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CLASSES OF QUANTUM INTEGRAL OPERATORS IN A COMPLEX DOMAIN

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ABSTRACT. Recently, the author introduced some interesting differential operators created by a class of analytic functions in the open unit disk. In this effort, we shall formulate their integral operators and define the convex formula together withe the corresponding differential operators (DO). By using the quantum calculus, we generalize these integral operators and study the geometric posses in the sequel.

1. INTRODUCTION

Integral operator (or transform) records an equality from its creative domain into another domain where it might be employed and disentangled greatly more straightforward than in the creative domain. The result is then charted posterior to the creative domain consuming the inverse of the integral operator (differential operator in this work).

Quantum calculus, occasionally named calculus without limits, is coordinated to conservative tiny calculus wanting of the impress of the limits (see [1]). It expresses "q-calculus" and "h-calculus", where h apparently, represents the Planck's constant while q indicates the quantum. The two parameters are associated with the formulation

$$q = e^{ih} = e^{2\pi i \hbar}$$

where $\hbar = \frac{h}{2\pi}$ is the condensed Planck constant. In the q-calculus, differentials of functions are formulated by

$$\delta_q(\chi(\zeta)) = \chi(q\zeta) - \chi(\zeta)\,.$$

For example, the q-derivative of the function ξ^n (for some positive integer n) is

$$\delta_q(\zeta^n) = \frac{q^n - 1}{q - 1}\zeta^{n-1} = [n]_q\, \zeta^{n-1}, \quad [n]_q = \frac{q^n - 1}{q - 1}.$$

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Jackson [2] presented the q- integral by the formulation

$$\int_0^{\zeta} \curlywedge(\tau) d_q\tau = \zeta(1-q) \sum_{m=0}^{\infty} q^m \curlywedge(\zeta q^m), \tag{1.1}$$

where the above series is convergence. Now for $\curlywedge(\zeta) = \zeta^m$, we conclude that

$$\int_0^{\zeta} \tau^m d_q\tau = \frac{\zeta^{m+1}}{[m+1]_q}, \quad \forall\, m \neq -1.$$

In this exploration, we express a new q-integral operators of complex coefficients and investigate the geometric behaviors based on the possessions of the theory of geometric functions. The recommended q- integral operators employ to define some q-subclasses of starlike functions. Quantum inequalities include the q-integral operator and some special functions are discussed.

2. Methodology

Let $\wedge$ be the family of analytic functions $\chi \in \cup$ and normalized by the conditions $\chi(0) = 0$ and $\chi'(0) = 1$, formulating by

$$\chi(\zeta) = \zeta + \sum_{n=2}^{\infty} \chi_n \zeta^n, \quad \zeta \in \cup. \tag{2.1}$$

A sub-class of $\wedge$ is the class of univalent functions denoting by (Ⓢ). Consequently, a function $\chi \in \wedge$ is starlike in $\cup$ ($\mathfrak{S}^*$) if and only if $\Re(\zeta\chi'(\zeta)/\chi(\zeta)) > 0$. In addition, a function $\chi \in \wedge$ is convex in $\cup$ ($\mathfrak{C}$) if and only if $1 + \Re(\zeta\chi''(\xi)/\chi'(\zeta)) > 0$.

Definition 1. *For two functions χ and $\mathfrak{X}$ in $\wedge$, are subordinated $\chi \prec \mathfrak{X}$, if a Schwarz function ς with $\varsigma(0) = 0$ and $|\varsigma(\zeta)| < 1$ satisfying $\chi(\zeta) = \mathfrak{X}(\varsigma(\zeta))$, $\zeta \in \cup$ (see* [3]*). Evidently, $\chi(\zeta) \prec \mathfrak{X}(\zeta)$ equivalents to $\chi(0) = \mathfrak{X}(0)$ and $\chi(\cup) \subset \mathfrak{X}(\cup)$.*

Definition 2. *A function $\chi \in \wedge$ is called major by $\mathfrak{X} \in \wedge$, if $|\chi(\zeta)| \leq |\mathfrak{X}(\zeta)|$.*

