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CERTAIN SUBCLASS OF MEROMORPHICALLY $P-$VALENT FUNCTIONS DEFINED BY LINEAR OPERATOR WITH POSITIVE AND FIXED SECOND COEFFICIENTS

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ABSTRACT. In this paper we consider the subclass $\Gamma_{\lambda}^{c}(n,\alpha,\beta)$ consisting of $p-$valent meromorphic functions analytic in $\mathbb{U}^{*}=\{z : z\in\mathbb{C} : 0<|z|<1\}$ and with fixed second coefficient. We obtain coefficient estimates, distortion theorems and closure theorems. Finally, we obtain radius of $p-$valent convexity for functions in this class.

1. INTRODUCTION

Let Σ_p be the class of functions of the form:

$$f(z)=z^{-p}+\sum_{k=p+1}^{\infty}a_k z^k \ \ (a_k\geq 0,\ p\in\mathbb{N}=\{1,2,...\}), \tag{1}$$

which are meromorphic and $p-$valent in the punctured unit disc $\mathbb{U}^{*}=\{z : z\in\mathbb{C} : 0<|z|<1\}$ $=\mathbb{U}\backslash\{0\}$. For $f\in\Sigma_p$, Saif and Kilicman [19] defined the linear operator D_{λ}^{n} as follows

$$\begin{aligned} D_{\lambda}^{0}f(z) &= f(z) \\ D_{\lambda}^{1}f(z) &= D_{\lambda}f(z), \\ &= (1+p\lambda)f(z)+\lambda z f'(z)\ (\lambda\geq 0) \\ &= z^{-p}+\sum_{k=p+1}^{\infty}\left[1+\lambda(p+k)\right]a_k z^k, \qquad (2) \\ D_{\lambda}^{2}f(z) &= D_{\lambda}\left(D_{\lambda}f(z)\right), \end{aligned}$$

and in general

$$\begin{aligned} D_{\lambda}^{n}f(z) &= D_{\lambda}\left(D_{\lambda}^{n-1}\right) \\ &= z^{-p}+\sum_{k=p+1}^{\infty}\left[1+\lambda(p+k)\right]^{n}a_k z^k\ (n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}). \qquad (3) \end{aligned}$$

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A function $f \in \Sigma_p$ is said to be in the class of meromorphically $p-$valent starlike of order η $\left(\Sigma_p^*(\eta)\right)$, if it satisfies (see [6, 13])

$$-Re\left\{\frac{zf^{'}(z)}{f(z)}\right\} > \eta \quad (z \in \mathbb{U}^*; \quad 0 \leq \eta < p). \tag{4}$$

Similary $f \in \Sigma_p$ is said to be in the class of meromorphically $p-$valent convex of order η $\left(\Sigma_p^c(\eta)\right)$, if it satisfies (see [16])

$$-Re\left\{1+\frac{zf^{''}(z)}{f^{'}(z)}\right\} > \eta \quad (z \in \mathbb{U}^*; \quad 0 \leq \eta < p). \tag{5}$$

For fixed $\alpha \geq 0$, $0 \leq \beta < 1$, $f \in \Sigma_p$ and by using the operator D_λ^n Saif and Kilicman [19] defined the class $\Gamma_\lambda(n,\alpha,\beta)$ as follows:

$$Re\left\{-\left(\frac{z\left(D_\lambda^n f(z)\right)^{'}}{p\left(D_\lambda^n f(z)\right)}+\beta\right)\right\} \geq \alpha\left|\frac{z\left(D_\lambda^n f(z)\right)^{'}}{p\left(D_\lambda^n f(z)\right)}+1\right| \ \left(n \in \mathbb{N}_0; \lambda > 0\right). \tag{6}$$

Meromorphic multivalent functions have been extensively studied by Aouf [1, 2, 3], Joshi and Srivastava [12], Mogra [14, 15], Owa et al. [17], Srivastava et al. [21], Joshi and Aouf [11], Aouf and Darwish [4, 5], Aouf and Srivastava [10], Raina and Srivastava [18], Uralegaddi and Ganigi [22] and Yang [23].

We begin by recalling the following lemma due to Saif and Kilicman [19].

