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ON A NEW SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING KOMATU INTEGRAL OPERATOR

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Abstract. The object of the paper is to study some properties for $K_c^\delta f(z)$ belonging to some class by applying Jack's lemma.

1. Introduction

Let A be denote the class of all analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \; n \in N = \{1, 2, 3, \cdots\} \tag{1}$$

which are analytic in the punctured unit disc $E = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Let S denote the subclass of A which consists of functions of the form (1) that are univalent and normalized by conditions $f(0) = 0$ and $f'(0) = 1$ in E.

Recently Komatu [6] introduced a certain integral operator $K_c^\delta f(z)$

$$K_c^\delta f(z) = \frac{c^\delta}{\Gamma(\delta)} \int_0^1 t^{c-2} \left(log \frac{1}{t} \right)^{c-1} f(tz) dt, \tag{2}$$

$c > 0, \delta \geq 0$ and $z \in E$.

Thus, if $f \in A$ is of the form (1) then it is easily seen from (2) that

$$K_c^\delta f(z) = z + \sum_{n=2}^{\infty} \left(\frac{c}{c+n-1} \right)^\delta a_n z^n, a > 0, \delta \geq 0. \tag{3}$$

We note that

(i). for $c = 1$ and $\delta = k$(k is any integer), the multiple transformation $K_1^\delta f(z) = I^k f(z)$ was studied by Flett [1].

(ii). for $c = 1$ and $\delta = -k (k \in \mathbb{N}_0)$, the differential operator $K_1^{-k} f(z) = D^k f(z)$ was studied by Salagean [7].

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(iii). for $c = 2$ and $\delta = k$(k is any integer), the operators $K_2^k f(z) = K^k f(z)$ was studied by Uralegaddi and Somanatha [9].

(iv). for $c = 2$, the multiple transformation $K_2^\delta f(z) = K^\delta f(z)$ was studied by Jung et al. [3].

In the following definition, we introduce a new class of analytic functions containing a integral operator of equation (3).

Definition 1.1. Let a function $f \in A$. Then $f \in K_c^\delta f(z)$ if and only if

$$Re\left\{\frac{z\left(K_c^\delta f(z)\right)'}{K_c^\delta f(z)}\right\} > \beta, z \in E, 0 \leq \beta \leq 1. \tag{4}$$

Let f and g be analytic in E. Then f is said to be subordinate to g if there exists an analytic function ω satisfying $\omega(0) = 0$ and $\omega(z) < 1$, such that $f(z) = g(\omega z), z \in E$. We denote this subordination as $f(z) \prec g(z)$ or $(f \prec g), z \in E$.

The basic idea in proving our result is the following lemma due to Jack [2] (also, due to Miller and Mocannu [4])

Lemma 1.2. *Let $\omega(z)$ be analytic in E with $\omega(0) = 0$. Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0]inE$ then we have $z_0\omega'(z) = k\omega(z_0)$, where $k \geq 1$ is a real number.*

2. Main Results

In the present paper, we follow similar works done by Shireishi and Owa [8] and Ochiai et al. [5], we derive the following result.

Theorem 2.1. *If $f \in A$ satisfies*

$$Re\left\{\frac{z\left(K_c^\delta f(z)\right)'}{K_c^\delta f(z)}\right\} < \frac{\beta - 3}{2(\beta - 1)},\ z \in E$$

for some $\beta(-1 < \beta \leq 0)$ then

$$\frac{K_c^\delta f(z)}{z} \prec \frac{1 + \beta z}{1 - z},\ z \in E.$$

This implies that

$$Re\left\{\frac{K_c^\delta f(z)}{z}\right\} > \frac{1 - \beta}{2}$$

Proof. Let us define the function $\omega(z)$ by

$$\frac{K_c^\delta f(z)}{z} = \frac{1 - \beta\omega(z)}{1 - \omega(z)},\ (\omega(z) \neq 1).$$