Ma and Minda [4] formulated different sub-classes of starlike and convex functions for which either of the expression $\frac{\zeta\chi'(\zeta)}{\chi(\zeta)}$ or $1 + \frac{\zeta\chi''(\zeta)}{\chi'(\zeta)}$ is subordinate to an additional common superordinate function. For this classes, they presented an analytic function Θ with positive real part in$\cup$, $\Theta(0) = 1, \Theta(0) > 0$, and Θ maps $\cup$ onto an area starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions contains of functions $\chi \in \wedge$ filling the subordination $\frac{\zeta\chi'(\zeta)}{\chi(\zeta)} \prec \Theta(\zeta)$. Likewise, the class of Ma-Minda convex functions involves of functions $\chi \in \wedge$ fluffing the subordination

$$1 + \frac{\zeta\chi''(\zeta)}{\chi'(\zeta)} \prec \Theta(\zeta).$$

Moreover, when $\Theta(\zeta) = \frac{1+\zeta}{1-\zeta}$, we obtain the main stalike and convex classes respectively. Ali et al. [5] combined the two classes in the class

$$\frac{\zeta\chi'(\zeta)}{\chi(\zeta)} + \frac{\zeta\chi''(\xi)}{\chi'(\zeta)} \prec \Theta(\zeta).$$

2.1. **Q-Integral operator.** Recently, Ibrahim and Darus introduced two different differential operators as follows [6, 7]

$$\begin{aligned}
&\Delta_\kappa^1 \chi(\zeta) = \zeta\chi'(\zeta) + \frac{\kappa}{2}\left(\chi(\zeta) - \chi(-\zeta) - 2\zeta\right), \quad \kappa \in \mathbb{R} \\
&\vdots \\
&\Delta_\kappa^m \chi(\zeta) = \Delta_\kappa(\Delta_\kappa^{m-1}\chi(\zeta)) \\
&\qquad = \zeta + \sum_{n=2}^{\infty}[n + \frac{\kappa}{2}(1 + (-1)^{n+1})]^m \chi_n \zeta^n, \\
&\qquad \left(\kappa \in \mathbb{R}, \zeta \in \cup, \chi \in \wedge\right).
\end{aligned} \tag{2.2}$$

And for q-differential operator [8]

$${}_q\Delta_\kappa^m \chi(\zeta) = \zeta + \sum_{n=2}^{\infty}[\,[n]_q + \frac{\kappa}{2}(1 + (-1)^{n+1})]^m \chi_n \zeta^n. \tag{2.3}$$

Moreover, they defined the following differential operator [9]

$$\begin{aligned}
&\mathcal{M}_\kappa^1 \chi(\zeta) = \kappa\zeta\chi'(\zeta) - (1-\kappa)\zeta\chi'(-\zeta) \\
&\qquad = \zeta + \sum_{n=2}^{\infty}[n\,(\kappa - (1-\kappa)(-1)^n)]\,\chi_n \zeta^n \\
&\vdots \\
&\mathcal{M}_\kappa^m \chi(\zeta) = \mathcal{M}_\kappa^1[\mathcal{M}_\kappa^{m-1}\chi(\zeta)] \\
&\qquad = \zeta + \sum_{n=2}^{\infty}[n\,(\kappa - (1-\kappa)(-1)^n)]^m \chi_n \zeta^n, \\
&\qquad \left(\kappa \in (0,1), \zeta \in \cup, \chi \in \wedge\right).
\end{aligned} \tag{2.4}$$

And for q-differential operator, it will become

$${}_q\mathcal{M}_\kappa^m \chi(\zeta) = \zeta + \sum_{n=2}^{\infty}[\,[n]_q\,(\kappa - (1-\kappa)(-1)^n)]^m \chi_n \zeta^n, \tag{2.5}$$

