Journal of Fractional Calculus and Applications
Vol. 12(1) Jan. 2021, pp. 90-100.
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

INVESTIGATION OF SOME STABILITY PROPERTIES OF SOLUTIONS FOR A CLASS OF NONLINEAR BOUNDARY VALUE FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. TIn this study, some stability properties of solutions for a class of nonlinear boundary value fractional differential equations were considered. First, Laplace transform method was used to show that the solution to the class of problem considered is Mittag-Leffler stable. Moreover, by applying the method of Lyapunov-like function approach, an equilibrium solution of the problem is proved to be asymptotically stable.

1. Introduction

In recent years, fractional calculus has become an interesting and important area of research due to its numerous applications in models of several phenomena in various fields of science and engineering. Indeed, a number of applications in areas such as biology [20], viscoelasticity [1], earthquake prediction [3, 15], signal processing [17], dynamical systems [2] and etc., abound in the literature. For more on the theories and applications of fractional calculus, (see [10, 14, 13, 21]) and the references therein.

The concept of stability is an important aspect of the qualitative theory of differential equations. Various stability types such as Hyers-Ulam-Rassias, Mittag-Leffler and Lyapunov-like direct method have been employed in the literature to study some properties of solutions of fractional differential equations. For instance, the author in [12] studied the stability with respect to part of the variables of nonlinear Caputo fractional differential equations. Sufficient conditions of stability, uniform stability, Mittag- Leffler stability and asymptotic uniform stability of this type were obtained within the method of Lyapunov-like functions.

Considering the Mittag-Leffller stability of fractional order nonlinear dynamic systems, the authors in [11] studied the fractional differential equation

$$ {}_{t_0}D_t^{\alpha}x(t) = f(t, x(t)), $$

with the initial value $x(t_0)$, where D^{α} denotes either Caputo or Riemann-Liouville fractional operator, $\alpha \in (0, 1)$, $f : [t_0, \infty) \times \Omega \longrightarrow \mathbb{R}^n$ is piece-wise continuous in

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2010 *Mathematics Subject Classification.* 34xx, 34Bxx, 34Dxx .

Key words and phrases. ?Fractional Calculus, Riemann-Liouville fractional derivative, Mittag-Leffler stability, Lyapunov-Like function.

Submitted Nov. 19, 2019. Revised April 28, 2020.

t and locally Lipschitz in x on $[t_0, \infty) \times \Omega$, and $\Omega \in \mathbb{R}^n$ is a domain that contains the origin $x = 0$.

On extension of Lyapunov direct method of the fractional nonautonomous systems with order lying in $(1, 2)$, the authors in [5] employed Lyapunov direct method in the study of stability problem of Caputo type nonautonomous systems. The work extended [16], which is an improvement of some results from the uniformly asymptotically stability of integer-order differential systems to fractional-order differential systems with order $p \in (0, 1)$, based on Lyapunov direct method.

Equally, on the systems of nonlinear fractional differential equations, [7] studied the stability and stabilization of a class of fractional-order nonlinear systems for $0 < \alpha < 2$. Based on the method of fractional order Lyapunov stability theorem, S-procedure and MittagLeffler function, the stability conditions that ensure local stability and stabilization of a class of fractional-order nonlinear systems under the Caputo derivative with $0 < \alpha < 2$. were proposed. Other works on stability of fractional differential equations can be found in [16, 4, 6, 18, 19] and the references therein.

In all the works considered, a continuously differentiable function V was used as a Lyapunov-like function, however in our own case, we shall assume that V is only continuous. To this end, we shall investigate some stability types of the nonlinear fractional boundary value problem

$$D^{\alpha}_{a_+} x(t) + k\, D^{\beta}_{a_+} x(t) + g(t, x(t)) = h(t),\ t \in [a, b], \tag{1}$$

$$D^{\alpha-1}_{a_+} x(a_+) = D^{\alpha-1}_{a_+} x(b_-), \tag{2}$$

$$I^{2-\alpha}_{a_+} x(a_+) = I^{2-\alpha}_{a_+} x(b_-), \tag{3}$$

$$I^{1-\beta}_{a_+} x(a_+) = I^{1-\beta}_{a_+} x(b_-), \tag{4}$$

in the Sobolev space

$$W^{\alpha,\beta}[a, b] = \left\{ x(t) \in C_{2-\alpha}[a, b] : D^{\alpha}_{a_+} x(t), D^{\beta}_{a_+} x(t) \in L^{\frac{1}{\beta}}[a, b] \right\},$$

where $0 < \beta < 1 < \alpha < 2$, $C_{2-\alpha}[a, b] = \left\{ x(t) : x(t)(t-a)^{2-\alpha} \in C^0[a, b] \right\}, k$ is a positive constant, $D^{\alpha}_{a_+} x(t)$ is understood here in the the Riemann-Liouville sense, $D^{\alpha-1}_{a_+} x(a_+) := \lim_{t \longrightarrow a^+} D^{\alpha-1}_{a_+} x(t)$, $g : [a, b] \times [0, \infty)$ is an $L_\infty-$ Carathеódory function, $h \in L^{\frac{1}{\beta}}[a, b]$.

