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OSCILLATION CRITERIA FOR A HALF LINEAR NEUTRAL TYPE FRACTIONAL DIFFERENCE EQUATION WITH DELAY

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Abstract. In this paper, sufficient conditions are established for the oscillatory and asymptotic behavior of the neutral type half linear fractional difference equation with delay of the form

$$\Delta(p(t)(\Delta_R^{\nu}(x(t)+q(t)x(t-\tau))^{\alpha})+r(t)x^{\beta}(t-\sigma)=0,\quad t\in\mathbb{N}_{t_0+1-\nu},$$

based on the assumption $\sum_{s=t_0}^{\infty}p^{-\frac{1}{\alpha}}(s)<\infty$, where Δ_R^{α} denotes the Riemann-Liouville difference operator of order $0<\upsilon\leq 1$ and $\alpha,\beta>0$ are quotient of odd positive integers, and obtained some oscillation criteria for the above equation by using Riccati transformation technique and some Hardy type inequalities. Some examples are provided to demonstrate the effectiveness of the main results.

1. Introduction

Qualitative analysis of the solutions of fractional difference equations has received great interest during the recent years. Fractional calculus finds significant application in the fields of viscoelasticity, capacitor theory, electrical circuits, electroanalytical chemistry, tumor growth models, neurology, control theory, statistics and a review on this direction, see $[17,18,22-24,26,31,32,34,36-38]$. Despite the qualitative analysis of solutions of many fractional differential equations, see $[4,8-16,19-21,25,28,29,33,35,42-48]$ the qualitative study of the solutions of fractional difference equations is very scarce, see $[1-3,5,6,27,30,39-41]$. In the qualitative study of the solutions of these scarce fractional difference equations, the various forms of the equation

$$\Delta(p(t)\Delta^{\nu}x(t))+r(t)f(x(t))=0,\quad t\in N_{\nu}$$

play a major role. All of these qualitative studies are based under the assumptions $\sum_{s=t_0}^{\infty}\frac{1}{p(s)}=\infty$ and $\Delta p(s)\geq 0$. The purpose of this paper is to relax these conditions and derive some oscillation and asymptotic criteria for half–linear fractional difference equation with delay of the form

$$\Delta(p(t)(\Delta^{\nu}\left[x(t)+q(t)x(t-\tau)\right])^{\alpha})+r(t)x^{\beta}(t-\sigma)=0,\quad t\in\mathbb{N}_{t_0+1-\nu},\qquad (1)$$

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with initial condition $\Delta^{\nu-1}x(t)\mid_{t=0}= x_0$, under the assumptions

$$\sum_{s=t_0}^{\infty}\frac{1}{p^{\frac{1}{\alpha}}(s)}<\infty \tag{2}$$

and without using that $\Delta p(t)\geq 0$. Also we don't restrict to take $\alpha=\beta$. Here $0<\nu\leq 1$ and Δ^{ν} denotes the Riemann left fractional difference operator of order ν and $\mathbb{N}_t=\{t,t+1,t+2,...\}$. Throughout the paper, we assume that α,β are the ratio of odd positive integers, $\beta\leq\alpha$, $p(t)>0$ for $t\geq t_0$, $q(t)$ is an oscillating sequence satisfying $\lim_{t\to\infty}q(t)=0$, $r(t)>0$ for $t\geq t_0$, σ and τ are positive integers with $\lim_{t\to\infty}(t-\sigma)=\lim_{t\to\infty}(t-\tau)=\infty$.

The sets of integer number and real numbers are denoted with $\mathbb{Z}$ and $\mathbb{R}$ respectively. By a solution of equation (1), we mean a nontrivial sequence $x(t):\mathbb{Z}\to\mathbb{R}$ which is defined for all $t\geq\min\{-\tau,-\sigma\}$ and satisfies equation (1) for sufficiently large t. We restrict our attention to those solutions of (1) which satisfy $\sup\{|x(t)|:t\geq T\}$ for all $T\geq T_x$. For our purpose, we assume that equation (1) possesses such a solution. As it is customary, a solution $x(t)$ of the equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called non-oscillatory.

2. Preliminaries

In this section, we present some preliminary definitions from discrete fractional calculus. We will make use of these results, throughout the paper.

Definition 1 [40] Let $\nu>0$. The ν-th fractional sum f is defined by

$$\Delta^{-\nu}f(t)=\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-s-1)^{\nu-1}f(s),$$

where f is defined for $s\equiv a \mod(1)$, $\Delta^{-\nu}f(t)$ is defined for $t\equiv(a+\nu)\mod(1)$ and $t^{(\nu)}=\frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$. The fractional sum $\Delta^{-\nu}f(t)$ maps functions defined in $\mathbb{N}_a$ to functions defined in $\mathbb{N}_{a+\nu}$.

