Journal of Fractional Calculus and Applications
Vol. 12(1) Jan. 2021, pp. 35-45.
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

EXISTENCE OF MILD SOLUTION FOR NONLINEAR HYBRID FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH CAPUTO DERIVATIVE

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Abstract. In this paper we prove an existence result for a boundary value problem of nonlinear hybrid fractional integrodifferential equation involving Caputo derivative. We employ a hybrid fixed point theorem of Dhage as a basic tool in the analysis of our problem. Finally, an example is given to illustrate the result.

1. Introduction

This note motivates some papers treating the fractional hybrid differential equations involving Riemann-Liouville differential operators of order $0 < \alpha < 1$.
In [18], Sitho et.al. discussed the following existence results for hybrid fractional integro-differential equations

$$\begin{cases} D^{\alpha}\left(\frac{x(t)-\sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t))}{f(t,x(t))}\right) = g(t,x(t)) \quad a.e. \quad t \in J = [0,T], \quad 0 < \alpha \leq 1 \\ x(0) = 0 \end{cases}$$

where D^{α} denotes the Riemann-Liouville fractional derivative of order α, $0 < \alpha \leq 1$, I^{ϕ} is the Riemann-Liouville fractional integral of order $\phi > 0$, $\phi \in \{\beta_1, \beta_2, ..., \beta_m\}$, $f \in C(J \times R, R \setminus \{0\})$, $g \in C(J \times R, R)$, with $h_i \in C(J \times R, R)$ with $h_i(0,0) = 0, i = 1, 2, \ldots, m$.
In [15], Khalid Hilal and Ahmed Kajouni considered boundary value problems for hybrid differential equations with fractional order (BVPHDEF of short) involving Caputo differential operators of order $0 < \alpha < 1$

$$\begin{cases} D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) \quad a.e. \quad t \in J = [0,T] \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c \end{cases}$$

2010 *Mathematics Subject Classification.* 34A12, 34A30, 34D20.
Key words and phrases. fractional integro-differential equation, mild result, hybrid fixed point theorem, Caputo derivative.
Submitted March 19, 2020. Revised April 15, 2020.

where $f \in C(J \times R, R\backslash\{0\})$, $g \in \mathcal{C}(J \times R, R)$ and a, b, c are real constants with $a + b \neq 0$.
Dhage and Lakshmikantham [5] discussed the following first order hybrid differential equation

$$\begin{cases} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t, x(t)) \quad a.e. \quad t \in J = [0, T] \\ \\ x(t_0) = x_0 \in R \end{cases}$$

where $f \in C(J \times R, R\backslash\{0\})$ and $g \in \mathcal{C}(J \times R, R)$. They established the existence, uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and a comparison results.
Zhao et.al. [19] are discussed the following fractional hybrid differential equations involving Riemann-Liouville differential operators

$$\begin{cases} D^q\left[\frac{x(t)}{f(t,x(t))}\right] = g(t, x(t)) \quad a.e. \quad t \in J = [0, T] \\ \\ x(0) = 0 \end{cases}$$

where $f \in C(J \times R, R\backslash\{0\})$ and $g \in \mathcal{C}(J \times R, R)$. They established the existence theorem for fractional hybrid differential equation, some fundamental differential inequalities are also established and the existence of extremal solutions.
Benchohra et al.[16] are discussed the following boundary value problems for differential equations with fractional order

$$\begin{cases} {}^cD^\alpha y(t) = f(t, y(t)), \quad for \quad each \quad t \in J = [0, T], \quad 0 < \alpha < 1 \\ ay(0) + by(T) = c \end{cases}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : [0, T] \times R \to R$, is a continuous function, a, b, c are real constants with $a + b \neq 0$.
Motivated by some recent studies on hybrid fractional differential equations see [15],[18], we consider the following boundary value problem problem :

$$\begin{cases} D^\alpha\left(\frac{x(t)-\sum_{i=1}^{m} I^{\beta_i} h_i(t,x(t))}{f(t,x(t))}\right) = g(t, x(t)) \quad a.e. \quad t \in J = [0, T], \quad 0 < \alpha < 1 \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c, \end{cases} \tag{1}$$

