d_qs.
\end{aligned}$$

Thus,

$$\begin{aligned}
|(Nu)(t_2) - (Nu)(t_1)| &= |(N_1u_1)(t_2) - (N_1u_1)(t_1)| + |(N_2u_2)(t_2) - (N_2u_2)(t_1)| \\
&\leq p_{1n}^* \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha_1-1)} - (t_1-qs)^{(\alpha_1-1)}|}{\Gamma_q(\alpha_1)} d_qs \\
&+ p_{2n}^* \int_0^{t_1} \frac{|(t_2-qs)^{(\alpha_2-1)} - (t_1-qs)^{(\alpha_2-1)}|}{\Gamma_q(\alpha_2)} d_qs \\
&+ p_{1n}^* \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha_1-1)}|}{\Gamma_q(\alpha_1)} d_qs + p_{2n}^* \int_{t_1}^{t_2} \frac{|(t_2-qs)^{(\alpha_1-1)}|}{\Gamma_q(\alpha_2)} d_qs.
\end{aligned}$$

As $t_1 \longrightarrow t_2$, the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3, together with the Arzelá-Ascoli theorem, we can conclude that N is continuous and compact. From an application of Theorem 2.10, we deduce that N has a fixed point (u_1, u_2) which is a solution of the problem (3).

Part 2. The diagonalization process.
Now, we use the following diagonalization process. For $k \in \mathbb{N}$ and $i = 1, 2$, we let $w_k = (w_{1k}, w_{2k})$, such that

$$\begin{cases} w_{ik}(t) = u_{in_k}(t); \ t \in [0, n_k], \\ w_{ik}(t) = u_{in_k}(n_k); \ t \in [n_k, \infty) \end{cases} ; \ i = 1, 2.$$

Here $\{n_k\}_{k\in\mathbb{N}^*}$ is a sequence of numbers satisfying

$$0 < n_1 < n_2 < \cdots n_k < \cdots \uparrow \infty.$$

Let $S = \{w_k\}_{k=1}^{\infty}$. Notice that

$$|w_{n_k}(t)| = \sum_{i=1}^{2} |w_{in_k}(t)| \leq R_n,\ \textit{for}\ t \in [0, n_1],\ k \in \mathbb{N}.$$

Also, if $k \in \mathbb{N}$ and $t \in [0, n_1]$, we have

$$w_{1n_k}(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)} f(s, w_{2n_k}(s)) d_q s,$$

and

$$w_{2n_k}(t) = u_0 + \int_0^t \frac{(t-qs)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} f(s, w_{1n_k}(s)) d_q s.$$

Thus, for $k \in \mathbb{N}$, and $t, x \in [0, n_1]$, we have

$$|w_{1n_k}(t) - w_{1n_k}(x)| \leq \int_0^{n_1} \frac{|(t-qs)^{(\alpha_1-1)} - (x-qs)^{(\alpha_1-1)}|}{\Gamma_q(\alpha_1)} |f_1(s, w_{2n_k}(s))| d_q s,$$

and

$$|w_{2n_k}(t) - w_{2n_k}(x)| \leq \int_0^{n_1} \frac{|(t-qs)^{(\alpha_2-1)} - (x-qs)^{(\alpha_2-1)}|}{\Gamma_q(\alpha_2)} |f_2(s, w_{1n_k}(s))| d_q s.$$

Hence

$$|w_{1n_k}(t) - w_{1n_k}(x)| \leq p^*_{1n_1} \int_0^{n_1} \frac{|(t-qs)^{(\alpha_1-1)} - (x-qs)^{(\alpha_1-1)}|}{\Gamma_q(\alpha_1)} d_q s,$$

and

$$|w_{2n_k}(t) - w_{2n_k}(x)| \leq p^*_{2n-1} \int_0^{n_1} \frac{|(t-qs)^{(\alpha_2-1)} - (x-qs)^{(\alpha_2-1)}|}{\Gamma_q(\alpha_2)} d_q s.$$

The Arzelà-Ascoli theorem guarantees that there is a subsequence $\mathbb{N}_1^*$ of $\mathbb{N}$ and a coupled function $z_1 := (z_{11}, z_{21}) \in X_{n_1}$ with $u_{n_k} \to z_1$ as $k \to \infty$ in X_{n_1} through $\mathbb{N}_1^*$. Let $\mathbb{N}_1 = \mathbb{N}_1^* \backslash \{1\}$.