Definition 2 [40] Let $\mu>0$ and $m-1<\mu<m$, where m is a positive integer, $m=\lceil\mu\rceil$. Set $\nu=m-\mu$. The μ-th order Riemann left fractional difference is defined as

$$\Delta^{\mu}f(t)=\Delta^{m-\nu}f(t)=\Delta^{m}\Delta^{-\nu}f(t),$$

where $\Delta^{-\nu}f(t)$ is ν-th fractional sum.

Theorem 1 (see[7]). Let f be a real-value function defined on N_a and μ, $\nu>0$, then the following equalities hold:

(i) $\Delta^{-\nu}[\Delta^{-\mu}f(t)]=\Delta^{-(\mu+\nu)}f(t)=\Delta^{-\mu}[\Delta^{-\nu}f(t)]$;

(ii) $\Delta^{-\nu}\Delta f(t)=\Delta\Delta^{-\nu}f(t)-\frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)}f(a)$.

3. Main results

To obtain our main results, we need following lemmas. For the sake of convenience, the function y is defined as

$$y(t)=x(t)+q(t)x(t-\tau)\ \text{ and }\ \xi(t)=:\frac{1}{\Gamma(\nu)}\sum_{s=t}^{\infty}(t-s-1)^{(\nu-1)}\frac{1}{p^{\frac{1}{\alpha}}(s)}. \tag{3}$$

$H_1)$ $\sum\limits_{s=t_0}^{t-\nu} \frac{1}{p^{\frac{1}{\alpha}}(s)} < \infty$ as $t \to \infty$.

$H_2)$ $\sum_{s=t_0}^{\infty} [k_1^{\beta} r(s)s^2 - c(2s+1)] = \infty, \ \ c > 0.$

$H_3)$ $\frac{y^{\beta}(t-\sigma)}{y^{\alpha}(t+1)} \geq A > 0$ and $\frac{\Delta y^{\alpha}(t)}{\Delta^{\nu} y(t+1) y^{\alpha-1}(t+1)} \geq B > 0$, for $y(t) \neq 0$ and $\Delta^{\alpha} y(t+1) \neq 0$ respectively.

Theorem 2 Assume that H_1 and H_2 are satisfied. Then every bounded solution of $Eq.(1)$ either oscillates or tends to zero.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory bounded solution of $Eq.(1)$. Without loss of generality, we can assume that $x(t)$ is an eventually positive bounded solution of $Eq.(1)$ (the proof is similar when $x(t)$ is eventually negative). Then there exists $t_1 > t_0$ such that $x(t) > 0$, $x(t-\tau) > 0$ and $x(t-\sigma) > 0$ for all $t \geq t_1 \geq t_0$. Further, suppose that $x(t)$ does not tend to zero as $n \to \infty$. Therefore by $Eq.(1)$ and (3), we have

$$\Delta(p(t)(\Delta^{\nu} y(t))^{\alpha}) = -r(t) x^{\beta}(t-\sigma) \leq 0, \quad t \geq t_1 \tag{4}$$

Thus $p(t)(\Delta^{\nu} y(t))^{\alpha}$ is an eventually nonincreasing sequence. Since $x(t)$ is bounded and does not tend to zero as $n \to \infty$, we have $\lim\limits_{t\to\infty} q(t)x(t-\tau) = 0$. Then, there exists an integer $t_2 \geq t_1$ such that $y(t) = x(t) + q(t)x(t-\tau) > 0$ and is bounded eventually for sufficiently large $t \geq t_2$. Next we show that $p(t)(\Delta^{\nu} y(t))^{\alpha}$ is eventually positive. Suppose that there exists an integer $t_3 \geq t_2$ and a constant $c_1 > 0$ such that $p(t_3)(\Delta^{\nu} y(t_3))^{\alpha} = -c_1 < 0$. Then we have

$$p(t)(\Delta^{\nu} y(t))^{\alpha} \leq p(t_3)(\Delta^{\nu} y(t_3))^{\alpha} = -c_1 < 0 \text{ for } \ t \geq t_3.$$

That is

$$\Delta^{\nu} y(t) < -\left(\frac{c_1}{p(t)}\right)^{\frac{1}{\alpha}}, \text{ for } \ t \geq t_3. \tag{5}$$