where D^α denotes the Caputo fractional derivative of order α, $0 < \alpha < 1$, I^ϕ is the Riemann-Liouville fractional integral of order $\phi > 0$, $\phi \in \{\beta_1, \beta_2, \ldots, \beta_m\}$, $f \in C(J \times R, R \setminus \{0\})$, $g \in \mathcal{C}(J \times R, R)$, a, b, c are real constants with $a + b \neq 0$, and $h_i \in C(J \times R, R)$ with $h_i(0, x(0)) = 0, i = 1, 2, \ldots, m$.
By a solution of the (1) we mean a function $x \in C(J, R)$ such that

(i) the function $t \longrightarrow \frac{x}{f(t,x)}$ is continuous for each $x \in R$, and
(ii) x satisfies the equations in (1).

This paper is arranged as follows. In Section 2, we recall some concepts and some fractional calculation law and establish preparation results. In Section 3, we study the existence of the boundary value problem (1), based on the Dhage fixed point theorem. In Section 4 an example is given to illustrate the result.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminaries facts which are used throughout this paper. By $E = C(J, R)$ we denote the Banach space of all continuous functions from $J = [0, T]$ into R with the norm

$$\|y\| = \sup\{|y(t)|, t \in J\}$$

The class $\mathcal{C}(J \times R, R)$ is called the Carathéodory class of functions on $J \times R$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on J .
By $L^1(J, R)$ denote the space of Lebesgue integrable real-valued functions on J equipped with the norm $\|.\|_{L^1}$ defined by
$\|x\|_{L^1} = \int_0^T |x(s)|ds$

Definition 1.([2]) The Riemann-Liouvelle fractional integral of the function $h \in L^1([a, b], R^+)$ of order $\alpha \in R^+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds$$

where Γ is the gamma function.

Definition 2.([2]) For a integral function h given on the interval $[a, b]$, the The Riemann-Liouville fractional-order derivative of h, is defined by

$$({}^cD_{a^+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)} (s)ds$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 3.([2]) For a function h given on the interval $[a, b]$, the Caputo fractional-order derivative of h, is defined by

$$({}^cD_{a^+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)} h^{(n)}(s)ds$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 1.([1]) Let $\alpha > 0$ and $x \in C(0, T) \cap L(0, T)$. Then the fractional differential equation

$$D^\alpha x(t) = 0$$

has a unique solution

$$x(t) = k_1 t^{\alpha-1} + k_2 t^{\alpha-2} + \ldots + k_n t^{\alpha-n},$$

where $k_i \in R$, $i = 1, 2, \ldots, n$, and $n - 1 < \alpha < n$.

Lemma 2. Let $\alpha > 0$. Then for $x \in C(0, T) \cap L(0, T)$ we have

$$I^\alpha D^\alpha x(t) = x(t) + c_0 + c_1 t + \ldots + c_{n-1} t^{n-1},$$

fore some $c_i \in R$, $i = 1, 2, \ldots, n-1$. where $n = [\alpha] + 1$.

3. Hybrid fractional integro-differential equations

In this section we consider the boundary value problem (1). The following hybrid fixed point theorem for three operators in a Banach algebra E, due to Dhage [8], will be used to prove the existence solution for the boundary value problem (1).

Lemma 3. Let S be a nonempty, closed convex and bounded subset of a Banach algebra E and let $A, C : E \longrightarrow E$ and $B : S \longrightarrow E$ be three operators satisfying:
(i) A and C are Lipschitzian with Lipschitz constants δ and ρ, respectively,
(ii) B is compact and continuous,
(iii) $x = AxBy + Cx \Longrightarrow x \in S$ for all $y \in S$,
(vi) $\delta M + \rho < 1$, where $M = \|B(S)\|$. Then the operator equation $x = AxBx+Cx$ has a solution.