Notice that

$$|w_{n_k}(t)| \leq R_n,\ \textit{for}\ t \in [0, n_2],\ k \in \mathbb{N}.$$

Also, if $k \in \mathbb{N}$, and $t, x \in [0, n_2]$, we have

$$|w_{1n_k}(t) - w_{1n_k}(x)| \leq p^*_{1n_2} \int_0^{n_2} \frac{|(t-qs)^{(\alpha_1-1)} - (x-qs)^{(\alpha-1-1)}|}{\Gamma_q(\alpha_1)} d_q s,$$

and

$$|w_{2n_k}(t) - w_{2n_k}(x)| \leq p^*_{2n_2} \int_0^{n_2} \frac{|(t-qs)^{(\alpha_2-1)} - (x-qs)^{(\alpha-2-1)}|}{\Gamma_q(\alpha_2)} d_q s.$$

The Arzelà-Ascoli theorem guarantees that there is a subsequence $\mathbb{N}_2^*$ of $\mathbb{N}_1$ and a function $z_2 := (z_{12}, z_{22}) \in X_{n_2}$ with $u_{n_k} \to z_2$ as $k \to \infty$ in X_{n_2} through $\mathbb{N}_2^*$. Note that $z_1 = z_2$ on $[0, n_1]$ since $\mathbb{N}_2^* \subset \mathbb{N}_1$. Let $\mathbb{N}_2 = \mathbb{N}_2^* \backslash \{2\}$. Proceed inductively to obtain for $m = 3, 4, \ldots$ a subsequence $\mathbb{N}_m^*$ of $\mathbb{N}_{m-1}$ and a function $z_m := (z_{1m}, z_{2m}) \in X_{n_m}$ with $u_{n_k} \to z_m$ as $k \to \infty$ in X_{n_m} through $\mathbb{N}_m^*$. Let $\mathbb{N}_m = \mathbb{N}_m^* \backslash \{m\}$.

Define a function u as follows. Fix $t \in (0,\infty)$ and let $m \in \mathbb{N}$ with $t \leq n_m$. Then define $u(t) := z_m(t)$. Thus $u = (u_1, u_2) \in X_\infty = C[0,\infty) \times C[0,\infty),\ u(0) = (u_{01}, u_{02})$ and $|u(t)| \leq R_n$, for $t \in [0,\infty)$.

Again fix $t \in (0,\infty)$ and let $m \in \mathbb{N}$ with $t \leq n_m$. Then for $n \in \mathbb{N}_m$ we have

$$u_{1n_k}(t) = u_{01} + \int_0^{n_m} \frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)} f_1(s, w_{2n_k}(s)) d_q s,$$

and

$$u_{2n_k}(t) = u_{02} + \int_0^{n_m} \frac{(t-qs)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} f_2(s, w_{1n_k}(s)) d_q s.$$

Let $n_k \to \infty$ through $\mathbb{N}_m$ to obtain

$$z_{1m}(t) = u_{01} + \int_0^{n_m} \frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)} f_1(s, z_{2m}(s)) d_q s,$$

and

$$z_{2m}(t) = u_{02} + \int_0^{n_m} \frac{(t-qs)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} f_2(s, z_{1m}(s)) d_q s.$$

Thus for $t \in [0, n_m]$,

$$({}^C D_q^{\alpha_1} u_1)(t) = f_1(t, u_2(t)),$$

and

$$({}^C D_q^{\alpha_2} u_2)(t) = f_2(t, u_1(t)).$$

Hence, the constructed function u is a solution of the coupled system (1). This completes the proof. □

4. Oscillation and nonoscillation results

Definition 4.1 [11] A solution u of problem (2) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise u is called nonoscillatory.

Definition 4.2 [11] A solution $u = (u_1, u_2)$ of the coupled system (1) is said to be strongly (weakly) oscillatory if each (at least one) of its components is oscillatory. Otherwise, it is said to be strongly (weakly) nonoscillatory if each (at least one) of its nontrivial components is nonoscillatory.

The following hypothesis will be used in the sequel.