Lemma 1. Let $f(z) \in \Sigma_p$. Then $f(z) \in \Gamma_\lambda(n,\alpha,\beta)$ if and only if

$$\sum_{k=p+1}^{\infty}\left[p(\alpha+\beta)+k\left(1+\alpha\right)\right]\left[\lambda\left(k+p\right)+1\right]^n a_k \leq p\left(1-\beta\right) \tag{7}$$

$$(\alpha \geq 0, \ 0 \leq \beta < 1, \ p \in p \in \mathbb{N}, \ n \in \mathbb{N}_0, \lambda > 0).$$

The result is sharp.

In view of Lemma 1, we can see that $f(z) \in \Gamma_\lambda(n,\alpha,\beta)$ satisfies the coefficient inequality

$$a_{p+1} \leq \frac{p(1-\beta)}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n}. \tag{8}$$

For $0 < c < 1$, we take

$$a_{p+1} = \frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n}. \tag{9}$$

Let $\Gamma_\lambda^c(n,\alpha,\beta)$ denote the subclass of $\Gamma_\lambda(n,\alpha,\beta)$ consisting of functions of form

$$f(z) = z^{-p}+\frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n}z^{p+1}+\sum_{k=p+2}^{\infty}a_k z^k \quad (a_k \geq 0 \ \text{ and } \ 0 < c < 1). \tag{10}$$

In this paper the techniques used are similar to those of Aouf and Darwish [4], Aouf and Joshi [8], Aouf et al. [7, 9], Owa et al. [17] and Silverman and Silvia [20].

Unless indicated, we assume that $\alpha \geq 0$, $0 \leq \beta < 1$, $p \in \mathbb{N}$, $n \in \mathbb{N}_0$, $0 < c < 1, \lambda > 0$, $f(z)$ defined by (10) and $z \in \mathbb{U}^*$.

2. COEFFICIENT ESTIMATES

In this section, we obtain coefficient estimates for functions in the class $\Gamma_{\lambda}^{c}(n,\alpha,\beta)$.

Theorem 1. A function $f(z)\in\Gamma_{\lambda}^{c}(n,\alpha,\beta)$ if and only if

$$\sum_{k=p+2}^{\infty}\left[p(\alpha+\beta)+k\left(1+\alpha\right)\right]\left[\left(k+p\right)\lambda+1\right]^{n}a_{k}\leq p\left(1-\beta\right)\left(1-c\right). \tag{11}$$

Proof. Substituting (9) in (7) and simplifying we get the result. □

Corollary 1. Let $f(z)\in\Gamma_{\lambda}^{c}(n,\alpha,\beta)$. Then

$$a_{k}\leq\frac{p(1-\beta)(1-c)}{[p(\alpha+\beta)+k(1+\alpha)][(k+p)\lambda+1]^{n}}\ (k\geq p+2). \tag{12}$$

The result is sharp for

$$f(z)=z^{-p}+\frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^{n}}z^{p+1}+\frac{p(1-\beta)(1-c)}{[p(\alpha+\beta)+k(1+\alpha)][(k+p)\lambda+1]^{n}}z^{k}(k\geq p+2). \tag{13}$$

3. DISTORTION THEOREMS

In this section, we obtain distortion theorems for functions in the class $\Gamma_{\lambda}^{c}(n,\alpha,\beta)$.

Theorem 2. Let $f(z)\in\Gamma_{\lambda}^{c}(n,\alpha,\beta)$, then for $|z|=r<1$, we have

$$\begin{aligned}&r^{-p}-\frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^{n}}r^{p+1}-\frac{p(1-\beta)(1-c)}{[p(\alpha+\beta)+(p+2)(1+\alpha)][(2p+2)\lambda+1]^{n}}r^{p+2}\\ \leq\ &|f(z)|\leq\\ &r^{-p}+\frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^{n}}r^{p+1}+\frac{p(1-\beta)(1-c)}{[p(\alpha+\beta)+(p+2)(1+\alpha)][(2p+2)\lambda+1]^{n}}r^{p+2}.\end{aligned} \tag{14}$$

The result is sharp for

$$f(z)=z^{-p}+\frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^{n}}z^{p+1}+\frac{p(1-\beta)(1-c)}{[p(\alpha+\beta)+(p+2)(1+\alpha)][(2p+2)\lambda+1]^{n}}z^{p+2}. \tag{15}$$