Lemma 4. Suppose that $0 < \alpha < 1$ and a, b, c are real constants with $a + b \neq 0$. Then, for any $h \in L^1(J, R)$, the mild solution $x \in C(J, R)$ of the problem

$$\begin{cases} D^\alpha\left(\frac{x(t)-\sum_{i=1}^m I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right) = h(t) \quad a.e. \quad t \in J = [0, T] \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c \end{cases} \tag{2}$$

is given by the hybrid integral equation

$$\begin{aligned} x(t) &= [f(t, x(t))]\Big[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds \\ &- \frac{1}{a+b}\Big(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds - c \\ &+ \frac{b\sum_{i=1}^m I^{\beta_i}h_i(T, x(T))}{f(T, x(T))}\Big)\Big] + \sum_{i=1}^m I^{\beta_i}h_i(t, x(t)) \quad ,t \in [0, T] \end{aligned} \tag{3}$$

Proof. Assume that x is a solution of the problem (3). By definition, $\frac{x(t)}{f(t,x(t))}$ is continuous. Applying the Caputo fractional operator of the order α, we obtain the first equation in (2). Again, substituting $t = 0$ and $t = T$ in (3) we have

$$\frac{x(0)}{f(0, x(0))} = \frac{-1}{a+b}\Big(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds - c + \frac{b\sum_{i=1}^m I^{\beta_i}h_i(T, x(T))}{f(T, x(T))}\Big) + \frac{\sum_{i=1}^m I^{\beta_i}h_i(0, x(0))}{f(0, x(0))}$$

$$\begin{aligned} \frac{x(T)}{f(T, x(T))} &= \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds - \frac{1}{a+b}\Big(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds \\ &- c + \frac{b\sum_{i=1}^m I^{\beta_i}h_i(T, x(T))}{f(T, x(T))}\Big) \\ &+ \frac{\sum_{i=1}^m I^{\beta_i}h_i(T, x(T))}{f(T, x(T))} \end{aligned}$$

then

$$\begin{aligned} a\frac{x(0)}{f(0, x(0))} + b\frac{x(T)}{f(T, x(T))} &= \frac{-ab}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds + \frac{ac}{a+b} \\ &+ \frac{b}{(\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds - \frac{b^2}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds \\ &+ \frac{bc}{a+b} + \Big(\frac{-ab}{a+b} - \frac{b^2}{a+b} + b\Big)\frac{\sum_{i=1}^m I^{\beta_i}h_i(T, x(T))}{f(T, x(T))} \end{aligned}$$

this implies that

$$a\frac{x(0)}{f(0,x(0))}+b\frac{x(T)}{f(T,x(T))}=c$$

In the forthcoming analysis, we need the following assumptions:

$(H1)$ The functions $f: J\times R\longrightarrow \times R\setminus\{0\}$ and $h_i: J\times R\longrightarrow \times R, h_i(0,x(0))=0, i=1,2,\dots,m$, are continuous and there exist two positive functions ϕ, ψ_i, $i=1,2,\dots,m$ with bound $\|\phi\|$ and $\|\psi\|$, $i=1,2,\dots,m$, respectively, such that

$$|f(t,x(t))-f(t,y(t))|\leq \phi(t)|x(t)-y(t)|$$

and

$$|h_i(t,x(t))-h_i(t,y(t))|\leq \psi(t)|x(t)-y(t)|,\quad i=1,2,\dots,m,$$

for $t\in J$ and $x,y\in R$.

$(H2)$ There exists a function $h\in L^1(J,R)$ such that .

$$|g(t,x)|\leq h(t)\quad a.e\quad t\in J$$

for all $x\in R$.

