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SUBORDINATION AND SUPERORDINATION RESULTS ASSOCIATED WITH FRASIN DIFFERENTIAL OPERATOR

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Abstract. Using a differential operator $D^{\alpha}_{m,\lambda}f(z)$ and making use of binomial series, we introduce and study some differential subordination and superordination results for analytic functions in the open unit disk. These results are obtained by investigating appropriate classes of admissible functions.

1. Introduction and definitions

Let $\mathcal{H}(\mathbb{D})$ denote the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{S}(\mathbb{D})$ denote the subclass of $\mathcal{H}(\mathbb{D})$ consisting of functions which are also univalent in $\mathbb{D}$. Further, for $a \in \mathbb{C}$ and $n \in \mathbb{N}$ consider $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(\mathbb{D})$ consisting of function of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$, with $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H}_1 \equiv \mathcal{H}[1, 1]$. Let $\mathcal{A}$ denote the class of the functions $\mathcal{H}[a, 1]$ which are normalized by the condition $f(0) = 0 = f'(0) - 1$ and have representation of the form

$$f(z) = z + \sum_{n\geq 2} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

Given two functions $f \in \mathcal{H}(\mathbb{D})$ and $g \in \mathcal{H}(\mathbb{D})$, we say that f is subordinated to g in $\mathbb{D}$ and write $f(z) \prec g(z)$, if there exists a Schwarz function w, analytic in $\mathbb{D}$, with $w(0) = 0, |w(z)| < |z|$, $z \in \mathbb{D}$, such that $f(z) = g(w(z))$ in $\mathbb{D}$. In particular, if $g(z)$ is univalent in $\mathbb{D}$, we have the following equivalence: $f(z) \prec g(z),\ z \in \mathbb{D} \Longleftrightarrow f(0) = g(0) \quad \text{and} f(\mathbb{D}) \subset g(\mathbb{D})$. We denote by $\mathcal{Q}$ the class of functions q that are analytic and injective on $\overline{\mathbb{D}} \setminus \mathrm{E}(q)$, where

$$\mathrm{E}(q) = \left\{ \zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} q(z) = \infty \right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus \mathrm{E}(q)$. Further let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$ with $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

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For a function f in $\mathcal{A}$ and making use of the binomial series

$$(1-\lambda)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \lambda^j \quad \big(m \in \mathbb{N} = \{1,2,...\},\ j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\big),$$

Frasin [5] (see also, [18]) defined the differential operator $D^{\alpha}_{m,\lambda} f(z)$ as follows:

$$D^0 f(z) = f(z), \tag{2}$$

$$D^1_{m,\lambda} f(z) = (1-\lambda)^m f(z) + (1-(1-\lambda)^m) z f'(z) \tag{3}$$

$$= D_{m,\lambda} f(z), \quad \lambda > 0;\ m \in \mathbb{N}, \tag{4}$$

$$D^{\alpha}_{m,\lambda} f(z) = D_{m,\lambda}(D^{\alpha-1} f(z)) \quad (\alpha \in \mathbb{N}). \tag{5}$$

If f is given by (1), then from (4) and (5) we see that

$$D^{\alpha}_{m,\lambda} f(z) = z + \sum_{n=2}^{\infty} \left(1 + (n-1) \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j \right)^{\alpha} a_n z^n, \quad \alpha \in \mathbb{N}_0. \tag{6}$$

Using the relation (6), it is easily verified that

$$C^m_j(\lambda) z (D^{\alpha}_{m,\lambda} f(z))' = D^{\alpha+1}_{m,\lambda} f(z) - (1 - C^m_j(\lambda)) D^{\alpha}_{m,\lambda} f(z), \tag{7}$$

where $C^m_j(\lambda) = \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j$.

Here we observe that for $m = 1$, we obtain the differential operator $D^{\alpha}_{1,\lambda} f(z)$ defined by Al-Oboudi [3] and for $m = \lambda = 1$, we get Salagean differential operator D^{α} [11].