(H_3) There exist continuous functions $q_{in} : I_n \to \mathbb{R}_+,\ n = 1,2$, and continuous, bounded and increasing real functions $g_i;\ i = 1,2$, such that, for a.e. $t \in I_n$, and each $u_i \in \mathbb{R}$, and $v \in \mathbb{R}^* := \mathbb{R}\backslash\{0\}$,

$$|f_1(t,u_2)| = q_{1n}(t) g_1(u_2),\ |f_2(t,u_1)| = q_{2n}(t) g_2(u_1),\ \textit{and}\ v g_i(v) > 0.$$

Remark 4.3 We can see that (H_3) implies (H_2) with $p_{in}(t) = M q_{in}(t)$, where

$$M = \sup_{v \in \mathbb{R}} |g_i(v)|.$$

The following theorem gives sufficient conditions to ensure the nonoscillation of solutions of the coupled system (1).

Theorem 4.4 Assume that (H_1) and (H_3) hold. If $u = (u_1, u_2)$ is a weakly nonoscillatory solution of (1), such that u_1 and u_2 have the same sign, then the first component u_1 is also nonoscillatory.

Proof. Assume to the contrary that u_1 is oscillatory but u_2 is eventually positive. Then in view of (H_3), there exists $n_m > 0$, such that $q_{1n}(t)g_1(u_2(t)) \geq 0$ for t larger than n_m. Thus, for all $t > n_m$,

$$u_1(t) = u_1(n_m) + \int_{n_m}^{n_m+1} \frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)} q_{1n}(s)g_1(u_2(s))d_qs > 0.$$

Hence, $u_1(t) > 0$ for all large t. This is a contradiction.

Analogously, the case when u_2 is an eventually negative is proved similarly. Indeed, if u_2 is an eventually negative, then from (H_3), there exists $n_m > 0$, such that for all $t > n_m$,

$$u_1(t) = u_1(n_m) + \int_{n_m}^{n_m+1} \frac{(t-qs)^{(\alpha_1-1)}}{\Gamma_q(\alpha_1)} q_{1n}(s)g_1(u_2(s))d_qs < 0.$$

Thus $u_1(t) < 0$ for all large t. This is again a contradiction. This means that u_1 is nonoscillatory. □

Corollary 4.5 Assume that (H_1) and (H_3) hold. If $u = (u_1, u_2)$ is a weakly nonoscillatory solution of (1), such that u_1 and u_2 have the same sign, then the second component u_2 is also nonoscillatory.

Corollary 4.6 Assume that (H_1) and (H_3) hold. If $u = (u_1, u_2)$ is a weakly nonoscillatory solution of (1), such that u_1 and u_2 have the same sign, then u is a strongly nonoscillatory solution of (1).

The following theorem presents the oscillatory result for the coupled system (1).

Theorem 4.7 Assume that (H_1) and (H_3) hold. If $u = (u_1, u_2)$ is a weakly oscillatory solution of (1), such that u_1 and u_2 have the same sign, then the first component u_1 is also oscillatory.

Proof. Assume to the contrary that u_1 is nonoscillatory. If u_1 is an eventually positive solution, then in view of (H_3), there exists $n_m > 0$, such that $q_{2n}(t)g_2(u_1(t)) \geq 0$ for t larger than n_m. Thus, for all $t > n_m$,

$$u_2(t) = u_2(n_m) + \int_{n_m}^{n_m+1} \frac{(t-qs)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} q_{2n}(s)g_2(u_1(s))d_qs > 0.$$

Hence, $u_2(t) > 0$ for all large t. This is a contradiction since u_2 is an oscillatory solution.

Analogously, the case when u_1 is an eventually negative is proved similarly. Indeed, if u_1 is an eventually negative, then from (H_3), there exists $n_m > 0$, such that for all $t > n_m$,

$$u_2(t) = u_2(n_m) + \int_{n_m}^{n_m+1} \frac{(t-qs)^{(\alpha_2-1)}}{\Gamma_q(\alpha_2)} q_{2n}(s) g_2(u_1(s)) d_q s < 0.$$

Thus $u_2(t) < 0$ for all large t. This is again a contradiction since u_2 is an oscillatory solution. This means that u_1 is oscillatory. □

Corollary 4.8 Assume that (H_1) and (H_3) hold. If $u = (u_1, u_2)$ is a weakly oscillatory solution of (1), such that u_1 and u_2 have the same sign, then u_2 is an oscillatory solution.

[**Corollary 4.9** Assume that (H_1) and (H_3) hold. If $u = (u_1, u_2)$ is a weakly oscillatory solution of (1), such that u_1 and u_2 have the same sign, then u is an oscillatory solution of (1).

