Journal of Fractional Calculus and Applications
Vol. 12(1) Jan. 2021, pp. 133-142.
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

A NOTE ON INFINITE PRODUCT OF BICOMPLEX NUMBERS

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Abstract. A necessary and sufficient condition for the General principle of convergence of infinite product of bicomplex number is derived in this paper.We also prove here some of its consequences.Several examples have been provided to ensure the validity of the theorems proved.

1. Introduction

The theory of bicomplex numbers is a matter of active research for quite a long time since seminal work of ([9] and [1]) in search of special algebra. the algebra of bicomplex numbers are widely use in the literature as it becomes viable commutative alternative [10] to the non skew field of quaternions introduced by Hamilton [4] (both are four dimensional and generalization of complex numbers).

2. Preliminaries

2.1. **The Bicomplex Numbers** [7]**.** A bicomplex number is defined as

$$\begin{aligned} z &= x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 \\ &= (x_1 + i_1 x_2) + i_2 (x_3 + i_1 x_4) \\ &= z_1 + i_2 z_2 \end{aligned}$$

where $x_i, i = 1, 2, 3, 4$ are all real numbers with $i_1^2 = i_2^2 = -1, i_1 i_2 = i_2 i_1, (i_1 i_2)^2 = 1$, and z_1, z_2 are complex numbers.

The set of all bicomplex numbers , complex numbers and real numbers are denoted by $\mathbb{C}_2, \mathbb{C}_1$ and $\mathbb{C}_0$ respectively.

2.2. **Algebra of Bicomplex Numbers** [7]**.** Addition is the operation on $\mathbb{C}_2$ defined by the function $\oplus : \mathbb{C}_2 \times \mathbb{C}_2 \to \mathbb{C}_2$,

$$(x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,\ y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4) = (x_1 + y_1) + i_1 (x_2 + y_2) + i_2 (x_3 + y_3) + i_1 i_2 (x_4 + y_4).$$

Scalar multiplication is the operation on $\mathbb{C}_2$ defined by the function $\odot : \mathbb{C}_0 \times \mathbb{C}_2 \to \mathbb{C}_2$,

$$(a, x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4) = (a x_1 + i_1 a x_2 + i_2 a x_3 + i_1 i_2 a x_4).$$

2010 *Mathematics Subject Classification.* 30D30, 30G35.
Key words and phrases. Bicomplex number, Bicomplex valued function, Analytic function, Idempotent Representation, Infinite product,General principle of convergence.
Submitted April 18, 2020.

The system $(\mathbb{C}_2, \oplus, \odot)$ is a linear space.

Here the norm is defined as

$$\begin{aligned} \|\ \| &: \mathbb{C}_2 \to \mathbb{R}_{\geq 0}, \\ \|x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4\| &= \left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{\frac{1}{2}}. \end{aligned}$$

So the system $(\mathbb{C}_2, \oplus, \odot, \|\ \|)$ is a normed linear space.

The space $\mathbb{C}_0^4$ with the Euclidean norm is known to be complete space. As $\mathbb{C}_2$ is embedded in $\mathbb{C}_0^4$ so that $x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4$ corresponds to (x_1, x_2, x_3, x_4) and for this reason the norm on $\mathbb{C}_2$ is the same as the norm of $\mathbb{C}_0^4$, then the normed linear space $(\mathbb{C}_2, \oplus, \odot, \|\ \|)$ is a complete Space. Hence $(\mathbb{C}_2, \oplus, \odot, \|\ \|)$ is a Banach Space.

The product on $\mathbb{C}_2$ is defined as

$$\otimes : \mathbb{C}_2 \times \mathbb{C}_2 \to \mathbb{C}_2,$$

$$(x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,\ y_1 + i_1 y_2 + i_2 y_3 + i_1 i_2 y_4) = \begin{pmatrix} x_1 y_1 - x_2 y_2 - x_3 y_3 + x_4 y_4 \\ + i_1 (x_1 y_2 + x_2 y_1 - x_3 y_4 - x_4 y_3) \\ + i_2 (x_1 y_3 - x_2 y_4 + x_3 y_1 - x_4 y_2) \\ + i_1 i_2 (x_1 y_4 + x_2 y_3 + x_3 y_2 + x_4 y_1) \end{pmatrix}.$$

Since,

$$\begin{aligned} (i)\ \|z(z_1 + i_2 z_2)\| &= |z| \cdot \|z_1 + i_2 z_2\|. \\ (ii)\ \|(z_1 + i_2 z_2)(w_1 + i_2 w_2)\| &\leq \sqrt[2]{2}\, \|z_1 + i_2 z_2\| \cdot \|w_1 + i_2 w_2\|. \end{aligned}$$

where $z \in \mathbb{C}_1, (z_1 + i_2 z_2)$ and $(w_1 + i_2 w_2) \in \mathbb{C}_2$.