From (5) we can write

$$\Delta(\Delta^{-(1-\nu)} y(t)) < -\left(\frac{c_1}{p(t)}\right)^{\frac{1}{\alpha}}, \text{ for } \ t \geq t_3. \tag{6}$$

Summing (6) from t_3 to $t-1$, we have

$$\Delta^{-(1-\nu)} y(t) < \Delta^{-(1-\nu)} y(t_3) - c_1^{\frac{1}{\alpha}} \sum_{s=t_3}^{t-\nu} \frac{1}{p^{\frac{1}{\alpha}}(s)} \tag{7}$$

Applying $\Delta^{(1-\nu)}$ to the both side of (7), we obtain

$$y(t) < -c_1^{\frac{1}{\alpha}} \Delta^{(1-\nu)} \left(\sum_{s=t_3}^{t-\nu} \frac{1}{p^{\frac{1}{\alpha}}(s)} \right).$$

Applying fractional sum $\Delta^{-\nu}$ to (5), by definition 1 and theorem 1 we obtain

$$\begin{aligned} y(t) &< \frac{t^{(\nu-1)}}{\Gamma(\nu)} c_2 - \frac{c_1^{\frac{1}{\alpha}}}{\Gamma(\nu)} \sum_{s=t_3}^{t-\nu} (t-s-1)^{(\nu-1)} \frac{1}{p^{\frac{1}{\alpha}}(s)} \\ &\leq \frac{1}{\Gamma(\nu)} \left(t^{(\nu-1)} c_2 - c_1^{\frac{1}{\alpha}} (t-t_3-1)^{(\nu-1)} \sum_{s=t_3}^{t-\nu} \frac{1}{p^{\frac{1}{\alpha}}(s)} \right) = -\infty \end{aligned}$$

as $t \to \infty$ by (H_2), where $c_2 = \Delta^{-\nu} y(0)$, which contradicts to the fact that $y(t) > 0$. Hence $p(t)(\Delta^{\nu} y(t))^{\alpha}$ is eventually positive. Now, since $y(t)$ is bounded, we can write

$\lim_{t\to\infty} y(t) = L$ $(-\infty < L < \infty)$. Assume that $0 \le L < +\infty$. Let $L > 0$. Then, there exists a constant $k > 0$ and a t_4 with $t_4 \ge t_3$ such that $y(t) > k > 0$ for $t \ge t_4$. Therefore, there exists a constant $k_1 > 0$ and a t_5 with $t_5 \ge t_4$ such that $x(t) = y(t) - q(t)x(t-\tau) > k_1 > 0$ for sufficiently large $t \ge t_5$. So, we can find a t_6 with $t_6 \ge t_5$ such that $x(t-\sigma) > k_1 > 0$ for $t \ge t_6$. Thus from (1) we have

$$\Delta(p(t)(\Delta^\nu y(t))^\alpha) \le -k_1^\beta r(t), \quad t \ge t_6. \tag{8}$$

If we multiply (8) by t^2 , and summing it from t_6 to $t-1$, we obtain

$$\begin{aligned} t^2 p(t)(\Delta^\nu y(t))^\alpha &\le c_3 + \sum_{s=t_6}^{t-1}[p(s+1)(\Delta^\nu y(s+1))^\alpha(2s+1) - k_1^\beta r(s)s^2] \\ &\le c_3 + \sum_{s=t_6}^{t-1}[p(s)(\Delta^\nu y(s))^\alpha(2s+1) - k_1^\beta r(s)s^2]. \end{aligned} \tag{9}$$

where $c_3 = t_6^2 p(t_6)(\Delta^\nu y(t_6))^\alpha$. Since $p(t)(\Delta^\nu y(t))^\alpha > 0$ and is nonincreasing, from (9) we have

$$t^2 p(t)(\Delta^\nu y(t))^\alpha \le c_3 - \sum_{s=t_6}^{t-1}[k_1^\beta r(s)s^2 - c_3(2s+1)].$$

as $t \to \infty$, virtue of by (H_2) this is a contradiction. So, $L > 0$ is impossible. Therefore, $L = 0$ is the only possible case. That is, $\lim_{t\to\infty} y(t) = 0$. Now, let us consider the case of $x(t) < 0$ for $t \ge t_1$. If we write $x(t) = -x(t)$, as in the proof of $x(t) > 0$, we can prove that $L = 0$. As for the rest, it is similar to the case of $x(t) > 0$. That is, $\lim_{t\to\infty} y(t) = 0$. This contradicts our assumption. Hence, the proof is completed.

Theorem 3. Assume that H_3 holds. Further, we suppose that the following condition holds.