Proof. Since $f(z)\in\Gamma_{\lambda}^{c}(n,\alpha,\beta)$, then Theorem 1 yields

$$\sum_{k=p+2}^{\infty}a_{k}\leq\frac{p(1-\beta)(1-c)}{[p(\alpha+\beta)+(p+2)(1+\alpha)][(2p+2)\lambda+1]^{n}}. \tag{16}$$

Thus, for $|z|=r<1$, we have

$$\begin{aligned}|f(z)|&\leq|z|^{-p}+\frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^{n}}|z|^{p+1}+\sum_{k=p+2}^{\infty}a_{k}|z|^{k}\\&\leq r^{-p}+\frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^{n}}r^{p+1}+r^{p+2}\sum_{k=p+2}^{\infty}a_{k}\\&\leq r^{-p}+\frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^{n}}r^{p+1}+\frac{p(1-\beta)(1-c)}{[p(\alpha+\beta)+(p+2)(1+\alpha)][(2p+2)\lambda+1]^{n}}r^{p+2}.\end{aligned}$$

Similarly we have

$$\begin{aligned}|f(z)|&\geq|z|^{-p}-\frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^{n}}|z|^{p+1}-\sum_{k=p+2}^{\infty}a_{k}|z|^{k}\\&\geq r^{-p}-\frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^{n}}r^{p+1}-\frac{p(1-\beta)(1-c)}{[p(\alpha+\beta)+(p+2)(1+\alpha)][(2p+2)\lambda+1]^{n}}r^{p+2}.\end{aligned}$$

This completes the proof. □

Theorem 3. Let $f(z) \in \Gamma_\lambda^c(n,\alpha,\beta)$, then for $|z| = r < 1$, we have

$$\begin{aligned} & pr^{-(p+1)} - \tfrac{p(p+1)(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} r^p - \tfrac{p(p+2)(1-\beta)(1-c)}{[p(\alpha+\beta)+(p+2)(1+\alpha)][(2p+2)\lambda+1]^n} r^{p+1} \\ \leq & \left|f'(z)\right| \leq \\ & pr^{-(p+1)} + \tfrac{p(p+1)(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} r^p + \tfrac{p(p+2)(1-\beta)(1-c)}{[p(\alpha+\beta)+(p+2)(1+\alpha)][(2p+2)\lambda+1]^n} r^{p+1}. \end{aligned} \tag{17}$$

The result is sharp for $f(z)$ given by (15).

Proof. In view of Theorem 1, it follows that

$$\sum_{k=p+2}^{\infty} k a_k \leq \tfrac{p(p+2)(1-\beta)(1-c)}{[p(\alpha+\beta)+(p+2)(1+\alpha)][(2p+2)\lambda+1]^n}. \tag{18}$$

Thus, for $|z| = r < 1$, and making use of (18), we obtain

$$\begin{aligned} \left|f'(z)\right| & \leq p|z|^{-(p+1)} + \tfrac{p(p+1)(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} |z|^p + \sum_{k=p+2}^{\infty} k a_k |z|^{k-1} \\ & \leq pr^{-(p+1)} + \tfrac{p(p+1)(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} r^p + r^{p+1} \sum_{k=p+2}^{\infty} k a_k \\ & \leq pr^{-(p+1)} + \tfrac{p(p+1)(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} r^p + \tfrac{p(p+2)(1-\beta)(1-c)}{[p(\alpha+\beta)+(p+2)(1+\alpha)][(2p+2)\lambda+1]^n} r^{p+1}. \end{aligned}$$

Similarly we have

$$\begin{aligned} \left|f'(z)\right| & \geq p|z|^{-(p+1)} - \tfrac{p(p+1)(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} |z|^p - \sum_{k=p+2}^{\infty} k a_k |z|^{k-1} \\ & \geq pr^{-(p+1)} - \tfrac{p(p+1)(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} r^p - \tfrac{p(p+2)(1-\beta)(1-c)}{[p(\alpha+\beta)+(p+2)(1+\alpha)][(2p+2)\lambda+1]^n} r^{p+1}. \end{aligned}$$

This completes the proof. □

4. CLOSURE THEOREMS

In this section, we obtain closure theorems for functions in the class $\Gamma_\lambda^c(n,\alpha,\beta)$.
Let $f_v(z)$ be defined, for $v = 1, 2, ..., m$, by

$$f_v(z) = z^{-p} + \tfrac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} z^{p+1} + \sum_{k=p+2}^{\infty} a_{k,v} z^k \quad (a_{k,v} \geq 0). \tag{19}$$