Ibrahim [10], for a non-zero complex number κ, formulated another differential operator as follows:

$$\begin{aligned}
&\Delta_\kappa^1 \chi(\zeta) = \left(\frac{\kappa}{\bar{\kappa}}\right)\zeta\,\chi'(\zeta) - \left(1 - \frac{\kappa}{\bar{\kappa}}\right)\zeta\,\chi'(-\zeta) \\
&\vdots \\
&\Delta_\kappa^m \chi(\zeta) = \Delta_\kappa(\Delta_\kappa^{m-1}\chi(\zeta)) \\
&\qquad = \zeta + \sum_{n=2}^{\infty}\left(n\left(\frac{\kappa}{\bar{\kappa}} - (1 - \frac{\kappa}{\bar{\kappa}})(-1)^n\right)\right)^m \chi_n \zeta^n, \\
&\qquad \left(\kappa \neq 0 \in \mathbb{C}, \zeta \in \cup, \chi \in \wedge\right).
\end{aligned} \tag{2.6}$$

And its q-differential equation

$$
\begin{aligned}
{}_q\Delta_\kappa^m \chi(\zeta) &= \Delta_\kappa(\Delta_\kappa^{m-1}\chi(\zeta)) \\
&= \zeta + \sum_{n=2}^{\infty} \left([n]_q \left(\frac{\kappa}{\bar{\kappa}} - (1 - \frac{\kappa}{\bar{\kappa}})(-1)^n \right) \right)^m \chi_n \zeta^n
\end{aligned} \tag{2.7}
$$

Here, we proceed to introduce the integral operator corresponding to the above differential operator, we have the following Table 2.1

TABLE 2.1. Q-integral operators

DO	Integral operator	q-integral operator
(2.2)	$I_\kappa^m \chi(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\chi_n \zeta^n}{[n + \frac{\kappa}{2}(1+(-1)^{n+1})]^m}$	${}_q I_\kappa^m \chi(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\chi_n \zeta^n}{[[n]_q + \frac{\kappa}{2}(1+(-1)^{n+1})]^m}$
(2.4)	$\mathcal{J}_\kappa^m \chi(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\chi_n \zeta^n}{[n(\kappa-(1-\kappa)(-1)^n)]^m}$	${}_q \mathcal{J}_\kappa^m \chi(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\chi_n \zeta^n}{[[n]_q(\kappa-(1-\kappa)(-1)^n)]^m}$
(2.6)	$L_\kappa^m \chi(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\chi_n \zeta^n}{\left(n\left(\frac{\kappa}{\bar{\kappa}}-(1-\frac{\kappa}{\bar{\kappa}})(-1)^n\right)\right)^m}$	${}_q L_\kappa^m \chi(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\chi_n \zeta^n}{\left([n]_q\left(\frac{\kappa}{\bar{\kappa}}-(1-\frac{\kappa}{\bar{\kappa}})(-1)^n\right)\right)^m}$

By using the q-integral operators we define the following q-starlike classes:

Definition 3. *A function $\chi \in \wedge$ is called q-starlike function in $\cup$ which is denoted by ${}_q\mathbf{S}_\kappa^m(\Theta)$ if one of the following q-inequality is held:*

- $\dfrac{\zeta\left({}_q I_\kappa^m \chi(\zeta)\right)'}{{}_q I_\kappa^m \chi(\zeta)} \prec \Theta(\zeta), \quad \kappa \in \mathbb{R};$
- $\dfrac{\zeta\left({}_q \mathcal{J}_\kappa^m \chi(\zeta)\right)'}{{}_q \mathcal{J}_\kappa^m \chi(\zeta)} \prec \Theta(\zeta), \quad \kappa \in (0, 1)$
- $\dfrac{\zeta\left({}_q L_\kappa^m \chi(\zeta)\right)'}{{}_q L_\kappa^m \chi(\zeta)} \prec \Theta(\zeta), \quad \kappa \neq 0 \in \mathbb{C},$

where $\Theta(0) = 1$, $\Theta'(0) > 0$ and $\Theta(\cup)$ is symmetric with respect to the real axis.

Note that all the q-integral operator are the generalized of q-Salagean operator [11]

$$
{}_q S^m \chi(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\chi_n \zeta^n}{\left([n]_q\right)^m}.
$$

We aim to study the geometric aspect of functions in ${}_q\mathbf{S}_\kappa^m$ in the next section.