So, $(\mathbb{C}_2, \oplus, \odot, \|\ \|, \otimes)$ is a Banach Algebra.

2.3. **Idempotent Representation of Bicomplex Numbers** [7]**.** There are four idempotent elements in $\mathbb{C}_2$. they are

$$0, 1, \frac{1 + i_1 i_2}{2}, \frac{1 - i_1 i_2}{2}.$$

We now denote two non trivial idempotent elements by

$$e_1 = \frac{1 + i_1 i_2}{2} \quad \text{and} \quad e_2 = \frac{1 - i_1 i_2}{2} \quad \text{in } \mathbb{C}_2.$$

where

$$e_1^2 = e_1, e_2^2 = e_2, e_1 e_2 = e_2 e_1 = 0, e_1 + e_2 = 1.$$

So, e_1 and e_2 are alternatively called orthogonal idempotents.

Every element $\xi : (z_1 + i_2 z_2) \in \mathbb{C}_2$ has the following unique representation

$$\begin{aligned} \xi &= (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2 \\ &= \xi_1 e_1 + \xi_2 e_2, \text{ where } \xi_1, \xi_2 \text{ are complex numbers.} \end{aligned}$$

This is known as idempotent representation of an element $\xi : (z_1 + i_2 z_2) \in \mathbb{C}_2$.

An element $\xi : (z_1 + i_2 z_2) \in \mathbb{C}_2$ is non-singular iff $\left|z_1^2 + z_2^2\right| \neq 0$ and it is singular iff $\left|z_1^2 + z_2^2\right| = 0$. The set of all singular elements is denoted by θ_2.

If $f(z)$ be a bicomplex valued function, then f can be represented as

$$f(z) = f_1(z_1) e_1 + f_2(z_2) e_2 \quad \text{where} \quad f_1(z_1), f_2(z_2) \in \mathbb{C}_1.$$

where $f_1(z_1), f_2(z_2)$ are both functions in $\mathbb{C}_1$.This type of decomposition is known as Ringleb decomposition in

2.4. **Bicomplex Exponential Function** [7]**.** If w be any bicomplex number then the sequence $\left(1+\frac{w}{n}\right)^n$ converges to a bicomplex number denoted by $\exp w$ or e^w, called the bicomplex exponential function.

$$i.e., e^w = \lim_{n\to\infty}\left(1+\frac{w}{n}\right)^n.$$

If $w = (z_1 + i_2 z_2)$, then we get the bicomplex version of Euler's formula

$$e^w = e^z\left(\cos z_2 + i_2 \sin z_2\right) = e^{|w|_{i_1}}\left(\cos \arg\ _{i_1} w + \sin \arg\ _{i_1} w\right)$$

where $e^w \notin \theta_2$.

2.5. **Bicomplex Logarithmic Function** [7]**.** Let ξ be a bicomplex number and w be another bicomplex number such that $w \notin \theta_2$. If $e^\xi = w$, then ξ is called logarithm of w.

Let $w = (z_1 + i_2 z_2) \notin \theta_2$. i.e., if $(z_1 - i_1 z_2) \neq 0$ and $(z_1 + i_1 z_2) \neq 0$ then

$$\begin{aligned} \log(z_1 + i_2 z_2) &= \{\log|z_1 - i_1 z_2| + i_1 \arg(z_1 - i_1 z_2) + 2n_1\pi i_1\} e_1 \\ &\quad + \{\log|z_1 + i_1 z_2| + i_1 \arg(z_1 + i_1 z_2) + 2n_2\pi i_1\} e_2. \end{aligned}$$

where $n_1, n_2 = 0, \pm1, \pm2, ...$

Also we can write,

$$\log(z_1 + i_2 z_2) = \log(z_1 - i_1 z_2) e_1 + \log(z_1 + i_1 z_2) e_2.$$

2.6. **Bicomplex Holomorphic Function** [7]**.** We start with a bicomplex valued function

$$f : \Omega \subset \mathbb{C}_2 \to \mathbb{C}_2.$$

The derivative of f at a point $\omega_0 \in \Omega$ is defined by

$$f'(\omega) = \lim_{h\to 0}\frac{f(\omega_0 + h) - f(\omega_0)}{h}$$

provided the limit exists and the domain is so chosen that

$$h = h_0 + i_1 h_1 + i_2 h_2 + i_1 i_2 h_3$$

is invertible. It is easy to prove that h is not invertible only for $h_0 = -h_3, h_1 = h_2$ or $h_0 = h_3, h_1 = -h_2.i.e.h \notin \theta_2.$