C) There is a sequence $\varphi(t) > 0$ which is defined on $N(t_o)$, such that

$$\lim_{t\to\infty}\sup\sum_{s=t_3}^{t-1}\varphi(s)r(s) = \infty$$

and

$$\lim_{t\to\infty}\sup\sum_{s=t_3}^{t-1}\left[\left(\frac{\alpha}{\Delta\varphi(s)}\right)^\alpha\left(\frac{B}{\alpha+1}\frac{\varphi(s)}{p(s+1)}\right)^{\alpha+1}\right] < \infty.$$

Then every bounded solution of $Eq.(1)$ is oscillatory.

Proof. Suppose to the contrary that $x(t)$ is a nonoscillatory solution of $Eq.(1)$. Without loss of generality, we can assume that $x(t)$ is an eventually positive solution of (1) (the proof is similar when $x(t)$ is eventually negative). Then there exists $t_1 > t_0$ such that $x(t) > 0$, $x(t-\tau) > 0$ and $x(t-\sigma) > 0$ for all $t \ge t_1 \ge t_0$. Since $x(t)$ is bounded, we have $\lim\limits_{t\to\infty} q(t)x(t-\tau) = 0$. Then, there exists an integer $t_2 \ge t_1$ such that $y(t)$ is also bounded for sufficiently large $t \ge t_2$. Then, there exists an integer $t_3 \ge t_2$ such that $x(t) = y(t) - q(t)x(t-\tau) \ge \frac{1}{2}y(t) > 0$ for $t \ge t_3$. Hence we can find a $t_4 \ge t_3$ such that

$$x(t-\sigma) \ge \frac{1}{2}y(t-\sigma) \quad \text{for } t \ge t_4.$$

Therefore by $Eq.$(1) and (3), we have

$$\Delta(p(t)(\Delta^{\nu}y(t))^{\alpha}) \leq -\frac{1}{2^{\beta}}r(t)y^{\beta}(t-\sigma), \quad t \geq t_4. \tag{10}$$

Define the function $w(t)$ by the Riccati substitution

$$w(t) = \frac{p(t)(\Delta^{\nu}y(t))^{\alpha}}{y^{\alpha}(t)}. \tag{11}$$

Since $p(t)(\Delta^{\nu}y(t))^{\alpha}$ and $y(t)$ are positive, $w(t) > 0$. If we apply the forward difference operator Δ to (11) we obtain

$$\begin{aligned}\Delta w(t) &= \Delta\left(\frac{p(t)(\Delta^{\nu}y(t))^{\alpha}}{y^{\alpha}(t)}\right) \\ &= \frac{y^{\alpha}(t)\Delta(p(t)(\Delta^{\nu}y(t))^{\alpha}) - p(t)(\Delta^{\nu}y(t))^{\alpha}\Delta y^{\alpha}(t)}{y^{\alpha}(t)y^{\alpha}(t+1)} \\ &= \frac{\Delta(p(t)(\Delta^{\nu}y(t))^{\alpha})}{y^{\alpha}(t+1)} - \frac{p(t)(\Delta^{\nu}y(t))^{\alpha}}{y^{\alpha}(t)}\frac{\Delta y^{\alpha}(t)}{y^{\alpha}(t+1)}\end{aligned} \tag{12}$$

By (H_1) and (10) from (12) we have

$$\begin{aligned}\Delta w(t) &= \frac{\Delta(p(t)(\Delta^{\nu}y(t))^{\alpha})}{y^{\alpha}(t+1)} - \frac{p(t)(\Delta^{\nu}y(t))^{\alpha}}{y^{\alpha}(t)}\frac{\Delta y^{\alpha}(t)}{y^{\alpha}(t+1)} \\ &\leq -\frac{1}{2^{\beta}}r(t)\frac{y^{\beta}(t-\sigma)}{y^{\alpha}(t+1)} - \frac{p(t)(\Delta^{\nu}y(t))^{\alpha}}{y^{\alpha}(t+1)}\frac{\Delta y^{\alpha}(t)}{y^{\alpha}(t)} \\ &\leq -\frac{1}{2^{\beta}}r(t)\frac{y^{\beta}(t-\sigma)}{y^{\alpha}(t+1)} - w(t)\frac{\Delta y^{\alpha}(t)}{y^{\alpha}(t+1)} \\ &= -\frac{1}{2^{\beta}}r(t)\frac{y^{\beta}(t-\sigma)}{y^{\alpha}(t+1)} - w(t)\frac{p^{\frac{1}{\alpha}}(t+1)\Delta^{\nu}y(t+1)}{y(t+1)}\frac{\Delta y^{\alpha}(t)}{p^{\frac{1}{\alpha}}(t+1)\Delta^{\nu}y(t+1)y^{\alpha-1}(t+1)} \\ &= -\frac{1}{2^{\beta}}r(t)A - w(t)w^{\frac{1}{\alpha}}(t+1)\frac{B}{p^{\frac{1}{\alpha}}(t+1)} \\ &\leq 0.\end{aligned} \tag{13}$$