This work is organized as follows. In section 2, we shall define some basic terms of fractional calculus and state some useful lemmas related to our work. In section three, we shall establish that the solution to the problem is Mittag-Leffler stable. Equally, by using Lypunov-like function approach, we shall show that an equilibrium solution of equations (1)-(4) is asymptotically stable.

2. Preliminaries

Definition 2.1.[9] The Riemann-Liouville fractional derivative of a function x, of order α, with lower limit a is defined as,

$$D^{\alpha}_{a_+} x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} x(s) ds, \tag{5}$$

with $n-1<\alpha<n,\ n=[\alpha]+1$, while the Riemann-Liouville fractional integral of a function x, of order $\alpha>0$, and denoted by $I^{\alpha}_{a_+}x(t)$ is defined by,

$$I^{\alpha}_{a_+}x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)ds. \tag{6}$$

Observe from equations (5) and (6) that,

$$D^{\alpha}_{a_+}x(t)=\frac{d^n}{dt^n}I^{n-\alpha}_{a_+}x(t). \tag{7}$$

Definition 2.2.[8] A function $f:[a,b]\times\mathbb{R}\longrightarrow\mathbb{R}$ is said to be a Carathéodory function if it satisfies the following conditions:

- $f(t,x)$ is Lebesgue measurable with respect to t in $[a,b]$,
- $f(t,x)$ is continuous with respect to x on $\mathbb{R}$

A function $f(t,x)$ defined on $[a,b]\times\mathbb{R}$ is said to be an L^p- Carathéodory function, $p\geq 1$, if it is a Carathéodory function and $\forall r>0$, there exists $h_r\in L^p(a,b)$, such that $\forall x\in[-r,r]$ and $\forall t\in[a,b]$, then $f(t,x)\leq h_r(t)$.

Definition 2.3.[14] A two-parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\ Re(\alpha),Re(\beta)>0,z\in\mathbb{C}. \tag{8}$$

Definition 2.4.[11] The solution of (1)-(4) is said to be Mittag-Leffler stable if

$$\|x(t)\|\leq\left\{m[x(t_0)]E_{\alpha-\beta}(-k(t-t_0)^{\alpha-\beta}\right\}^b, \tag{9}$$

where t_0 is the initial time, $m(0)=0, m(x)>0, b>0$ and $m(x)$ is locally Lipschitz with respect to $x\in W^{\alpha,\beta}[t_0,b]$ with Lipschitz constant m_0.

Definition 2.5.[5] A continuous function $\beta:[0,a)\times[0,\infty)\longrightarrow[0,\infty)$ is said to belong to class KL functions, if for each fixed s, the mapping $\beta(r,s)$ belongs to class K with respect to r, and for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s)\longrightarrow 0$, as $s\longrightarrow\infty$.

Lemma 2.6. [10] The space $AC^n[a,b]$ consists of those and only function f, which can be represented in the form

$$f(x)=I^n_{a_+}\varphi(x)+\sum_{k=0}^{n-1}c_k(x-a)^k, \tag{10}$$

where $\varphi\in L_1(a,b),\ c_k(k=0,1,2,\cdots,n-1)$ are arbitrary constants.

Lemma 2.7.[10] If $f\in L_1(a,b)$ and $I^{n-\alpha}_{a_+}f(t)\in AC^n[a,b]$, then

$$I^{\alpha}_{a_+}(D^{\alpha}_{a_+}f(x))=f(x)-\sum_{j=1}^{n}\frac{D^{\alpha-j}_{a_+}f(a_+)(x-a)^{\alpha-j}}{\Gamma(\alpha-j+1)}, \tag{11}$$

holds. A particular case, where $0<\beta<1<\alpha<2$, then according to [8], we have that

$$I^{\alpha}_{a_+}D^{\beta}_{a_+}x(t)=I^{\alpha-\beta}_{a_+}x(t)-\frac{1}{\Gamma(\alpha)}(t-a)^{\alpha-1}I^{1-\beta}_{a_+}x(a_+). \tag{12}$$