$(H3)$ There exists a number $r>1$ such that

$$r\geq\frac{F_0\Big[\Big(1+\frac{|b|}{|a+b|}\Big)\Big(\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}\Big)+\frac{|c|}{|a+b|}+\Big|\frac{b\sum_{i=1}^m I^{\beta_i}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big]+K_0\sum_{i=1}^m\frac{T^{\beta_i}}{\Gamma(\beta_i+1)}}{1-\|\phi\|\Big[\Big(1+\frac{|b|}{|a+b|}\Big)\Big(\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}\Big)+\frac{|c|}{|a+b|}+\Big|\frac{b\sum_{i=1}^m I^{\beta_i}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big]-\sum_{i=1}^m\frac{\|\psi_i\|T^{\beta_i}}{\Gamma(\beta_i+1)}}$$

where $F_0=\sup_{t\in J}|f(t,0)|$ and $K_0=\sup_{t\in J}|h_i(t,0)|$, $i=1,2,\dots,m$,

Theorem 1. Assume that the conditions $(H1)-(H3)$ hold. Then the boundary value problem (1) has at least one mild solution on J provided that

$$\begin{aligned}\|\phi\|\Big[\Big(1+\frac{|b|}{|a+b|}\Big)\Big(\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}\Big)&+\frac{|c|}{|a+b|}+\Big|\frac{b\sum_{i=1}^m I^{\beta_i}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big] \qquad (4)\\ &+\sum_{i=1}^m\frac{\|\phi\|T^{\beta_i}}{\Gamma(\beta_i+1)}<1.\end{aligned}$$

Proof. Set $E=C(J,R)$ and define a subset S of E as

$$S=\{x\in E:\|x\|\leq r\}$$

where r satisfies inequality (3).

Clearly S is closed, convex, and bounded subset of the Banach space E. By Lemma 3, problem (1) is equivalent to the integral equation (3). Now we define three operators;

$\mathcal{A}: E\longrightarrow E$ by

$$\mathcal{A}x(t)=f(t,x(t)),\quad t\in J,$$

$\mathcal{B}: S\longrightarrow E$ by

$$\begin{aligned}\mathcal{B}x(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s,x(s))ds-\frac{1}{a+b}\Big(\frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}g(s,x(s))ds-c\\&+\frac{b\sum_{i=1}^m I^{\beta_i}h_i(T,x(T))}{f(T,x(T))}\Big),\quad t\in J,\end{aligned}$$

and $\mathcal{C}: E \longrightarrow E$ by

$$\mathcal{C}x(t)=\sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t))=\sum_{i=1}^{m}\int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)}h_i(s,x(s))ds, \quad t\in J$$

We shall show that the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfy all the conditions of Lemma 3. This will be achieved in the following series of steps.

Step 1: We first show that $\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian on E.

Let $x,y\in E$. Then by $(H1)$, for $t\in J$ we have

$$\begin{aligned}|\mathcal{A}x(t)-\mathcal{A}y(t)| &= |f(t,x(t))-f(t,y(t))|\\ &\le \phi(t)|x(t)-y(t)|\le \|\phi\|\|x-y\|\end{aligned}$$

which implies $\|\mathcal{A}x-\mathcal{A}y\|\le\|\phi\|\|x-y\|$ for all $x,y\in E$. Therefore, $\mathcal{A}$ is a Lipschitzian on E with Lipschitz constant $\|\phi\|$.

Analogously, for any $x,y\in E$, we have

$$\begin{aligned}|\mathcal{C}x(t)-\mathcal{C}y(t)| &= \Big|\sum_{i=1}^{m} I^{\beta_i}h_i(t,x(t))-\sum_{i=1}^{m} I^{\beta_i}h_i(t,y(t))\Big|\\ &\le \sum_{i=1}^{m}\int_0^t \frac{(t-s)^{\beta_i-1}}{\Gamma(\beta_i)}\psi_i(s)|x(s)-y(s)|ds\\ &\le \sum_{i=1}^{m}\frac{\|\psi_i\|T^{\beta_i}}{\Gamma(\beta_i+1)}\|x-y\|\end{aligned}$$

This means that

$$\|\mathcal{C}x-\mathcal{C}y\|\le\sum_{i=1}^{m}\frac{\|\psi_i\|T^{\beta_i}}{\Gamma(\beta_i+1)}\|x-y\|$$

Thus, $\mathcal{C}$ is a Lipschitzian on E with Lipschitz constant $\sum_{i=1}^{m}\frac{\|\psi_i\|T^{\beta_i}}{\Gamma(\beta_i+1)}$.