If the bicomplex derivative of f exists at each point of its domain then in similar to complex functiond, f will be a bicomplex holomorphic function in Ω. Indeed if f can be expressed as

$$\begin{aligned} f(\omega) &= g_1(z_1, z_2) + i_2 g_2(z_1, z_2) \\ \omega &= z_1 + i_2 z_2 \in \Omega \end{aligned}$$

then f will be holomorphic if and only if g_1, g_2 are both complex holomorphic in z_1, z_2 and

$$\frac{\partial g_1}{\partial z_1} = \frac{\partial g_2}{\partial z_2}, \frac{\partial g_1}{\partial z_2} = -\frac{\partial g_2}{\partial z_1}.$$

Moreover,

$$f'(\omega) = \frac{\partial g_1}{\partial z_2} + i_2\frac{\partial g_2}{\partial z_1}.$$

2.7. **Bicomplex Entire Function** [7]**.** A function f is said to be a bicomplex entire function if f is analytic in the whole bicomplex plane $\mathbb{C}_2$.

2.8. **Bicomplex Meromorphic Function** [7]**.** A function f is said to be bicomplex meromorphic function in an open set $\Omega \subseteq T$ if f is a quotient $\frac{g}{h}$ of two functions which are bicomplex holomorphic in Ω where $h \notin \theta_2$.

If $f(z)$ be a bicomplex meromorphic function, then f can be represented as

$$f(z) = f_1(z_1)\, e_1 + f_2(z_2)\, e_2 \quad \text{where} \quad f_1(z_1), f_2(z_2) \in \mathbb{C}_1 \ .$$

where $f_1(z_1), f_2(z_2)$ are both meromorphic functions in $\mathbb{C}_1$.

Now, we introduce the the infinite series and infinite product of bicomplex numbers.

2.9. **Infinite Series of Bicomplex Numbers.** $\sum_{k=0}^{\infty} \xi_k \quad \forall\ k,\ \xi_k \in \mathbb{C}_2$ is called infinite series in $\mathbb{C}_2$. Define the sequence $S : \mathbb{N} \to \mathbb{C}_2$ by

$$S_n = \sum_{k=0}^{n} \xi_k \qquad \forall\ n \in \mathbb{N}.$$

Then the infinite sum converges iff $\lim_{n\to\infty} S_n$ exists and diverges if the limit does not exists.

If $\lim_{n\to\infty} S_n = \xi^*$ then ξ^* is called sum of the series and we write $\sum_{k=0}^{\infty} \xi_k = \xi^*$.

$$(x_1 + i_1x_2 + i_2x_3 + i_1i_2x_4,\ y_1 + i_1y_2 + i_2y_3 + i_1i_2y_4) = \begin{pmatrix} x_1y_1 - x_2y_2 - x_3y_3 + x_4y_4 \\ +i_1(x_1y_2 + x_2y_1 - x_3y_4 - x_4y_3) \\ +i_2(x_1y_3 - x_2y_4 + x_3y_1 - x_4y_2) \\ +i_1i_2(x_1y_4 + x_2y_3 + x_3y_2 + x_4y_1) \end{pmatrix}.$$

Since,

$$\begin{aligned} (i)\ \|z(z_1 + i_2z_2)\| &= |z| \cdot \|z_1 + i_2z_2\| \\ (ii)\ \|(z_1 + i_2z_2)(w_1 + i_2w_2)\| &\leq \sqrt[2]{2}\, \|z_1 + i_2z_2\| \cdot \|w_1 + i_2w_2\| \end{aligned}$$

where $z \in \mathbb{C}_1, (z_1 + i_2z_2)$ and $(w_1 + i_2w_2) \in \mathbb{C}_2$.

So, $(\mathbb{C}_2, \oplus, \odot, ||\ \ ||, \otimes)$ is a Banach Algebra.