Theorem 4. Let $f_v(z) \in \Gamma_\lambda^c(n,\alpha,\beta)$ for $v = 1, 2, ..., m$. Then

$$g(z) = z^{-p} + \tfrac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} z^{p+1} + \sum_{k=p+2}^{\infty} b_k z^k \quad (b_k \geq 0), \tag{20}$$

is also in the same class, where

$$b_k = \frac{1}{m} \sum_{v=1}^{m} a_{k,v}. \tag{21}$$

Proof. Since $f_v(z) \in \Gamma_\lambda^c(n,\alpha,\beta)$ for $v = 1,2,...,m$, it follows from Theorem 1 that

$$\sum_{k=p+2}^{\infty} [p(\alpha+\beta)+k(1+\alpha)]\left[(k+p)\lambda+1\right]^n a_{k,v} \leq p(1-\beta)(1-c). \tag{22}$$

Hence

$$\begin{aligned} & \sum_{k=p+2}^{\infty} [p(\alpha+\beta)+k(1+\alpha)]\left[(k+p)\lambda+1\right]^n b_k \\ = & \sum_{k=p+2}^{\infty} [p(\alpha+\beta)+k(1+\alpha)]\left((k+p)\lambda+1\right)^n \left(\frac{1}{m}\sum_{v=1}^{m} a_{k,v}\right) \\ = & \frac{1}{m}\sum_{v=1}^{m}\sum_{k=p+2}^{\infty} [p(\alpha+\beta)+k(1+\alpha)]\left[(k+p)\lambda+1\right]^n a_{k,v} \\ \leq & \; p(1-\beta)(1-c), \end{aligned} \tag{23}$$

and the result follows. □

Theorem 5. Let

$$f_{p+1}(z) = z^{-p} + \frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} z^{p+1} \tag{24}$$

and

$$\begin{aligned} f_k(z) &= z^{-p} + \frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} z^{p+1} + \frac{p(1-\beta)(1-c)}{[p(\alpha+\beta)+k(1+\alpha)][(k+p)\lambda+1]^n} z^k \\ (k &\geq p+2). \end{aligned} \tag{25}$$

Then $f(z) \in \Gamma_\lambda^c(n,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=p+1}^{\infty} \mu_k f_k(z), \tag{26}$$

where $\mu_k \geq 0$ $(k \geq p+1)$ and $\sum\limits_{k=p+1}^{\infty} \mu_k = 1$.

Proof. Suppose that

$$f(z) = \sum_{k=p+1}^{\infty} \mu_k f_k(z)$$

$$= z^{-p} + \frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} z^{p+1} + \sum_{k=p+2}^{\infty} \frac{p(1-\beta)(1-c)}{[p(\alpha+\beta)+k(1+\alpha)][(k+p)\lambda+1]^n} \mu_k z^k. \tag{27}$$

Then it follows that

$$\begin{aligned} & \sum_{k=p+2}^{\infty} \frac{[p(\alpha+\beta)+k(1+\alpha)][(k+p)\lambda+1]^n}{p(1-\beta)(1-c)} \cdot \frac{p(1-\beta)(1-c)}{[p(\alpha+\beta)+k(1+\alpha)][(k+p)\lambda+1]^n} \mu_k \\ = & \sum_{k=p+2}^{\infty} \mu_k = (1-\mu_{p+1}) \leq 1. \end{aligned} \tag{28}$$

So, by Theorem 1, $f(z) \in \Gamma_\lambda^c(n,\alpha,\beta)$.

Conversely, assume that $f(z) \in \Gamma_\lambda^c(n,\alpha,\beta)$. Then

$$a_k \leq \frac{p(1-\beta)(1-c)}{[p(\alpha+\beta)+k(1+\alpha)][(k+p)\lambda+1]^n} \quad (k \geq p+2). \tag{29}$$

Putting

$$\mu_k = \frac{[p(\alpha+\beta)+k(1+\alpha)][(k+p)\lambda+1]^n}{p(1-\beta)(1-c)} a_k \quad (k \geq p+2), \tag{30}$$

and

$$\mu_{p+1} = 1 - \sum_{k=p+2}^{\infty} \mu_k, \tag{31}$$

we see that $f(z)$ can be expressed in the form (26). This completes the proof. □

5. RADIUS OF $p-$VALENT CONVEXITY

In this section, we obtain radius of $p-$valent convexity for functions in the class $\Gamma_\lambda^c(n, \alpha, \beta)$.