Step 2: The operator $\mathcal{B}$ is completely continuous on S.

We first show that the operator $\mathcal{B}$ is continuous on E. Let $\{x_n\}$ be a sequence in S converging to a point $x\in S$. Then by the Lebesgue dominated convergence theorem, for all $t\in J$, we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x_n(s))ds &= \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\lim_{n\to\infty}g(s,x_n(s))ds\\ &= \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x(s))ds\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}g(s,x_n(s))ds &= \frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\lim_{n\to\infty}g(s,x_n(s))ds\\ &= \frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}g(s,x(s))ds\end{aligned}$$

In consequence, we have

$$\lim_{n\to\infty}\mathcal{B}x_n=\mathcal{B}x$$

pointwise on J. Further, it can be shown as below the sequence $f(\mathcal{B}x_n)$ of function is an equicontinuous set in E. Therefore, $\mathcal{B}x_n\longrightarrow\mathcal{B}x$ uniformly. As a result, $\mathcal{B}$ is a continuous operator on S.

Next we will prove that the set $\mathcal{B}(S)$ is a uniformly bounded in S. For any $x \in S$, we have

$$
\begin{aligned}
|\mathcal{B}x(t)| &= \Big| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,x(s))ds - \frac{1}{a+b}\Big(\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s,x(s))ds - c \\
&+ \frac{b\sum_{i=1}^m I^{\beta_i} h_i(T,x(T))}{f(T,x(T))} \Big) \Big| \\
&\leq \|h\|_{L^1} \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{|b|}{|a+b|} \frac{T^\alpha}{\Gamma(\alpha+1)} \|h\|_{L^1} + \frac{|c|}{|a+b|} + \Big| \frac{b\sum_{i=1}^m I^{\beta_i} h_i(T,x(T))}{(a+b)f(T,x(T))} \Big| \\
&\leq \Big(1 + \frac{|b|}{|a+b|}\Big)(\|h\|_{L^1} \frac{T^\alpha}{\Gamma(\alpha+1)}) + \frac{|c|}{|a+b|} + \Big| \frac{b\sum_{i=1}^m I^{\beta_i} h_i(T,x(T))}{(a+b)f(T,x(T))} \Big| = K_1
\end{aligned}
$$

for all $t \in J$. Therefore, $\|\mathcal{B}\| \leq K_1$, which shows that $\mathcal{B}$ is uniformly bounded on S.

Step 3: Now, we will show that $\mathcal{B}(S)$ is an equicontinuous set in E. Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $x \in S$. Then we have

$$
\begin{aligned}
|\mathcal{B}x(\tau_2) - \mathcal{B}x(\tau_1)| &= \Big| \int_0^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,x(s))ds - \int_0^{\tau_1} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,x(s))ds \Big| \\
&\leq \int_0^{\tau_1} \frac{|(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}|}{\Gamma(\alpha)} |g(s,x(s))|ds + \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} |g(s,x(s))|ds \\
&\leq \int_0^{\tau_1} \frac{|(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}|}{\Gamma(\alpha)} \|h\|_{L^1} ds + \int_{\tau_1}^{\tau_2} \frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)} \|h\|_{L^1} ds
\end{aligned}
$$

which is independent of $x \in S$. As $\tau_1 \longrightarrow \tau_2$, the right-hand side of the above inequality tends to zero. Therefore, it follows from the Arzelá-Ascoli theorem that $\mathcal{B}$ is a completely continuous operator on S.

Step 4: The hypothesis (iii) of Lemma 3 is satisfied.