3. GEOMETRIC ASPECT

We have the existence solution in the following result:

Theorem 3.1. *Let $\chi \in \wedge$ with positive connections. If $\Theta(\zeta), \zeta \in \cup$ is univalent convex in $\cup$ then $\chi \in_q \mathbf{S}_\kappa^m(\Theta)$.*

Proof. It is enough to show one of the following inequalities:

- $\Re\left(\frac{\zeta({}_qI_\kappa^m\chi(\zeta))'}{{}_qI_\kappa^m\chi(\zeta)}\right) > 0, \quad \kappa \in \mathbb{R}^+;$
- $\Re\left(\frac{\zeta({}_q\mathcal{J}_\kappa^m\chi(\zeta))'}{{}_q\mathcal{J}_\kappa^m\chi(\zeta)}\right) > 0, \quad \kappa \in (0,1);$
- $\Re\left(\frac{\zeta({}_qL_\kappa^m\chi(\zeta))'}{{}_qL_\kappa^m\chi(\zeta)}\right) > 0, \quad \kappa \neq 0 \in \mathbb{C}.$

Let us consider the first q-integral ${}_qI_\kappa^m$ then we have the next conclusion

$$\begin{aligned}
&\Re\left(\frac{\zeta({}_qI_\kappa^m\chi(\zeta))'}{{}_qI_\kappa^m\chi(\zeta)}\right) > 0\\
&\Leftrightarrow \Re\left(\frac{\zeta + \sum_{n=2}^\infty n/\left([n,q] + \frac{\kappa}{2}(1+(-1)^{n+1})\right)^m \chi_n\zeta^n}{\zeta + \sum_{n=2}^\infty 1/\left([n]_q + \frac{\kappa}{2}(1+(-1)^{n+1})\right)^m \chi_n\zeta^n}\right) > 0\\
&\Leftrightarrow \Re\left(\frac{1 + \sum_{n=2}^\infty n/\left([n,q] + \frac{\kappa}{2}(1+(-1)^{n+1})\right)^m \chi_n\zeta^{n-1}}{1 + \sum_{n=2}^\infty 1/\left([n]_q + \frac{\kappa}{2}(1+(-1)^{n+1})\right)^m \chi_n\zeta^{n-1}}\right) > 0\\
&\Leftrightarrow \left(\frac{1 + \sum_{n=2}^\infty n/\left([n,q] + \frac{\kappa}{2}(1+(-1)^{n+1})\right)^m \chi_n}{1 + \sum_{n=2}^\infty 1/\left([n]_q + \frac{\kappa}{2}(1+(-1)^{n+1})\right)^m \chi_n}\right) > 0, \quad \zeta \to 1^+\\
&\Leftrightarrow \left(1 + \sum_{n=2}^\infty n/\left([n]_q + \frac{\kappa}{2}(1+(-1)^{n+1})\right)^m \chi_n\right) > 0.
\end{aligned}$$

Similarly for ${}_q\mathcal{J}_\kappa^m$. Now, we consider the q-integral ${}_qL_\kappa^m$, we have the following computation:

$$\begin{aligned}
&\Re\left(\frac{\zeta({}_qL_\kappa^m\chi(\zeta))'}{{}_qL_\kappa^m\chi(\zeta)}\right) > 0\\
&\Leftrightarrow \Re\left(\frac{\zeta + \sum_{n=2}^\infty n/\left([n]_q\ \left(\frac{\kappa}{\bar\kappa} - (1-\frac{\kappa}{\bar\kappa})(-1)^n\right)\right)^m \chi_n\zeta^n}{\zeta + \sum_{n=2}^\infty 1/\left([n]_q\ \left(\frac{\kappa}{\bar\kappa} - (1-\frac{\kappa}{\bar\kappa})(-1)^n\right)\right)^m \chi_n\zeta^n}\right) > 0\\
&\Leftrightarrow \Re\left(\frac{1 + \sum_{n=2}^\infty n/\left([n]_q\ \left(\frac{\kappa}{\bar\kappa} - (1-\frac{\kappa}{\bar\kappa})(-1)^n\right)\right)^m \chi_n\zeta^{n-1}}{1 + \sum_{n=2}^\infty 1/\left([n]_q\ \left(\frac{\kappa}{\bar\kappa} - (1-\frac{\kappa}{\bar\kappa})(-1)^n\right)\right)^m \chi_n\zeta^{n-1}}\right) > 0\\
&\Leftrightarrow \left(\frac{1 + \sum_{n=2}^\infty n/\left([n,q]\right)^m \chi_n}{1 + \sum_{n=2}^\infty 1/\left([n]_q\right)^m \chi_n}\right) > 0, \quad \zeta \to 1^+\\
&\Leftrightarrow \left(1 + \sum_{n=2}^\infty n/\left([n]_q\right)^m \chi_n\right) > 0.
\end{aligned}$$