2.10. **Infinite Product of Bicomplex Numbers.** If we multiply an infinite number of factors according to some definite law then the product so obtained is called an infinite product. Let $\{u_k\}$be the sequence of bicomplex numbers. Thus the product $u_1u_2u_3$....of infinite number of factors is denoted symbolically as $\prod_{k=1}^{\infty} u_k$ and in case the factors be finite we write it as

$$P_n = \prod_{k=1}^{n} u_k$$

It is also clear from above that

$$\frac{P_n}{P_{n-1}} = u_n \quad \text{and} \quad \frac{P_{n+p}}{P_n} = u_{n+1}u_{n+2}........u_{n+p}.$$

For the sake of convenience we will choose the factors to be of the form $(1 + u_k)$,

$$\prod_{k=1}^{\infty} (1 + u_k) = (1 + u_1)(1 + u_2)(1 + u_3)......$$

The product of n factors is written as

$$\prod_{k=1}^{n}(1+u_k)=(1+u_1)(1+u_2)(1+u_3)\,.......\,(1+u_n)\,.$$

In this paper we wish to find out infinite product theorem and some example under the treatment of bicomplex analysis.We do not explain the standerd definitions and notatios of the theories of bicomplex valued entire function as those are available in ([7],[2],.[5]and[6])

3. Theorems.

In this section we present the main results of our paper.

3.1. **General Principle of Convergence of Infinite Product.**

Theorem 3.1. *The necessary and sufficient condition for the convergence of infinite product P is that corresponding to any $\varepsilon>0,\exists \quad m\in\mathbb{N}$ such that*

$$\left\|\frac{P_{n+t}}{P_n}-1\right\|<\varepsilon \;\;\forall\; n>m,\quad t\;\in\;\mathbb{N}.$$

where $P_n=\prod_{k=1}^{n}(u_k+1)$ *and* $P_{n+t}=\prod_{k=1}^{n+t}(1+u_k)\,,\{u_k\}$ *is sequence of bicomplex numbers.*

Proof. Let $\{P_n\}$ be convergent to a non zero finite limit $l\in\mathbb{C}_2$.

Since $\{P_n\}$ is a sequence of bicomplex numbers

$$\begin{aligned}P_n=\prod_{k=1}^{n}(u_k+1)&=\left(\prod_{k=1}^{n}\left(1+u_k^{'}\right)\right)e_1+\left(\prod_{k=1}^{n}\left(1+u_k^{''}\right)\right)e_2\\&=P_n^{'}e_1+P_n^{''}e_2\end{aligned}$$

$$\begin{aligned}\text{where } u_k&=u_k^{'}e_1+u_k^{''}e_2\\P_n^{'}&=\prod_{k=1}^{n}\left(1+u_k^{'}\right)\\P_n^{''}&=\prod_{k=1}^{n}\left(1+u_k^{''}\right).\end{aligned}$$

Since $\{P_n\}$ converges to $l\in\mathbb{C}_2$

Since, $l=l_1e_1+l_2e_2$ where $l_1,l_2\in\mathbb{C}$.

Hence $\left\{P_n^{'}\right\}$ converges to l_1 and $\left\{P_n^{''}\right\}$ converges to l_2. [2]Also by general principle of convergence of infinite product in complex numbers we can say that $\left\{P_{n+t}^{'}\right\}$ converges to l_1 and $\left\{P_{n+t}^{''}\right\}$ converges to l_2.

Now,

$$\begin{aligned}\lim_{n\to\infty}\frac{P_{n+t}}{P_n}&=\lim_{n\to\infty}\frac{P_{n+t}^{'}e_1+P_{n+t}^{''}e_2}{P_n^{'}e_1+P_n^{''}e_2}\\&=\frac{l_1e_1+l_2e_2}{l_1e_1+l_2e_2}=1.\end{aligned}$$

Since for $\varepsilon > 0, \exists \quad m \in \mathbb{N}$ such that

$$\left\| \frac{P_{n+t}}{P_n} - 1 \right\| < \varepsilon \ \ \forall \ n \geq m.$$