Lemma 2.9. [8] If $x\in W^{\alpha,\beta}_{2-\alpha}[a,b]$, then x satisfies the relations (1)- (4) if, and only if, x satisfies the Voterra-integral integral equation

$$x(t)-\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}D^{\alpha-1}_{a_+}x(b_-)-\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)}I^{2-\alpha}_{a_+}x(a_+)+kI^{\alpha-\beta}_{a_+}x(t)-k\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}I^{1-\beta}_{a_+}x(b_-)$$

$$= I_{a_+}^{\alpha}[h(t) - g(t, x(t))].$$

Lemma 2.10.[5]. Let $x(t), k(t)$ be nonnegative continuous functions on $[t_0, b]$ and let $a(t)$ be a nondecreasing function on $[t_0, b]$; further let $g(u)$ be a nondecreasing continuous function for $u \geq 0$ and $g(u) > 0$ for $u > 0$, then the inequality

$$x(t) \leq a(t) + \int_{t_0}^{t} k(s)g(x(s))ds, \ (t_0 \leq t \leq b),$$

implies the inequality

$$x(t) \leq \Omega^{-1}\left[\Omega(a(t)) + \int_{t_0}^{t} k(s)ds\right], \ (t_0 \leq t \leq b' \leq b),$$

where $\Omega(u) = -\int_{\epsilon}^{u} \frac{ds}{g(s)}$, $(\epsilon > 0, u > 0)$ and $b' = \max(t_0 \leq \tau \leq b) : \Omega(a(\tau)) + \int_{t_0}^{\tau} k(s)ds \leq \Omega(\infty)$ lies within the domain of definition $\Omega^{-1}(u)$, for $t_0 \leq t \leq b'$.

Lemma 2.11. [5]. Let $x(t)$ be a continuous and nonnegative function defined on a real interval $t_0 \leq t \leq T$ (T could be infinity) and let $a(t)$ be a nonnegative and monotonously nondecreasing function on the given interval. If

$$x(t) \leq a(t) + M\int_{t_0}^{t} (t-s)^{p-1}x(s)ds, \ (p > 0), \tag{13}$$

where M is a positive constant, then

$$x(t) \leq a(t)E_p(M\Gamma(p)(t-t_0)^p), \ (t_0 \leq t \leq T). \tag{14}$$

3. Existence results

First, we shall show that the solution to equations (1)-(4) is Mittag-Leffler stable.

Theorem 3.1. Suppose that there exists a nonnegative and nonincreasing function $V : [0, b] \times \mathbb{R} \longrightarrow \mathbb{R}^+$ that is locally Lipschitz with respect to x such that

$$h(t) - g(t, x(t)) \leq V(t, x(t))t^{\alpha-\beta}, \tag{15}$$

and

$$D_{0_+}^{\alpha-1}x(0_+) + I_{0_+}^{2-\alpha}x(0_+) + I_{0_+}^{1-\beta}x(0_+) \leq V(0, x(0)) = 0, \tag{16}$$

then the solution of equations (1)-(4) is Mittag-Leffler stable.

Proof.

Now, taking the Laplace transform of both sides of equation (1) and making use of the boundary conditions (2) - (4), we have

$$\mathcal{L}\left\{D_{0_+}^{\alpha}x(t) + k\ D_{0_+}^{\beta}x(t) + g(t, x(t)) = h(t)\right\}$$

$$\Longrightarrow s^{\alpha}X(s) - D_{0_+}^{\alpha}x(0_+) - sI_{0_+}^{2-\alpha}x(0_+) + ks^{\beta}X(s) - kI_{0_+}^{1-\beta}x(0_+)) + G(s, X(s)) = H(s)$$

$$\Longrightarrow (s^{\alpha} + ks^{\beta})X(s) = D_{0_+}^{\alpha-1}x(0_+) + sI_{0_+}^{2-\alpha}x(0_+) + kI_{0_+}^{1-\beta}x(0_+) + H(s) - G(s, X(s))$$

$$\Longrightarrow X(s) = \frac{1}{(s^{\alpha} + ks^{\beta})}\left[D_{0_+}^{\alpha-1}x(0_+) + sI_{0_+}^{2-\alpha}x(0_+) + kI_{0_+}^{1-\beta}x(0_+))\right] + \frac{1}{(s^{\alpha} + ks^{\beta})}[H(s) - G(s, X(s))]$$

$$\Longrightarrow X(s) = \frac{s^{-\beta}}{s^{\alpha-\beta} + k}D_{0_+}^{\alpha-1}x(0_+) + \frac{s^{1-\beta}}{s^{\alpha-\beta} + k}I_{0_+}^{2-\alpha}x(0_+)$$

$$+\frac{s^{-\beta}}{s^{\alpha-\beta}+k}I_{0_+}^{1-\beta}x(0_+)+\frac{H(s)-G(s,X(s))s^{-\beta}}{s^{\alpha-\beta}+k}$$