Let $x \in E$ and $y \in S$ be arbitrary elements such that $x = \mathcal{A}x\mathcal{B}y + \mathcal{C}x$. Then we have

$$\begin{aligned}
|x(t)| &\leq |\mathcal{A}x(t)||\mathcal{B}y(t)| + |\mathcal{C}x(t)| \\
&\leq |f(t,x(t)|\Big|\Big[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x(s))ds + \frac{1}{a+b}\Big(\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s,x(s))ds + c \\
&+ \frac{b\sum_{i=1}^m I^{\beta_i} h_i(T,x(T))}{f(T,x(T))}\Big)\Big]\Big| + \Big|\sum_{i=1}^m I^{\beta_i} h_i(s,s(t))\Big| \\
&\leq (|f(t,x(t)-f(t,0)| + |f(t,0)|)\Big[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|h\|_{L^1} ds \\
&+ \frac{1}{|a+b|}\Big(\frac{|b|}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\|h\|_{L^1} ds + |c| \\
&+ \Big|\frac{b\sum_{i=1}^m I^{\beta_i} h_i(T,x(T))}{f(T,x(T))}\Big|\Big)\Big] + \sum_{i=1}^m I^{\beta_i}(|h_i(s,x(s)) - h_i(s,0) + |h_i(s,0)|) \\
&\leq (r\|\phi\| + F_0)\Big[\Big(1 + \frac{|b|}{|a+b|}\Big)\Big(\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}\Big) + \frac{|c|}{|a+b|} + \Big|\frac{b\sum_{i=1}^m I^{\beta_i} h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big] \\
&+ \sum_{i=1}^m \frac{(r\|\psi_i\| + k_0)T^{\beta_i}}{\Gamma(\beta_i+1)}
\end{aligned}$$

which leads to

$$\begin{aligned}
\|x\| &\leq (r\|\phi\| + F_0)\Big[\Big(1 + \frac{|b|}{|a+b|}\Big)\Big(\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}\Big) + \frac{|c|}{|a+b|} \\
&+ \Big|\frac{b\sum_{i=1}^m I^{\beta_i} h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big] + \sum_{i=1}^m \frac{(r\|\psi_i\| + k_0)T^{\beta_i}}{\Gamma(\beta_i+1)} \leq r
\end{aligned}$$

Therefore, $x \in S$.
Step 4. Finally we show that $\delta M + \rho < 1$, that is, (vi) of Lemma 3 holds.
Since

$$\begin{aligned}
M = \|\mathcal{B}(S)\| &= \sup_{x\in S}\{\sup_{t\in J}|\mathcal{B}x(t)|\} \\
&\leq \Big(1 + \frac{|b|}{|a+b|}\Big)\Big(\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}\Big) \\
&+ \frac{|c|}{|a+b|} + \Big|\frac{b\sum_{i=1}^m \frac{T^{\beta_i}}{\Gamma(\beta_i+1)} I^{\beta_i} h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|
\end{aligned}$$

and by (H_3) we have

$$\|\phi\|M + \sum_{i=1}^m \frac{T^{\beta_i}}{\Gamma(\beta_i+1)}\|\psi_i\| < 1.$$

with $\delta = \|\phi\|$ and $\rho = \sum_{i=1}^m \frac{T^{\beta_i}}{\Gamma(\beta_i+1)}\|\psi_i\|$.
Thus all the conditions of Lemma 3 are satisfied and hence the operator equation $x = \mathcal{A}x\mathcal{B}x + \mathcal{C}x$ has a solution in S. In consequence, problem (2) has a mild solution on J. This completes the proof.

4. Exemple

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional boundary value problem:

$$\begin{cases} D^{\frac{3}{4}}\left(\frac{x(t)-\sum_{i=1}^{4} I^{\beta_i}h_i(t,x(t))}{f(t,x(t))}\right) = \frac{(t-1)^2}{35(13-t^2)}(7|x(t)|+15) \quad a.e. \quad t \in J=[0,1] \\ \frac{x(0)}{f(0,x(0))} + \frac{x(1)}{f(1,x(1))} = \frac{\pi}{2}, \end{cases} \tag{5}$$