Clearly, $\omega(z)$ is analytic in E and $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$ in E. Since

$$\frac{z\left(K_c^\delta f(z)\right)'}{K_c^\delta f(z)} = \frac{-\beta z\omega'(z)}{1 - \beta\omega(z)} + \frac{z\omega'(z)}{1 - \omega(z)} + 1,$$

we see that

$$Re\left\{\frac{z\left(K_c^\delta f(z)\right)'}{K_c^\delta f(z)}\right\} = Re\left\{\frac{-\beta z\omega'(z)}{1 - \beta\omega(z)} + \frac{z\omega'(z)}{1 - \omega(z)} + 1\right\}$$
$$< \frac{\beta - 3}{2(\beta - 1)},\ (z \in E)$$

for $-1 < \beta \leq 0$. If there exists a point $z_0 \in E$ such that

$$\max_{|z|\leq|z_0|} |\omega(z)| = |\omega(z_0)| = 1,$$

then Lemma 1.2, gives us that $\omega(z_0) = e^{i\theta}$ and $z_0\omega'(z_0) = k\omega(z_0), k \geq 1$.

Thus we have

$$\begin{aligned}\frac{z_0\left(K_c^\delta f(z_0)\right)'}{K_c^\delta f(z_0)} &= \frac{-\beta z_0\omega'(z_0)}{1-\beta\omega(z_0)} + \frac{z_0\omega'(z_0)}{1-\omega(z_0)} + 1\\ &= 1 + \frac{k}{1-e^{i\theta}} - \frac{k}{1-\beta e^{i\theta}}.\end{aligned}$$

It follows that

$$Re\left\{\frac{1}{1-\omega(z_0)}\right\} = Re\left\{\frac{1}{1-e^{i\theta}}\right\} = \frac{1}{2}$$

$$\text{and } Re\left\{\frac{1}{1-\beta\omega(z_0)}\right\} = Re\left\{\frac{1}{1-\beta e^{i\theta}}\right\} = \frac{1}{2} - \frac{1-\beta^2}{2(1+\beta^2-2\beta cos\theta)}.$$

Therefore, we have

$$Re\left\{\frac{z_0\left(K_c^\delta f(z_0)\right)'}{K_c^\delta f(z_0)}\right\} = 1 - \frac{k(\beta^2-1)}{2(1+\beta^2-2\beta cos\theta)}.$$

This implies that $-1 < \beta \leq 0$,

$$Re\left\{\frac{z_0\left(K_c^\delta f(z_0)\right)'}{K_c^\delta f(z_0)}\right\} \geq 1 + \frac{(1-\beta^2)}{2(\beta-1)^2} = \frac{\beta-3}{2(\beta-1)}.$$

This contradicts the condition in the theorem. Then there is no $z_0 \in E$ such that $|\omega(z_0)| = 1$ for all $z \in E$, that is

$$\frac{K_c^\delta f(z)}{z} < \frac{1+\beta z}{1-z},\ z \in E.$$

Further more, since

$$\omega(z) = \frac{\frac{K_c^\delta f(z)}{z} - 1}{\frac{K_c^\delta f(z)}{z} - \beta}, z \in E$$

and $|\omega(z)| < 1,\ (z \in E)$, we conclude that

$$Re\left\{\frac{K_c^\delta f(z)}{z}\right\} > \frac{1-\beta}{2}.$$

□

Taking $\beta = 0$ in the Theorem 2.1, we have the following corollary.

Corollary 2.2. *If $f \in A$ satisfies*

$$Re\left\{\frac{z(K_c^\delta f(z))'}{K_c^\delta f(z)}\right\} > \frac{3}{2},\ z \in E$$

then

$$\frac{K_c^\delta f(z}{z} \prec \frac{1}{1-z},\ z \in E$$

and

$$Re\left\{\frac{K_c^\delta f(z)}{z}\right\} > \frac{1}{2},\ z \in E$$

Theorem 2.3. *If $f \in A$ satisfies*
$$Re\left\{\frac{z(K_c^\delta f(z))'}{K_c^\delta f(z)}\right\} > \frac{3\beta-1}{2(\beta-1)},\ z \in E$$
for some $\beta(-1 < \beta \leq 0)$ then
$$\frac{z}{K_c^\delta f(z)} \prec \frac{1+z}{1-z},\ z \in E$$
and
$$\left|\frac{K_c^\delta f(z)}{z} - \frac{1}{1-\beta}\right| < \frac{1}{1-\beta},\ z \in E.$$
This implies that $Re\left\{\frac{K_c^\delta f(z)}{z}\right\} > 0,\ z \in E$.