Conversely let,

$$Let \ \left\| \frac{P_{n+t}}{P_n} - 1 \right\| < \varepsilon \ \forall \ n \geq m.$$

$$i.e., \lim_{n \to \infty} \frac{P_{n+t}}{P_n} = 1$$

Now the idempotent representation shows that

$$\begin{aligned} \frac{P_{n+t}}{P_n} &= \frac{P'_{n+t}}{P'_n} e_1 + \frac{P''_{n+t}}{P''_n} e_2 \\ i.e. \lim_{n \to \infty} \left(\frac{P'_{n+t}}{P'_n} e_1 + \frac{P''_{n+t}}{P''_n} e_2 \right) &= 1 = e_1 + e_2. \end{aligned}$$

Since we can write,

$$\lim_{n \to \infty} \frac{P'_{n+t}}{P'_n} = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{P''_{n+t}}{P''_n} = 1.$$

$Therefore$, $P'_{n+t}, P''_{n+t}, P'_n$ and P''_n satisfy general principle of convergence of infinite product in $\mathbb{C}_1$[2].

Hence $P'_{n+t}, P''_{n+t}, P'_n$ and P''_n are all convergent in $\mathbb{C}_1$.

Now, $\{P_n\}$ is convergent in $\mathbb{C}_2$.

This completes the proof of the theorem. □

Remark 3.1 The following corollaries are evident from Theorem 3.1 as we see below.

Corollary 3.1 The infinite product $\prod (1 + a_n)$ where $a_n \in \mathbb{C}_2 \ \forall \ n$, is absolutely convergent iff the series $\sum \log (1 + a_n)$ is absolutely convergent i.e., iff $\sum a_n$ is absolutely convergent where $(1 + a_n) \notin \theta_2$ otherwise $\log (1 + a_n)$ is not defined.

Proof. It follows from the theorem 3.1 and the theorem which states that if a_n is real and non negative then the series $\sum a_n$ and the product $\prod (1 + a_n)$ converges and diverges together. □

Corollary 3.2 If $\sum\limits_{n=1}^{\infty} \|a_n\|$ $(a_n \in \mathbb{C}_2$ and $(1 + a_n) \notin \theta_2)$ converges, then infinite product $\prod\limits_{n=1}^{\infty} (1 + a_n)$ converges.

Proof. It follows from Corollary 3.1 and Theorem 3.1. □

Remark 3.2 The following example ensures the validity of Corollary 3.1.

Example 3.1 The product

$$\prod_{n=1}^{\infty} \left(1 + \frac{z}{n\pi}\right) e^{-\frac{z}{n\pi}} \quad \text{is absolutely convergent series of bicomplex numbers.}$$

Let $\prod\limits_{n=1}^{\infty}(1+a_n(z))$ denote the given product where $z \in \mathbb{C}_2$.Then

$$\begin{aligned}\prod_{n=1}^{\infty}(1+a_n(z)) &= \prod_{n=1}^{\infty}\left(1+\frac{z}{n\pi}\right)e^{-\frac{z}{n\pi}} \\ &= \prod_{n=1}^{\infty}\left(1+\frac{z}{n\pi}\right)\left(1-\frac{z}{n\pi}+\frac{1}{2!}\frac{z^2}{n^2\pi^2}-......\right) \\ &= \prod_{n=1}^{\infty}\left(1-\frac{1}{2}\cdot\frac{z^2}{n^2\pi^2}-\frac{1}{3}\cdot\frac{z^3}{n^3\pi^3}-.......\right)\end{aligned}$$

so that

$$a_n(z) = -\frac{1}{2}\cdot\frac{z^2}{n^2\pi^2}-\frac{1}{3}\cdot\frac{z^3}{n^3\pi^3}-........$$

Let $b_n = \frac{1}{n^2}$ then,

$$\lim\left\|\frac{a_n(z)}{b_n(z)}\right\| = \frac{1}{2\pi^2}\|z\|^2, \quad \text{is a finite quantity for all } z.$$

But the series $\sum \frac{1}{n^2}$ is convergent. Hence bycomparison test $\sum \|a_n(z)\|$ is also convergent. Hence, $\prod\limits_{n=1}^{\infty}\left(1+\frac{z}{n\pi}\right)e^{-\frac{z}{n\pi}}$ is absolutely convergent.