where

$$\begin{aligned} \sum_{i=1}^{4} I^{\beta_i}h_i(t,x(t)) &= I^{\frac{1}{3}}\frac{3te^{-3t}}{15(3+t)}\Big(\frac{x^2(t)+9|x(t)|}{|x(t)|+5}+\frac{12e^{3t}}{5}\Big) \\ &+ I^{\frac{7}{4}}\frac{t\sin t}{7(4+e^t)}\Big(\frac{x^2(t)+4|x(t)|}{|x(t)|+3}+\cos t\Big) \\ &+ I^{\frac{10}{3}}\frac{2\sin \pi t}{9+(5+t)^2)}\Big(\frac{x^2(t)+8|x(t)|}{|x(t)|+5}+\frac{4}{5}\Big) \\ &+ I^{\frac{29}{6}}\frac{3t\cos t+4t\sin t}{10(4-t)^2}\Big(\frac{x^2(t)+5|x(t)|}{|x(t)|+4}+\frac{t}{t+2}\Big) \end{aligned}$$

and

$$f(t,x(t)) = \frac{3(\cos \pi t+2t)}{5(2+10t)^2}\Big(\frac{x^2(t)+5|x(t)|}{3+|x(t)|}\Big)+\frac{8-2^{2-t}}{27},$$

Here $\alpha=\frac{3}{4}$, $T=1$, $a=b=1$, $c=\frac{\pi}{2}$, $m=4$, $\beta_1=\frac{1}{3}$, $\beta_2=\frac{7}{4}$, $\beta_3=\frac{10}{3}$, and $\beta_4=\frac{29}{6}$. We can show that

$$|f(t,x)-f(t,y)| \le \Big(\frac{1+2t}{(2+10t)^2}\Big)|x-y|$$

$$|h_1(t,x)-h_1(t,y)| \le \Big(\frac{18t}{75(3+t)}\Big)|x-y|$$

$$|h_2(t,x)-h_2(t,y)| \le \Big(\frac{4t}{21(4+e^t)}\Big)|x-y|$$

$$|h_3(t,x)-h_3(t,y)| \le \Big(\frac{16}{45+5(5+t)^2)}\Big)|x-y|$$

$$|h_4(t,x)-h_4(t,y)| \le \Big(\frac{5t}{8(4-t)^2)}\Big)|x-y|$$

It follows that $\phi(t)=(1+2t)(2+10t)^{-2}$, $\psi_1(t)=\frac{18t(3+t)^{-1}}{75}$, $\psi_2(t)=\frac{4t(4+e^t)^{-1}}{21}$, $\psi_3(t)=16(45+5(5+t)^2)^{-1}$, and $\psi_4(t)=\frac{5t(4-t)^{-2}}{8}$, which give norms $\|\phi\|=\frac{1}{4}$, $\|\psi_1\|=\frac{3}{50}$, $\|\psi_2\|=\frac{4(28+7e)^{-1}}{3}$, $\|\psi_3\|=\frac{8}{85}$, and $\|\psi_4\|=\frac{5}{72}$. Since

$$|g(t,x(t))| = \Big|\frac{(t-1)^2+3}{35(13+t^2)}(7+|x|+15)\Big| \le \Big(\frac{t-1)^2+3}{13-t^2}\Big)\Big(\frac{|x|}{5}+\frac{3}{7}\Big)$$

It is easy to verify that $\|p\|=\frac{4}{13}$, $F_0=\sup\limits_{t\in[0,2]}|f(t,0)|=\frac{4}{27}$, and $F_0=\sup\limits_{t\in[0,2]}|h_i(t,0)|=\frac{4}{27}$, $i=1,2,3,4$. We see that condition $(H3)$ is followed with a number $r \in$

$[0.187454, 131.292851]$.

$$\|\phi\|\Big[\Big(1+\frac{|b|}{|a+b|}\Big)\Big(\|h\|_{L^1}\frac{T^\alpha}{\Gamma(\alpha+1)}\Big) + \frac{|c|}{|a+b|}+\Big|\frac{b\sum_{i=1}^4 I^{\beta_i}h_i(T,x(T))}{(a+b)f(T,x(T))}\Big|\Big] + \sum_{i=1}^{4}\frac{\|\phi\|T^{\beta_i}}{\Gamma(\beta_i+1)} \simeq 0.18957628293 < 1.$$

Consequently all conditions in Theorem 1, are satisfied. Therefore, problem (5) has at least one solution on $[0,1]$.

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