On taking the inverse Laplace transform of the preceding, we obtain that

$$x(t)=t^{\alpha-1}E_{\alpha-\beta,\alpha}(-kt^{\alpha-\beta})D_{0_+}^{\alpha-1}x(0_+)$$

$$+t^{\alpha-2}E_{\alpha-\beta,\alpha-1}(-kt^{\alpha-\beta})I_{0_+}^{2-\alpha}x(0_+)+t^{\alpha-1}E_{\alpha-\beta,\alpha}(-kt^{\alpha-\beta})I_{0_+}^{1-\beta}x(0_+)$$

$$+\int_0^t(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}[-k(t-s)^{\alpha-\beta}](h(s)-g(s,x(s))ds$$

$$\Longrightarrow |x(t)|t^{2-\alpha}\le tE_{\alpha-\beta,\alpha}(-kt^{\alpha-\beta})|D_{0_+}^{\alpha-1}x(0_+)|$$

$$+E_{\alpha-\beta,\alpha-1}(-kt^{\alpha-\beta})|I_{0_+}^{2-\alpha}x(0_+)|+tE_{\alpha-\beta,\alpha}(-kt^{\alpha-\beta})|I_{0_+}^{1-\beta}x(0_+)|$$

$$+t^{2-\alpha}\int_0^t(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}[-k(t-s)^{\alpha-\beta}]|(h(s)-g(s,x(s))|ds$$

$$\le tE_{\alpha-\beta,\alpha}(-kt^{\alpha-\beta})|D_{0_+}^{\alpha-1}x(0_+)|+E_{\alpha-\beta,\alpha-1}(-kt^{\alpha-\beta})|I_{0_+}^{2-\alpha}x(0_+)|+$$

$$tE_{\alpha-\beta,\alpha}(-kt^{\alpha-\beta})|I_{0_+}^{1-\beta}x(0_+)|$$

$$+t^{2-\alpha}\int_0^t(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}[-k(t-s)^{\alpha-\beta}]s^{\alpha-\beta}V(s,x(s))ds$$

But,

$$\int_0^t(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}[-k(t-s)^{\alpha-\beta}]s^{\alpha-\beta}ds$$

$$=\sum_{n=0}^{\infty}\frac{-k^n}{\Gamma[\alpha(n+1)-\beta n]}\int_0^t(t-s)^{\alpha(1+n)-\beta n-1}s^{\alpha-\beta}ds$$

By making change of variable $s=tv$, we have that

$$\int_0^t(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}[-k(t-s)^{\alpha-\beta}]s^{\alpha-\beta}ds$$

$$=\sum_{n=0}^{\infty}\frac{-k^n}{\Gamma[\alpha(n+1)-\beta n]}\int_0^1 t^{\alpha(2+n)-\beta(n+1)}(1-v)^{\alpha(1+n)-\beta n-1}v^{\alpha-\beta}dv$$

$$=\sum_{n=0}^{\infty}\frac{-k^n}{\Gamma[\alpha(n+1)-\beta n]}\frac{t^{\alpha(2+n)-\beta(n+1)}\Gamma[(\alpha(1+n)-\beta n]\Gamma(\alpha-\beta+1)}{\Gamma[\alpha(2+n)-\beta(n+1)+1]}$$

$$=\sum_{n=0}^{\infty}\frac{-k^nt^{\alpha(2+n)-\beta(n+1)}\Gamma(\alpha-\beta+1)}{\Gamma[\alpha(2+n)-\beta(n+1)+1]}$$

$$=\Gamma(\alpha-\beta+1)t^{2\alpha-\beta}E_{\alpha-\beta,2\alpha-\beta+1}(-kt^{\alpha-\beta}).$$

Thus,

$$|x(t)|t^{2-\alpha}\le tE_{\alpha-\beta,\alpha}(-kt^{\alpha-\beta})|D_{0_+}^{\alpha-1}x(0_+)|$$

$$+E_{\alpha-\beta,\alpha-1}(-kt^{\alpha-\beta})|I_{0_+}^{2-\alpha}x(0_+)|+tE_{\alpha-\beta,\alpha}(-kt^{\alpha-\beta})|I_{0_+}^{1-\beta}x(0_+)|$$

$$+V(0,x(0))t^{2+\alpha-\beta}\Gamma(\alpha-\beta+1)E_{\alpha-\beta,2\alpha-\beta+1}(-kt^{\alpha-\beta}).$$