Moreover, by the definition of q-integrals, we inform that

$$({}_qI_\kappa^m\chi)(0) = ({}_q\mathcal{J}_\kappa^m\chi)(0) = ({}_qL_\kappa^m\chi)(0) = 0.$$

Since, $\Theta \in \mathcal{P}$ with all positive real connections including the above connections, that is

$$\left|\frac{\zeta({}_qI_\kappa^m\chi(\zeta))'}{{}_qI_\kappa^m\chi(\zeta)}\right| \le |\Theta(\zeta)|, \quad \left|\frac{\zeta({}_q\mathcal{J}_\kappa^m\chi(\zeta))'}{{}_q\mathcal{J}_\kappa^m\chi(\zeta)}\right| \le |\Theta(\zeta)|, \quad \left|\frac{\zeta({}_qL_\kappa^m\chi(\zeta))'}{{}_qL_\kappa^m\chi(\zeta)}\right| \le |\Theta(\zeta)|$$

But Θ is convex univalent in $\cup$ therefore, yields that

$$\chi(\zeta) \in {}_q\mathbf{S}^m_\kappa(\Theta).$$

$\square$

Now, we discuss the consequences when $\chi \in \mathbf{S}^m_\kappa$.

Theorem 3.2. *Assume that $\chi \in {}_q\mathbf{S}^m_\kappa(\Theta)$, where Θ is convex univalent function in $\cup$. Then every q-integral operators in Table 2.1 satisfies the inequality*

$${}_qI^m_\kappa\chi(\zeta) \prec \zeta \exp\Big(\int_0^\zeta \frac{\Theta(\eth(\omega))-1}{\omega}dw\Big),$$

$${}_q\mathcal{J}^m_\kappa\chi(\zeta) \prec \zeta \exp\Big(\int_0^\zeta \frac{\Theta(\eth(\omega))-1}{\omega}dw\Big),$$

and

$${}_qL^m_\kappa\chi(\zeta) \prec \zeta \exp\Big(\int_0^\zeta \frac{\Theta(\eth(\omega))-1}{\omega}dw\Big),$$

where $\eth(\zeta)$ is analytic in $\cup$, with $\eth(0)=0$ and $|\eth(\zeta)|<1$. Moreover, for $|\zeta|=\xi$, then every q-integral operator fulfills the formula inequality

$$\exp\Big(\int_0^1 \frac{\Theta\eth(-\xi))-1}{\xi}\Big)d\xi \le \Big|\frac{{}_qI^m_\kappa\chi(\zeta)}{\zeta}\Big| \le \exp\Big(\int_0^1 \frac{\Theta(\eth(\xi))-1}{\xi}\Big)d\xi,$$

$$\exp\Big(\int_0^1 \frac{\Theta\eth(-\xi))-1}{\xi}\Big)d\xi \le \Big|\frac{{}_q\mathcal{J}^m_\kappa\chi(\zeta)}{\zeta}\Big| \le \exp\Big(\int_0^1 \frac{\Theta(\eth(\xi))-1}{\xi}\Big)d\xi$$

and

$$\exp\Big(\int_0^1 \frac{\Theta\eth(-\xi))-1}{\xi}\Big)d\xi \le \Big|\frac{{}_qL^m_\kappa\chi(\zeta)}{\zeta}\Big| \le \exp\Big(\int_0^1 \frac{\Theta(\eth(\xi))-1}{\xi}\Big)d\xi.$$