5. Examples

Example 1. Consider the following problem of fractional $\frac{1}{4}$−difference coupled system

$$\begin{cases} ({}^cD_{\frac{1}{4}}^{\frac{1}{2}} u)(t) = f(t, v(t)) \\ ({}^cD_{\frac{1}{4}}^{\frac{1}{2}} v)(t) = g(t, u(t)) \\ u(0) = v(0) = 0; \ u \text{ and } v \text{ are bounded on } \mathbb{R}_+, \end{cases} ; \ t \in \mathbb{R}_+, \tag{8}$$

where

$$f(t, v) = \frac{c_n t^{\frac{5}{4}} \sin t}{64(1+\sqrt{t})(1+|u|)}; \ v \in \mathbb{R}, \ t \in [0, n]; \ n \in \mathbb{N}^*$$

and

$$g(t, u) = \frac{c_n t^{\frac{5}{4}} \sin t}{64(1+\sqrt{t})(1+|v|)}; \ u \in \mathbb{R}, \ t \in [0, n]; \ n \in \mathbb{N}^*$$

with

$$c_n = 16 n^{-\frac{7}{4}} \Gamma_{\frac{1}{4}}\left(\frac{3}{2}\right); \ n \in \mathbb{N}^*.$$

Since

$$|f(t, u)| \leq \frac{t^{\frac{5}{4}} c_n}{64}; \ t \in \mathbb{R}_+, \ n \in \mathbb{N}^*,$$

and

$$|g(t, v)| \leq \frac{t^{\frac{5}{4}} c_n}{64}; \ t \in \mathbb{R}_+, \ n \in \mathbb{N}^*,$$

then the hypothesis (H_2) is satisfied with

$$p_{in}(t) = \frac{t^{\frac{5}{4}} c_n}{64}; \ i = 1, 2, \ t \in \mathbb{R}_+, \ n \in \mathbb{N}^*.$$

So; for any $n \in \mathbb{N}^*$, we have

$$p_{in}^* = \frac{n^{\frac{5}{4}} c_n}{64}.$$

A simple computation shows that all conditions of Theorem 3.2 are satisfied. Hence, the problem (8) has at least one bounded solution defined on $\mathbb{R}_+$.

Example 2. Consider now the following problem of fractional $\frac{1}{4}-$difference coupled system

$$\begin{cases} ({}^cD^{\frac{1}{2}}_{\frac{1}{4}}u)(t) = w_1(t, v(t)) \\ ({}^cD^{\frac{1}{2}}_{\frac{1}{4}}v)(t) = w_2(t, u(t)) \\ u(0) = v(0) = 0; \ u \text{ and } v \text{ are bounded on } \mathbb{R}_+, \end{cases} ; \ t \in \mathbb{R}_+, \tag{9}$$

where

$$w_1(t, v) = \frac{2ut^{\frac{5}{4}} \sin t}{64(1 + \sqrt{t})(1 + |u|)}; \ u, v \in \mathbb{R},$$

and

$$w_2(t, u) = \frac{2vt^{\frac{5}{4}} \sin t}{64(1 + \sqrt{t})(1 + |v|)}; \ u, v \in \mathbb{R},$$

for each $t \in [0, n]; \ n \in \mathbb{N}^*$. Since

$$|w_1(t, u)| \leq \frac{2vt^{\frac{5}{4}}}{64(1 + |v|)}; \ t \in \mathbb{R}_+, \ n \in \mathbb{N}^*,$$

and

$$|w_2(t, v)| \leq \frac{2ut^{\frac{5}{4}}}{64(1 + |u|)}; \ t \in \mathbb{R}_+, \ n \in \mathbb{N}^*,$$

then the hypothesis (H_3) is satisfied with $g_i(x) = \frac{2x}{1+|x|}; \ x \in \mathbb{R}$, and

$$q_{in}(t) = \frac{t^{\frac{5}{4}}}{64}; \ i = 1, 2, \ t \in \mathbb{R}_+, \ n \in \mathbb{N}^*.$$

So;

$$M = \sup_{x \in \mathbb{R}} |g_i(x)| = 2.$$

A simple computation shows that all conditions of Theorem 4.4 are satisfied. If $w = (w_1, w_2)$ is a weakly nonoscillatory solution of (9), such that w_1 and w_2 have the same sign, then the first component w_1 is also nonoscillatory.

Also, from Theorem 4.7, if $u = (u_1, u_2)$ is a weakly oscillatory solution of (9), such that u_1 and u_2 have the same sign, then the first component u_1 is also oscillatory.

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