From the statement of the theorem, we have that

$$|x(t)|t^{2-\alpha}\le bE_{\alpha-\beta,\alpha}(-kt^{\alpha-\beta})V(0,x(0))$$

$$+V(0,x(0))b^{2+\alpha-\beta}\Gamma(\alpha-\beta+1)E_{\alpha-\beta,2\alpha-\beta+1}(-kt^{\alpha-\beta}).$$

$$\Longrightarrow \|x(t)\|\le V(0,x(0))b^{2+\alpha-\beta}\Gamma(\alpha-\beta+1)E_{\alpha-\beta}(-kt^{\alpha-\beta}).$$

Therefore, the solution of (1)-(4) is Mittag-Leffler stable.

Next, by using Lyapunov-like direct method, we shall establish that an equilibrium solution of (1) - (4) is asymptotically stable about an equilibrium point. First, we consider the comparison theory below.

Theorem 3.2.(Comparison Theorem) Assume that $x \in W^{\alpha,\beta}[a,b]$ satisfies

$$D^{\alpha}_{a_+}x(t) + k\ D^{\beta}_{a_+}x(t) \leq 0, \tag{17}$$

and

$$D^{\alpha-1}_{a_+}x(a_+), I^{2-\alpha}_{a_+}x(a_+), I^{1-\beta}_{a_+}x(a_+) \leq 0, \tag{18}$$

then $x(t) \leq 0,\ \forall t \in [a,b]$.

Proof. Suppose the conclusion of our theorem is false, then there exists some $t_1, t_2 \in [a,b]$ such that $x(t) \leq 0,\ \forall t \in [a,t_1]$ and $x(t) > 0,\ \forall t \in (t_1,t_2]$. Let

$$x(t_0) = \max_{t_1 \leq t \leq t_2} x(t).$$

Operating $I^{\alpha}_{a_+}$ to both sides of (15), we have

$$I^{\alpha}_{a_+}(D^{\alpha}_{a_+}x(t) + k\ D^{\beta}_{a_+}x(t)) \leq 0$$
$$\Longrightarrow x(t) - \frac{(t-a)^{\alpha-1}D^{\alpha-1}_{a_+}x(a_+)}{\Gamma(\alpha)} - \frac{(t-a)^{\alpha-2}I^{2-\alpha}_{a_+}x(a_+)}{\Gamma(\alpha-1)} + kI^{\alpha-\beta}_{a_+}x(t)$$
$$-\frac{k(t-a)^{\alpha-1}I^{1-\beta}_{a_+}x(a_+)}{\Gamma(\alpha)} \leq 0,$$

then from equation (16), we have that

$$x(t) + kI^{\alpha-\beta}_{a_+}x(t) \leq 0.$$

So,

$$x(t_0) + kI^{\alpha-\beta}_{a_+}x(t_0) \leq 0$$

But,

$$I^{\alpha-\beta}_{a_+}x(t_0) = \frac{1}{\Gamma(\alpha-\beta)}\int_a^{t_0}(t_0-s)^{\alpha-\beta-1}x(s)ds$$
$$= \frac{1}{\Gamma(\alpha-\beta)}\int_a^{t_1}(t_1-s)^{\alpha-\beta-1}x(s)ds + \frac{1}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_0}(t_0-s)^{\alpha-\beta-1}x(s)ds$$
$$\geq \frac{1}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_0}(t_0-s)^{\alpha-\beta-1}x(s)ds$$
$$\geq \frac{1}{\Gamma(\alpha-\beta)}\int_{t_1}^{t_0}(t_0-s)^{\alpha-\beta-1}x(t_0)ds$$
$$= \frac{x(t_0)(t_0-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}$$

Thus,

$$x(t_0) + \frac{kx(t_0)(t_0-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \leq x(t_0) + kI^{\alpha-\beta}_{a_+}x(t_0) \leq 0$$
$$\Longrightarrow x(t_0)(1 + \frac{k(t_0-t_1)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)})$$
$$\Longrightarrow x(t_0) \leq 0,$$

which is a contradiction. Therefore,

$$x(t) \leq 0,\ \forall t \in [a,b]$$

Theorem 3.3 Let $x = 0$ be an equilibrium point for the fractional differential equation (1) - (4), and let D be a domain in $W^{\alpha,\beta}[a,b]$ that contains $x = 0$. Suppose that there exists a continuous function $V \in D$ such that $V(t, x(t)) : [0,\infty) \times D \longrightarrow \mathbb{R}$, and a class of K functions $\lambda_i, (i = 1, 2, 3)$ such that

$$\lambda_1(\|x\|) \leq t^{2-\alpha} V(t, x(t)) \leq \lambda_2(\|x\|), \tag{19}$$

$$D^{\alpha}_{0_+} V(t, x(t)) + D^{\beta}_{0_+} V(t, x(t)) \leq -\lambda_3(\|x\|), \tag{20}$$

$\forall t > 0$ and $\forall x \in D$. Then $x = 0$ is uniformly asymptotically stable.