Proof. Since $\chi \in {}_q\mathbf{S}^m_\kappa(\Theta)$, we have one of the following subordination inequalities:

$$\Big(\frac{\zeta({}_qI^m_\kappa\chi(\zeta))'}{{}_qI^m_\kappa\chi(\zeta)}\Big) \prec \Theta(\zeta), \quad \zeta\in\cup,$$

$$\Big(\frac{\zeta({}_q\mathcal{J}^m_\kappa\chi(\zeta))'}{{}_q\mathcal{J}^m_\kappa\chi(\zeta)}\Big) \prec \Theta(\zeta), \quad \zeta\in\cup,$$

or

$$\Big(\frac{\zeta({}_qL^m_\kappa\chi(\zeta))'}{{}_qL^m_\kappa\chi(\zeta)}\Big) \prec \Theta(\zeta), \quad \zeta\in\cup,$$

which leads to exist a Schwarz function with $\eth(0)=0$ and $|\eth(\zeta)|<1$ fulfilling

$$\Big(\frac{\zeta({}_qI^m_\kappa\chi(\zeta))'}{{}_qI^m_\kappa\chi(\zeta)}\Big) = \Theta(\eth(\zeta)), \quad \zeta\in\cup.$$

A computation leads to

$$\Big(\frac{({}_qI^m_\kappa\chi(\zeta))'}{{}_qI^m_\kappa\chi(\zeta)}\Big) - \frac{1}{\zeta} = \frac{\Theta(\eth(\zeta))-1}{\zeta}.$$

By integrating both sides, we obtain

$$\log\big({}_qI^m_\kappa\chi(\zeta)\big) - \log\zeta = \int_0^\zeta \frac{\Theta(\eth(\omega))-1}{\omega}d\omega.$$

Consequently, we attain

$$\log\Big(\frac{{}_qI_\kappa^m\chi(\zeta)}{\zeta}\Big) = \int_0^\zeta \frac{\Theta(\eth(\omega))-1}{\omega}d\omega. \tag{3.1}$$

By utilizing the meaning of subordination, we conclude that

$${}_qI_\kappa^m\chi(\zeta) \prec \zeta\exp\Big(\int_0^\zeta \frac{\Theta(\eth(\omega))-1}{\omega}d\omega\Big).$$

But Θ plots the disk $0 < |\zeta| < \xi < 1$ onto a domain which is convex and symmetric with respect to the real axis, which implies that

$$\Theta(-\xi|\zeta|) \le \Re(\Theta(\eth(\xi\zeta))) \le \Theta(\xi|\zeta|), \quad \xi \in (0,1),$$

that is

$$\Theta(-\xi) \le \Theta(-\xi|\zeta|), \quad \Theta(\xi|\zeta|) \le \Theta(\xi)$$

and

$$\int_0^1 \frac{\Theta(\eth(-\xi|\zeta|))-1}{\xi}d\xi \le \Re\Big(\int_0^1 \frac{\Theta(\eth(\xi))-1}{\xi}d\xi\Big) \le \int_0^1 \frac{\Theta(\eth(\xi|\zeta|))-1}{\xi}d\xi.$$

By taking the Eq. (3.1), we receive

$$\int_0^1 \frac{\Theta(\eth(-\xi|\zeta|))-1}{\xi}d\xi \le \log\Big|\frac{{}_qI_\kappa^m\chi(\zeta)}{\zeta}\Big| \le \int_0^1 \frac{\Theta(\eth(\xi|\zeta|))-1}{\xi}d\xi,$$

which leads to

$$\exp\Big(\int_0^1 \frac{\Theta(\eth(-\xi|\zeta|))-1}{\xi}d\xi\Big) \le \Big|\frac{{}_qI_\kappa^m\chi(\zeta)}{\zeta}\Big| \le \exp\Big(\int_0^1 \frac{\Theta(\eth(\xi|\zeta|))-1}{\xi}d\xi\Big).$$

Hence, we have

$$\exp\Big(\int_0^1 \frac{\Theta(\eth(-\xi))-1}{\zeta}\Big)d\xi \le \Big|\frac{{}_qI_\kappa^m\chi(\zeta)}{\zeta}\Big| \le \exp\Big(\int_0^1 \frac{\Theta(\eth(\xi))-1}{\xi}\Big)d\xi.$$