Proof. Let us define the function $\omega(z)$ by
$$\frac{z}{K_c^\delta f(z)} = \frac{1-\beta\omega(z)}{1-\omega(z)},\ \omega(z) \neq 1. \tag{5}$$
Then, we have $\omega(z)$ is analytic in E and $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$ in E. Differenting equation (5), we obtain
$$\frac{z(K_c^\delta f(z))'}{K_c^\delta f(z)} = \frac{-z\omega'(z)}{1-\omega(z)} + \frac{\beta z\omega'(z)}{1-\beta\omega(z)} + 1$$
$$\Rightarrow Re\left\{\frac{z(K_c^\delta f(z))'}{K_c^\delta f(z)}\right\} = Re\left\{\frac{-z\omega'(z)}{1-\omega(z)} + \frac{\alpha z\omega'(z)}{1-\beta\omega(z)} + 1\right\}$$
$$> \frac{3\beta-1}{2(\beta-1)},\ z \in E,$$
for $(-1 < \beta \leq 0)$. If there exists a point $(z_0 \in E)$ such that Lemma 1.2, gives us that $\omega(z_0) = e^{i\theta}$ and $z_0\omega'(z_0) = k\omega(z_0), k \geq 1$. Thus we have
$$\frac{z_0(K_c^\delta f(z_0))'}{K_c^\delta f(z_0)} = \frac{-z_0\omega'(z_0)}{1-\omega(z_0)} + \frac{\beta z_0\omega'(z_0)}{1-\beta\omega(z_0)} + 1$$
$$= 1 - \frac{k}{1-e^{i\theta}} + \frac{k}{1-\beta e^{i\theta}}.$$
Therefore, we have
$$Re\left\{\frac{z_0(K_c^\delta f(z_0))'}{K_c^\delta f(z_0)}\right\} = 1 + \frac{k(\beta^2-1)}{2(1+\beta^2-2\beta cos\theta)}.$$
This implies that, for $-1 < \alpha \leq 0$,
$$Re\left\{\frac{z_0(K_c^\delta f(z_0))'}{K_c^\delta f(z_0)}\right\} = 1 - \frac{k(1-\alpha^2)}{2(1+\alpha^2-2\alpha cos\theta)}$$
$$\leq \frac{3\alpha-1}{2(\alpha-1)}.$$
This contradicts the condition in the theorem.

Hence, there is no $z_0 \in E$ such that $|\omega(z_0)| = 1$ for all $z \in E$, that is
$$\frac{z}{K_c^\delta f(z)} \prec \frac{1+z}{1-z},\ z \in E.$$

Furthermore, since

$$\omega(z) = \frac{1 - \frac{K_c^\delta f(z)}{z}}{1 - \frac{\beta K_c^\delta f(z)}{z}},\ z \in E$$

and $|\omega(z)| < 1, (z \in E)$ we conclude that

$$\left| \frac{K_c^\delta f(z)}{z} - \frac{1}{1-\beta} \right| < \frac{1}{1-\beta},\ z \in E$$

which implies that

$$Re\left\{ \frac{K_c^\delta f(z)}{z} \right\} > 0,\ z \in E.$$

We complete the proof of the theorem. □

By setting $\beta = 0$ in Theorem 2.3, we readily obtain the following

Corollary 2.4. *If $f \in A$ satisfies*

$$Re\left\{ \frac{z\left(K_c^\delta f(z)\right)'}{K_c^\delta f(z)} \right\} > \frac{1}{2}, z \in E$$

then

$$\frac{z}{K_c^\delta f(z)} \prec \frac{1+z}{1-z},\ z \in E$$

and

$$\left| \frac{K_c^\delta f(z)}{z} - 1 \right| < 1,\ z \in E.$$

Theorem 2.5. *If $f \in A$ satisfies*

$$Re\left\{ \frac{z\left(K_c^\delta f(z)\right)'}{K_c^\delta f(z)} \right\} < \frac{\beta(2-\gamma) - (2+\gamma)}{2(\beta - 1)}, z \in E$$

for some β $(-1 < \beta \leq 0$ and $0 < \beta \leq 1$ then

$$\left(\frac{K_c^\delta f(z)}{z} \right)^{\frac{1}{\gamma}} \prec \frac{1+\beta z}{1-z},\ z \in E.$$