Proof: First we pick an open ball $B_r(x)$ in the domain D with centre at x and radius $r > 0$. We denote δ to be

$$\delta = \min_{\|x\|=r} \lambda_1(\|x\|),$$

then for any $\theta \in (0, \delta)$, let $\Omega_\theta = \left\{x \in B_r(x) : t^{2-\alpha} V(t, x(t)) \leq \theta\right\}$.
Now, since $t^{2-\alpha} V(t, x(t)) \leq \theta$, then it follows from (17) that $\lambda_1(\|x\|) \leq \theta$.
By implication, Ω_θ is a subset of $\{x \in B_r(x) : \lambda_1(\|x\|) \leq \theta\}$.

Similarly, it can be shown that Ω_θ contains the set

$$\{x \in B_r(x) : \lambda_2(\|x\|) \leq \theta\}.$$

Next, since $V(t, x(t))$ is a monotone decreasing function by theorem 3.2 and (18), it follows that any solution starting in Ω_θ for any initial time $t \geq t_0$ stays in Ω_θ for all future time. Therefore any trajectory starting in $\{x \in B_r(x) : \lambda_2(\|x\|) \leq \theta\}$ remains in Ω_θ, and also in the set $\{x \in B_r(x) : \lambda_1(\|x\|)\}$, for all $t \geq t_0$. From our assumptions (17) and (18), it follows that

$$D^{\alpha}_{0_+} V(t, x(t)) + D^{\beta}_{0_+} V(t, x(t)) \leq -\lambda V(t, x(t), \tag{21}$$

where $\lambda = \frac{\lambda_3 t^{2-\alpha}}{\lambda_2}$

Operating $I^{\alpha}_{0_+}$ to both sides of (19), we have

$$I^{\alpha}_{0_+}\left(D^{\alpha}_{0_+} V(t, x(t)) + D^{\beta}_{0_+} V(t, x(t)) \leq -\lambda V(t, x(t)\right)$$

$$\Longrightarrow V(t, x(t)) \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} D^{\alpha-1}_{0_+} V(t, x(t)) \mid_{t=0} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} I^{2-\alpha}_{0_+} V(t, x(t)) \mid_{t=0}$$

$$+ k\frac{t^{\alpha-1}}{\Gamma(\alpha)} I^{1-\beta}_{0_+} V(t, x(t)) \mid_{t=0} - kI^{\alpha-\beta}_{0_+} V(t, x(t)) - I^{\alpha}_{0_+}(\lambda V(t, x(t)))$$

$$\leq \frac{t^{\alpha-1}}{\Gamma(\alpha)} D^{\alpha-1}_{0_+} V(t, x(t)) \mid_{t=0} + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} I^{2-\alpha}_{0_+} V(t, x(t)) \mid_{t=0}$$

$$+ k\frac{t^{\alpha-1}}{\Gamma(\alpha)} I^{1-\beta}_{0_+} V(t, x(t)) \mid_{t=0}$$

$$+ \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} V(s, x(s)) \left[-k - \lambda(t-s)^{\beta}\right] ds.$$

This implies that

$$t^{2-\alpha} V(t, x(t)) \leq \frac{t}{\Gamma(\alpha)} D^{\alpha-1}_{0_+} V(t, x(t)) \mid_{t=0} + \frac{I^{2-\alpha}_{0_+} V(t, x(t)) \mid_{t=0}}{\Gamma(\alpha-1)}$$

$$+ \frac{kt}{\Gamma(\alpha)} I^{1-\beta}_{0_+} V(t, x(t)) \mid_{t=0}$$

$$+\frac{t^{2-\alpha}}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}V(s,x(s))\left[-k(t-s)^{-\beta}-s\right]ds.$$

Set $h(t)=t^{2-\alpha}V(t,x(t)),\ h_0=\frac{I_{0_+}^{2-\alpha}V(t,x(t))|_{t=0}}{\Gamma(\alpha-1)},\ h_1=\frac{D_{0_+}^{\alpha-1}V(t,x(t))|_{t=0}}{\Gamma(\alpha)}+\frac{kI_{0_+}^{1-\beta}V(t,x(t))|_{t=0}}{\Gamma(\alpha)}$ and $\mu=k+\lambda(t-s)^{\beta}$, then we have that

$$h(t)\le h_0+th_1+\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}\left(-\mu h(s)\right)ds.$$