3.2. **A Sufficient Condition for the Convergence of Infinite Product.**

Theorem 3.2. *The infinite product $\prod(1+a_n)$ will be convergent if the two series $\sum a_n$ and $\sum \|a_n\|^2$ are both convergent where $\{a_n\}$ is a sequence of bicomplex numbers and $(1+a_n) \notin \theta_2$.*

Proof. Let $\sum a_n$ and $\sum \|a_n\|^2$ be convergent series of bicomplex numbers.

$$a_n = a_n^{'}\, e_1 + a_n^{''}\, e_2 \quad \text{where } a_n^{'}, a_n^{''} \in \mathbb{C}_1.$$

Now,

$$1+a_n = \left(a_n^{'}+1\right)e_1 + \left(a_n^{''}+1\right)e_2.$$

Since, $(1+a_n) \notin \theta_2$ so, $\log(1+a_n)$ is defined.
Hence $\left(a_n^{'}+1\right) \neq 0$ and $\left(a_n^{''}+1\right) \neq 0$.
Therefore $\log\left(a_n^{'}+1\right)$ and $\log\left(a_n^{''}+1\right)$ are both defined.

$$\log(1+a_n) = \log\left(a_n^{'}+1\right)e_1 + \log\left(a_n^{''}+1\right)e_2.$$

Since $\sum a_n$ and $\sum \|a_n\|^2$ are convergent series of bicomplex numbers, It follows that $\sum a_n^{'}, \sum \left|a_n^{'}\right|^2, \sum a_n^{''}$ and $\sum \left|a_n^{''}\right|^2$ are convergent series of complex numbers [2].

From the sufficient condition for convergence of infinite product complex numbers, we can say that

$$\prod_{n=1}^{\infty}\left(1+a_n^{'}\right) \quad \text{and} \quad \prod_{n=1}^{\infty}\left(1+a_n^{''}\right).$$

Clearly,

$$\prod_{n=1}^{\infty}(1+a_n)=\left(\prod_{n=1}^{\infty}\left(1+a_n^{'}\right)\right)e_1+\left(\prod_{n=1}^{\infty}\left(1+a_n^{''}\right)\right)e_2.$$

Hence $\prod\limits_{n=1}^{\infty}(1+a_n)$ is convergent.

This proves the theorem. □

Theorem 3.3. *Every absolutely convergent infinite product of bicomplex numbers is convergent.*

Proof. Let the product $\prod(1+a_n)$ be absolutely convergent.

Then by general principle of convergence of infinite product of bicomplex numbers we say that for $\varepsilon>0,\exists\quad m\in\mathbb{N}$ such that

$$(1+\|a_1\|)\,.......\,(1+\|a_{n+t}\|)-1<\varepsilon\ \ \forall\ n\geq m,\quad t=1,2,3,......$$

$$\text{Therefore, } 1+\|a_1\|\geq\|1+a_1\|\,,........,1+\|a_{n+t}\|\geq\|1+a_{n+t}\|$$

We have

$$\|(1+a_{n+1})\,......\,(1+a_{n+t})-1\|\leq\|(1+\|a_1\|)\,.......\,(1+\|a_{n+t}\|)\|<\varepsilon\ \forall\ n\geq m,\quad t=1,2,3,......$$

Hence $\prod(1+a_n)$ is convergent.Thus the theorem is established. □

Now we wish to prove the convergence of infinite product using the notion of logarithm of bicomplex valued function as we see in the following theorem.

Theorem 3.4. *A necessary and sufficient condition for* $\sum\|\log(1+a_n)\|$ *to be convergent series of bicomplex numbers is that* $\sum\|a_n\|$ *is convergent, where* $\log(1+a_n)$ *is defined when* $(1+a_n)\notin\theta_2$.

Proof.

$$Let\sum\|\log(1+a_n)\|\quad\text{be convergent.}$$

$$i.e., \lim_{n\to\infty}\log(1+a_n)=0$$

$$\begin{aligned}
i.e., \lim_{n\to\infty}(1+a_n) &= 1\\
i.e., \lim_{n\to\infty}a_n &= 0\\
&\ \text{Since } a_n \text{ is bicomplex variable}\\
\text{Therefore, } a_n &= a_n^{'}\ e_1+a_n^{''}\ e_2\qquad\text{where } a_n^{'},a_n^{''}\in\mathbb{C}_1.
\end{aligned}$$

$$\lim_{n\to\infty}\left(a_n^{'}\ e_1+a_n^{''}\ e_2\right)=0\cdot e_1+0\cdot e_2$$

$$i.e., \lim_{n\to\infty}a_n^{'}=0\qquad\text{and}\qquad\lim_{n\to\infty}a_n^{'}=0.$$

Let $\varepsilon=\frac{1}{2}$, we can find $K_1,K_2\in\mathbb{N}$ such that $\left|a_n^{'}\right|<\frac{1}{2}\ \forall\ n\geq K_1$ and $\left|a_n^{''}\right|<\frac{1}{2}\ \forall\ n\geq K_2$.