Then implies that

$$\left(\frac{K_c^\delta f(z)}{z} \right)^{\frac{1}{\gamma}} > \frac{1-\beta}{2},\ z \in E.$$

Proof. Let us define the function $\omega(z)$ by

$$\frac{K_c^\delta f(z)}{z} = \left(\frac{1 - \beta\omega(z)}{1 - \omega(z)} \right)^\gamma,\ \omega(z) \neq 1.$$

Clearly, $\omega(z)$ is analytic in E and $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$ in E. Since

$$\frac{z\left(K_c^\delta f(z)\right)'}{K_c^\delta f(z)} = \gamma \left(\frac{z\omega'(z)}{1-\omega(z)} - \frac{\beta z\omega'(z)}{1 - \beta\omega(z)} \right) + 1.$$

We see that

$$Re\left\{\frac{z\left(K_c^\delta f(z)\right)'}{K_c^\delta f(z)}\right\} = Re\left\{\gamma\left(\frac{z\omega'(z)}{1-\omega(z)} - \frac{\beta z\omega'(z)}{1-\beta\omega(z)}\right)+1\right\}$$
$$< \frac{\beta(2-\gamma)-(2+\gamma)}{2(\beta-1)},\ z\in E,$$

for $\beta(-1<\beta\leq 0)$ and $0<\gamma\leq 1$. If there exists a point $(z_0\in E)$ such that

$$\max_{|z|<|z_0|}|\omega(z)| = |\omega(z_0)| = 1$$

then by Lemma 1.2, gives us that $\omega(z_0)=e^{i\theta}$ and $z_0\omega'(z_0)=k\omega(z_0),\ k\geq 1$.

Thus we have

$$\frac{z_0\left(K_c^\delta f(z_0)\right)'}{K_c^\delta f(z_0)} = \gamma\left(\frac{z_0\omega'(z_0)}{1-\omega(z_0)} - \frac{\beta z_0\omega'(z_0)}{1-\beta\omega(z_0)}\right)+1$$
$$= 1 + \frac{k}{1-e^{i\theta}} - \frac{k}{1-\beta e^{i\theta}}.$$

Therefore, we have

$$Re\left\{\frac{z_0\left(K_c^\delta f(z_0)\right)'}{K_c^\delta f(z_0)}\right\} = 1 + \frac{\gamma k(1-\beta^2)}{2(1+\beta^2-2\beta cos\theta)}.$$

Thus implies that, for $\beta(-1<\beta\leq 0)$ and $0<\gamma\leq 1$

$$Re\left\{\frac{z_0\left(K_c^\delta f(z_0)\right)'}{K_c^\delta f(z_0)}\right\} \geq \frac{\beta(2-\gamma)-(2+\gamma)}{2(\beta-1)}.$$

This contradicts the condition in the theorem.

Hence, there is no $z_0\in E$ such that $|\omega(z_0)|=1$ for all $z\in E$, that is

$$\left(\frac{K_c^\delta f(z)}{z}\right)^{\frac{1}{\gamma}} \prec \frac{1-\beta z}{1-z},\ z\in E.$$

Furthermore, since

$$\omega(z) = \frac{\left(\frac{K_c^\delta f(z)}{z}\right)^{\frac{1}{\gamma}}-1}{\left(\frac{K_c^\delta f(z)}{z}\right)^{\frac{1}{\gamma}}-\beta}$$

and $|\omega(z)|<1,\ (z\in E)$, we conclude that

$$\left(\frac{K_c^\delta f(z)}{z}\right)^{\frac{1}{\gamma}} > \frac{1-\beta}{2},\ z\in E,$$

we complete the proof of the theorem. □

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