Since $w(t)$is nonincreasing, $w(t+1) \leq w(t)$.From (13) we have

$$\Delta w(t) \leq -\frac{1}{2^{\beta}}r(t)A - \frac{B}{p^{\frac{1}{\alpha}}(t+1)}w^{1+\frac{1}{\alpha}}(t+1). \tag{14}$$

Multiplying the inequality (14) by a sequence $\varphi(t) > 0$ and summing up it from t_3 to $t-1$, we obtain

$$\varphi(t)w(t) \leq \varphi(t_3)w(t_3) + \sum_{s=t_3}^{t-1} w(s+1)\Delta\varphi(s) - \sum_{s=t_3}^{t-1}[\frac{A}{2^{\beta}}\varphi(s)r(s) - \frac{B}{p^{\frac{1}{\alpha}}(s+1)}w^{1+\frac{1}{\alpha}}(s+1)\varphi(s)]$$

or

$$\varphi(t)w(t+1) \leq \varphi(t_3)w(t_3) - \sum_{s=t_3}^{t-1}\frac{A}{2^{\beta}}\varphi(s)r(s) + \sum_{s=t_3}^{t-1}[\Delta\varphi(s)w(s+1) - \frac{B\varphi(s)}{p^{\frac{1}{\alpha}}(s+1)}w^{1+\frac{1}{\alpha}}(s+1)]. \tag{15}$$

Get $F(w) = Mw - Nw^{1+\frac{1}{\alpha}}$ in (15) where $M = \Delta\varphi(s) > 0$ and $N = \frac{B\varphi(s)}{p^{\frac{1}{\alpha}}(s+1)} > 0$. The function F has the maximum value at $w = \left(\frac{\alpha M}{(\alpha+1)N}\right)^{\alpha}$ such that $F_{\max}(w) = \left(\frac{\alpha}{N}\right)^{\alpha}\left(\frac{M}{\alpha+1}\right)^{\alpha+1}$. Therefore from (15) we can write

$$-\varphi(t_3)w(t_3) \le -\sum_{s=t_3}^{t-1}\frac{A}{2^{\beta}}\varphi(s)r(s) + \sum_{s=t_3}^{t-1}\left[\left(\frac{\alpha}{\Delta\varphi(s)}\right)^{\alpha}\left(\frac{B}{\alpha+1}\frac{\varphi(s)}{p^{\frac{1}{\alpha}}(s+1)}\right)^{\alpha+1}\right]$$

which contradicts with the condition (C), when $t \to \infty$. Therefore $x(t)$ can not be positive. Hence the proof is completed.

Example 1 Consider the equation

$$\Delta(t^3(\Delta^{\nu}(x(t) + \left(-\frac{1}{2}\right)^{t} x(t-1)))^{\alpha}) + r(t)x^{\beta}(s-2) = 0, \quad t \in \mathbb{N}_{t_0+1-\nu}, \tag{16}$$

where $p(t) = t^3$, $q(t) = \left(-\frac{1}{2}\right)^t$, $\alpha = \frac{1}{3}$, $\beta = \frac{1}{5}$, $\tau = 1$, $r(t) = t-2$ and $\sigma = 2$. Choosing $\varphi(t) = t^2$ we have

$$\lim_{t\to\infty}\sup\sum_{s=t_3}^{t-1}\varphi(s)r(s) = \lim_{t\to\infty}\sup\sum_{s=t_3}^{t-1}s^2(s-2) = \infty$$

and

$$\lim_{t\to\infty}\sup\sum_{s=t_3}^{t-1}\left[\left(\frac{\alpha}{\Delta\varphi(s)}\right)^{\alpha}\left(\frac{B}{\alpha+1}\frac{\varphi(s)}{p^{\frac{1}{\alpha}}(s+1)}\right)^{\alpha+1}\right] = B^{\frac{4}{3}}\lim_{t\to\infty}\sup\sum_{s=t_3}^{t-1}\frac{3(s-1)^{\frac{4}{3}}}{8s^3} = B^{\frac{4}{3}}(0.289\,47) < \infty.$$

Hence all the conditions of Theorem 3 are provided. Therefore every bounded solution of the equation (16) is oscillatory.

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