Let $K=\max\{K_1,K_2\}$

$\left|a_n^{'}\right|<\frac{1}{2}$ and $\left|a_n^{''}\right|<\frac{1}{2}\ \forall\ n\geq K.$

$$\begin{aligned}
\log\left(1+a_n^{'}\right) &= a_n^{'}-\frac{1}{2}a_n^{'2}+\frac{1}{3}a_n^{'3}-................\\
\log\left(1+a_n^{''}\right) &= a_n^{''}-\frac{1}{2}a_n^{''2}+\frac{1}{3}a_n^{''3}-................
\end{aligned}$$

$$\begin{aligned}\left|\frac{\log\left(1+a_n^{'}\right)}{a_n^{'}}-1\right| &= \left|-\frac{1}{2}a_n^{'}+\frac{1}{3}a_n^{'2}-..........\right| \\ &\leq \frac{1}{2^2}+\frac{1}{2^3}+.......... \\ &= \frac{\frac{1}{2^2}}{1-\frac{1}{2}}=\frac{1}{2}.\end{aligned}$$

Similarly

$$\left|\frac{\log\left(1+a_n^{''}\right)}{a_n^{''}}-1\right| \leq \frac{1}{2}.$$

$$\frac{1}{2}\leq\left|\frac{\log\left(1+a_n^{'}\right)}{a_n^{'}}-1\right|\leq\frac{3}{2} \quad \text{and} \quad \frac{1}{2}\leq\left|\frac{\log\left(1+a_n^{''}\right)}{a_n^{''}}-1\right|\leq\frac{3}{2}. \tag{1}$$

$$i.e., \left|a_n^{'}\right| \leq 2\left|\log\left(1+a_n^{'}\right)\right| \text{ and } \left|a_n^{''}\right| \leq 2\left|\log\left(1+a_n^{''}\right)\right| \quad \forall\, n\geq K.$$

So by comparison test,

$$\sum\left|a_n^{'}\right| \text{ and } \sum\left|a_n^{''}\right| \text{ both convergent.}$$

We know that $\sum_{k=0}^{\infty}(a_k+i_2b_k)$ converges absolutely if and only if $\sum_{k=0}^{\infty}(a_k-i_1b_k)$ and $\sum_{k=0}^{\infty}(a_k+i_1b_k)$ converges absolutely.

So, $\sum\|a_n\|$ is convergent where a_n's are bicomplex numbers. □

Conversely let, $\sum\|a_n\|$ be convergent series of bicomplex numbers. Then

$$\lim a_n=0.$$

From previous part we can write,

$$\lim a_n^{'}=0 \text{ and } \lim a_n^{''}=0 \qquad \text{where } a_n^{'},a_n^{''} \text{ are complex numbers.}$$

From equation (1) we can write

$$\left|\log\left(1+a_n^{'}\right)\right|\leq\frac{3}{2}\left|a_n^{'}\right| \text{ and } \left|\log\left(1+a_n^{''}\right)\right|\leq\frac{3}{2}\left|a_n^{''}\right| \quad \forall\, n\geq K.$$

By Comparison test,

$$\sum\left|\log\left(1+a_n^{'}\right)\right| \quad \text{and} \quad \sum\left|\log\left(1+a_n^{''}\right)\right| \text{ both convergent.}$$

Therefore $\sum\|\log(1+a_n)\|$ is convergent.This completes the proof of the theorem.

Future Prospect : In the line of the works as carried out in the paper one may think of the formation of infinite product of n - dimensional bicomplex numbers with the help of idempotents 0, 1, $\frac{1+i_1i_2}{2}$, $\frac{1-i_1i_2}{2}$, $\frac{1+i_1i_3}{2}$, $\frac{1-i_1i_3}{2}$, $\frac{1+i_2i_3}{2}$, $\frac{1-i_2i_3}{2}$,...., $\frac{1+i_{n-1}i_n}{2}$ and $\frac{1-i_{n-1}i_n}{2}$ in $\mathbb{C}_n$. As a consequence the derivation of relevant results in this area is still a virgin area of research.

Acknowledgement : The first and fourth author sincerely acknowledge the financial support rendered by DST-FIST 2019.

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