It is worthy of observation that μ is a strictly increasing function and h_0,h_1 are nonnegative functions, then by lemma 2.10. we have that

$$\begin{aligned} h(t)&\le G^{-1}\left[G(h_0+h_1t+\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}ds\right]\\ &=G^{-1}\left[G(h_0+h_1t+\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right], \end{aligned}$$

where $G(h)=-\int_g^h\frac{d\tau}{\mu(\tau)},\ 0<g<r$.
The function $G(h)$ is a strictly decreasing differentiable function on $(0,r)$. In addition, for all $t\ge 0,\ h(t)$ is monotonously nonincreasing function whenever $h(t)>0$ and $\lim_{h\longrightarrow 0}G(h)=\infty$.

We suppose that $a=\lim_{h\longrightarrow r}\int_g^h\frac{d\tau}{\mu(\tau)}$, then the function $G(h)\in(-a,\infty)$. Since G is strictly decreasing, its inverse G^{-1} is defined on $(-a,\infty)$. We define a function $\nu(p,q)$ by

$$\nu(p,q)=\begin{cases} G^{-1}\left[G(p+h_1q)+\frac{q^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right] & p>0\\ 0 & p=0\end{cases}$$

Then from the above, we have that

$$h(t)\le\nu(h_0,t)$$

for all $t\ge 0$.

We show that ν is a $\mathcal{KL}$ function. Observe that G and G^{-1} are continuous on $(-a,\infty)$.

ν is strictly increasing with respect to p for each q, since

$$\frac{\partial\nu(p,q)}{\partial p}=\frac{\mu(\nu(p,q))}{\mu(p+h_1q)}>0.$$

Also, whenever $q^{\alpha-\beta-1}>\frac{h_1\Gamma(\alpha-\beta)}{\mu(p+h_1q)}$ and for a fixed p, ν is strictly decreasing with respect to q, since

$$\frac{\partial\nu(p,q)}{\partial q}=-\mu(\nu(p,q))\left[\frac{q^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}-\frac{h_1}{\mu(p+h_1q)}\right]$$

. Finally, $\nu(p,q)\longrightarrow 0$, as $q\longrightarrow\infty$. Hence, ν is a $\mathcal{KL}$ function. Following the above process, we have that

$$t^{2-\alpha}V(t,x(t))\le\nu(\frac{I_{0_+}^{2-\alpha}V(t,x(t))\mid_{t=0}}{\Gamma(\alpha-1)},t)$$

Hence any solution starting in $\{x\in B_r(x):\lambda_2(\|x\|)\le\theta\}$ must satisfy the inequality

$$\|x\|\le\lambda_1t^{2-\alpha}V(t,x(t))$$

$$\lambda_1\nu(t^{2-\alpha}\|x(0)\|,t)$$
$$:=\gamma(\|x(0)\|,t),$$

where γ belongs to a class of $\mathcal{KL}$ functions. Therefore, whenever $t\longrightarrow\infty$, $\|x(t)\|\longrightarrow 0$. This implies that the zero solution is asymptotically stable.

Corollary 3.4. Let $x=0$ be an equilibrium point for the fractional differential equation (1)-(4) and let $D\subset W^{\alpha,\beta}[a,b]$ be a domain that contains $x=0$. Suppose that there exists a continuous function $V(t,x):[0,\infty)\times D\longrightarrow\mathbb{R}$ such that

$$t^{1-\beta}(\|x\|)\le t^{2-\alpha}V(t,x(t))\le t^{3-\alpha}(\|x\|), \tag{22}$$

$$D^{\alpha}_{0_+}V(t,x(t))+D^{\beta}_{0_+}V(t,x(t))\le -t(\|x\|), \tag{23}$$

then the equilibrium point $x=0$ is uniformly asymptotically stable.