Theorem 6. Let $f(z) \in \Gamma_\lambda^c(n, \alpha, \beta)$. Then $f(z)$ is $p-$valent meromorphically convex in $|z| < r = r(n, p, \alpha, \beta, c)$, where $r(n, p, \alpha, \beta, c)$ is the largest value for which

$$\frac{p(p+1)(3p+1)(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} r^{2p} + \frac{p(1-\beta)(1-c)k(k+2p)}{[p(\alpha+\beta)+k(1+\alpha)][(k+p)\lambda+1]^n} r^{k+p} = p^2. \tag{32}$$

The result is sharp for

$$f_k(z) = z^{-p} + \frac{p(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} z^{p+1} + \frac{p(1-\beta)(1-c)}{[p(\alpha+\beta)+k(1+\alpha)][(k+p)\lambda+1]^n} z^k \quad \text{for some } k. \tag{33}$$

Proof. We must show that

$$\left| \frac{\left(zf'(z)\right)' + pf'(z)}{f'(z)} \right| \leq p \quad for \quad |z| < r = r(n, p, \alpha, \beta, c).$$

Note that

$$\left| \frac{\left(zf'(z)\right)' + pf'(z)}{f'(z)} \right| \leq \frac{\frac{p(p+1)(2p+1)(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} r^{2p} + \sum\limits_{k=p+2}^{\infty} k(k+p)a_k r^{k+p}}{p - \frac{p(p+1)(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1^n]} r^{2p} - \sum\limits_{k=p+2}^{\infty} k a_k r^{k+p}} \leq p, \tag{34}$$

for $|z| < r$ if and only if

$$\begin{aligned} &\frac{p(p+1)(3p+1)(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} r^{2p} + \sum_{k=p+2}^{\infty} k(k+2p)a_k r^{k+p} \\ \leq \quad & p^2. \end{aligned} \tag{35}$$

Since $f(z) \in \Gamma_\lambda^c(n, \alpha, \beta)$, then from (11) we may take

$$a_k = \frac{p(1-\beta)\mu_k}{[p(\alpha+\beta)+k(1+\alpha)][(k+p)\lambda+1]^n}, \qquad \sum_{k=p+2}^{\infty} \mu_k \leq 1. \tag{36}$$

For each fixed r, we choose the positive integer $k_0 = k_0(r)$ for which $\frac{k(k+2p)}{[p(\alpha+\beta)+k(1+\alpha)][(k+p)\lambda+1]^n} r^{k+p}$ is maximal. Then it follows that

$$\sum_{k=p+2}^{\infty} k(k+2p)a_k r^{k+p} \leq \frac{p(1-\beta)(1-c)k_0(k_0+2p)}{[p(\alpha+\beta)+k_0(1+\alpha)][(k_0+p)\lambda+1]^n} r^{k_0+p}. \tag{37}$$

Hence $f(z)$ is $p-$valent meromorphically convex in $|z| < r(n,p,\alpha,\beta,c)$ provided that

$$\frac{p(p+1)(3p+1)(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} r^{2p} + \frac{p(1-\beta)(1-c)k_0(k_0+2p)}{[p(\alpha+\beta)+k_0(1+\alpha)][(k_0+p)\lambda+1]^n} r^{k_0+p} \leq p^2. \quad (38)$$

We find the value $r_0 = r_0(n,\alpha,\beta,c)$ and the corresponding integer $k_0(r_0)$ so that

$$\frac{p(p+1)(3p+1)(1-\beta)c}{[p(\alpha+\beta)+(p+1)(1+\alpha)][(2p+1)\lambda+1]^n} r_0^{2p} + \frac{p(1-\beta)(1-c)k_0(k_0+2p)}{[p(\alpha+\beta)+k_0(1+\alpha)][(k_0+p)\lambda+1]^n} r_0^{k_0+p} = p^2. \quad (39)$$

Then this value r_0 is the radius of $p-$valent meromorphically convexity for $f(z) \in \Gamma_\lambda^c(n,\alpha,\beta)$. □

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