Similarly we obtain

$$\exp\Big(\int_0^1 \frac{\Theta(\eth(-\xi))-1}{\zeta}\Big)d\xi \le \Big|\frac{{}_q\mathcal{J}_\kappa^m\chi(\zeta)}{\zeta}\Big| \le \exp\Big(\int_0^1 \frac{\Theta(\eth(\xi))-1}{\xi}\Big)d\xi$$

and

$$\exp\Big(\int_0^1 \frac{\Theta(\eth(-\xi))-1}{\zeta}\Big)d\xi \le \Big|\frac{{}_qL_\kappa^m\chi(\zeta)}{\zeta}\Big| \le \exp\Big(\int_0^1 \frac{\Theta(\eth(\xi))-1}{\xi}\Big)d\xi.$$

□

Corollary 3.3. [6, 9, 10] *Let $q \to 1$ in Theorem 3.2. Then*

$$I_\kappa^m\chi(\zeta) \prec \zeta\exp\Big(\int_0^\zeta \frac{\Theta(\eth(\omega))-1}{\omega}d\omega\Big),$$

$$\mathcal{J}_\kappa^m\chi(\zeta) \prec \zeta\exp\Big(\int_0^\zeta \frac{\Theta(\eth(\omega))-1}{\omega}d\omega\Big),$$

and

$$L_\kappa^m\chi(\zeta) \prec \zeta\exp\Big(\int_0^\zeta \frac{\Theta(\eth(\omega))-1}{\omega}d\omega\Big),$$

Example 3.4. Consider the following data $\Theta(\zeta) = \Big(\frac{1+\zeta}{1-\zeta}\Big)$, and $(\kappa, q) = (1, 0.5)$.

Then we have

$$ {}_qI_\kappa^m\chi(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\chi_n\zeta^n}{[2/3]^m} $$

$$ {}_q\mathcal{J}_\kappa^m\chi(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\chi_n\zeta^n}{[1/2]^m} $$

$$ {}_qL_\kappa^m\chi(\zeta) = \zeta + \sum_{n=2}^{\infty} \frac{\chi_n\zeta^n}{[1/2]^m}. $$

Since

$$ \Theta(\zeta) = 1 + 2\zeta + 2\zeta^2 + 2\zeta^3 + 2\zeta^4 + O(\zeta^5), \quad \zeta \in \cup $$

which implies that $\chi \in {}_q\mathbf{S}_\kappa^m(\Theta)$ with the connections bound

$$ |\chi_n| \le \frac{2[2/3]^m}{n}, \quad \forall n > 1,\, m \ge 1 $$

for ${}_qI_\kappa^m\chi(\zeta)$ and

$$ |\chi_n| \le \frac{2[1/2]^m}{n}, \quad \forall n > 1,\, m \ge 1 $$

for ${}_q\mathcal{J}_\kappa^m$ and ${}_qL_\kappa^m$.

Moreover, for sum $\eth(\omega) = \omega, \omega \in \cup$, we have

$$ \zeta \exp\Big(\int_0^\zeta \frac{\Theta(\eth(\omega)) - 1}{\omega} dw \Big) = \zeta + 2\zeta^2 + 3\zeta^3 + 4\zeta^4 + 5\zeta^5 + 6\zeta^6 + O(\zeta^7). $$

Hence, by Theorem 3.2,

$$ {}_qI_\kappa^m\chi(\zeta) \prec \zeta \exp\Big(\int_0^\zeta \frac{\Theta(\eth(\omega)) - 1}{\omega} dw \Big), $$

$$ {}_q\mathcal{J}_\kappa^m\chi(\zeta) \prec \zeta \exp\Big(\int_0^\zeta \frac{\Theta(\eth(\omega)) - 1}{\omega} dw \Big), $$

and

$$ {}_qL_\kappa^m\chi(\zeta) \prec \zeta \exp\Big(\int_0^\zeta \frac{\Theta(\eth(\omega)) - 1}{\omega} dw \Big). $$

4. Conclusion

From above, we conclude that a class of q-integrals operators in the open unit disk satisfied q-starlike functions. This presentation can be extended in view of other classes of analytic functions (multivalent class and meromorphic class and harmonic class).

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