Proof. From the proof of theorem 3.3, we have that

$$t^{2-\alpha}V(t,x(t))\le\frac{t}{\Gamma(\alpha)}D^{\alpha-1}_{0_+}V(t,x(t))\mid_{t=0}+\frac{I^{2-\alpha}_{0_+}V(t,x(t))\mid_{t=0}}{\Gamma(\alpha-1)}$$
$$+\frac{kt}{\Gamma(\alpha)}I^{1-\beta}_{0_+}V(t,x(t))\mid_{t=0}-\frac{kt^{2-\alpha}}{\Gamma(\alpha-\beta)}\int_a^t(t-s)^{\alpha-\beta-1}V(s,x(s))ds$$
$$-\frac{t^{2-\alpha}}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}V(s,x(s))ds$$
$$\le\frac{t}{\Gamma(\alpha)}D^{\alpha-1}_{0_+}V(t,x(t))\mid_{t=0}+\frac{I^{2-\alpha}_{0_+}V(t,x(t))\mid_{t=0}}{\Gamma(\alpha-1)}$$
$$+\frac{kt}{\Gamma(\alpha)}I^{1-\beta}_{0_+}V(t,x(t))\mid_{t=0}$$
$$-\frac{t^{2-\alpha}}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}V(s,x(s))ds.$$

Then by lemma 2.11 , we have that

$$t^{2-\alpha}V(t,x(t))\le\left(\frac{I^{2-\alpha}_{0_+}V(t,x(t))\mid_{t=0}}{\Gamma(\alpha-1)}\right)E_\alpha(-\Gamma(\alpha)t^\alpha)$$
$$+\left(\frac{t}{\Gamma(\alpha)}D^{\alpha-1}_{0_+}V(t,x(t))\mid_{t=0}+\frac{kt}{\Gamma(\alpha)}I^{1-\beta}_{0_+}V(t,x(t))\mid_{t=0}\right)E_\alpha(-\Gamma(\alpha)t^\alpha)$$
$$\Longrightarrow t^{2-\alpha}V(t,x(t))\le\left(\frac{I^{2-\alpha}_{0_+}V(t,x(t))\mid_{t=0}}{\Gamma(\alpha-1)}\right)e^{-\Gamma(\alpha)t^\alpha}$$
$$+\left(\frac{t}{\Gamma(\alpha)}D^{\alpha-1}_{0_+}V(t,x(t))\mid_{t=0}+\frac{kt}{\Gamma(\alpha)}I^{1-\beta}_{0_+}V(t,x(t))\mid_{t=0}\right)e^{-\Gamma(\alpha)t^\alpha}$$
$$:=\rho(\frac{I^{2-\alpha}_{0_+}V(t,x(t))\mid_{t=0}}{\Gamma(\alpha-1)},t)$$

Thus,

$$t^{2-\alpha}V(t,x(t))\le\rho(\frac{I^{2-\alpha}_{0_+}V(t,x(t))\mid_{t=0}}{\Gamma(\alpha-1)},t) \tag{24}$$

It is easy to see that ρ is in the class of $\mathcal{KL}$ functions. To show this, we set $a_0 = \frac{I_{0_+}^{2-\alpha}V(t,x(t))|_{t=0}}{\Gamma(\alpha-1)}$, $a_1 = \frac{1}{\Gamma(\alpha)}D_{0_+}^{\alpha-1}V(t,x(t))\ |_{t=0}$, $a_2 = \frac{k}{\Gamma(\alpha)}I_{0_+}^{1-\beta}V(t,x(t))\ |_{t=0}$ Now for each fixed t, we have that

$$\frac{\partial\rho(\frac{I_{0_+}^{2-\alpha}V(t,x(t))|_{t=0}}{\Gamma(\alpha-1)},t)}{\partial a_0} = e^{-\Gamma(\alpha)t^\alpha} > 0.$$

Equally for a fixed a_0, we have that

$$\begin{aligned}\frac{\partial\rho(\frac{I_{0_+}^{2-\alpha}V(t,x(t))|_{t=0}}{\Gamma(\alpha-1)},t)}{\partial t} &= -\Gamma(\alpha+1)t^{\alpha-1}e^{-\Gamma(\alpha)t^\alpha}\,(a_1t + a_2t + \ a_0)\\ &\quad + (a_1+a_2)\,e^{-\Gamma(\alpha)t^\alpha} < 0.,\end{aligned}$$

for all t positive. Thus, ρ is a strictly decreasing function with respect to any $t > 0$. Also, $\rho \longrightarrow 0$ as $t \longrightarrow \infty$. Therefore, ρ is a $\mathcal{KL}$ function. It follows from (22) and (24) that

$$\|x\| \le \Phi(\frac{I_{0_+}^{2-\alpha}V(t,x(t))\ |_{t=0}}{\Gamma(\alpha-1)},t)).$$

Finally, $\|x\| \longrightarrow 0$ as $t \longrightarrow \infty$. Thus, $x = 0$ is uniformly asymptotically stable.

4. Conclusion

We have succeeded in establishing some stability results for a class of nonlinear boundary value fractional differential equations in a Sobolev space.

Acknowledgement

The author is grateful to the anonymous referees for carefully checking of the details and for helpful comments that improved this paper.

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