Let Ω and Δ be any set in $\mathbb{C}$, let p be analytic function in $\mathbb{D}$ with $p(0) = a$ and let $\psi(r,s,t;z) : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$, then Miller and Mocanu [8] studied implication of the form

$$\big\{\psi(p(z), zp'(z), z^2 p''(z);\ z) :\ z \in \mathbb{D}\big\} \subset \Omega \implies p(\mathbb{D}) \subset \Delta. \tag{8}$$

If Δ is a simply connected domain containing the point a and $\Delta \neq \mathbb{C}$, then the Riemann mapping theorem ensure that there is a conformal mapping q of $\mathbb{D}$ onto Δ such that $q(0) = a$. In this case (8) can be rewritten as

$$\big\{\psi(p(z), zp'(z), z^2 p''(z);\ z) :\ z \in \mathbb{D}\big\} \subset \Omega \implies p(z) \prec q(z). \tag{9}$$

Further, if Ω is a simply connected domain and $\Omega \neq \mathbb{C}$, then there is a conformal mapping h of $\mathbb{D}$ onto Ω such that $h(0) = \psi(a,0,0;0)$. If in addition, the function $\psi(p(z), zp'(z), z^2 p''(z);\ z)$ is analytic in $\mathbb{D}$, then (8) can be rewritten as

$$\psi(p(z), zp'(z), z^2 p''(z);\ z) \prec h(z) \implies p(z) \prec q(z). \tag{10}$$

In this article, for suitable defined classes of admissible functions, involving the differential operator $D^{\alpha}_{m,\lambda} f(z)$ with binomial series, we study implications of the form (9) and (10). Through the simple algebraic check of admissible functions, we get various subordination, superordination and differential inequalities that would be difficult to obtain directly. Aghalary *et al.* [1], Ali *et al.* [2], Baricz *et al.* [4], Prajapat *et al.* [10] and Soni *et al.*[14] have considered similar problem for various linear and multiplier operators. To investigate our results, we need the following definitions and lemmas.

Definition 1.1. ([8], Definition 2.3a, p.27). Let Ω be a set in $\mathbb{C}$, $q(z) \in \mathcal{Q}$

and $n \in \mathbb{N}$. The class of admissible functions $\Psi_n(\Omega, q)$ consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r, s, t; z) \notin \Omega,$$

whenever

$$r = q(\zeta), \quad s = k\zeta\, q'(\zeta) \quad \text{and} \quad \Re\Big(\frac{t}{s} + 1\Big) \geq k\,\Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\Big),$$

for $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D} \setminus E(q)$ and $k \geq n$. In particular, $\Psi_1(\Omega, q) \equiv \Psi(\Omega, q)$.

Definition 1.2. ([9], Definition 3, p.817). Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n(\Omega, q)$ consists of those function $\psi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r, s, t; \zeta) \in \Omega,$$

whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m} \quad \text{and} \quad \Re\Big(\frac{t}{s} + 1\Big) \leq \frac{1}{m}\,\Re\Big(\frac{zq''(z)}{q'(z)} + 1\Big),$$

for $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$ and $m \geq n \geq 1$. In particular, $\Psi'_1(\Omega, q) \equiv \Psi'(\Omega, q)$.

Lemma 1.1. ([8],Theorem 2.3b, p.28). Let $\psi \in \Psi_n(\Omega, q)$ with $q(0) = a$. If $p \in \mathcal{H}[a, n]$ satisfies

$$\psi\Big(p(z), zp'(z), z^2p''(z); z\Big) \in \Omega,$$

then $p(z) \prec q(z)$.

Lemma 1.2. ([9], Theorem 1, p.818). Let $\psi \in \Psi'_n(\Omega, q)$ with $q(0) = a$. If $p \in \mathcal{Q}(a)$ and $\psi\Big(p(z), zp'(z), z^2p''(z); z\Big) \in \mathcal{S}(\mathbb{D})$, then

$$\Omega \subset \Big\{\psi\Big(p(z), zp'(z), z^2p''(z); z\Big) : \ z \in \mathbb{D}\ \Big\},$$

implies $q(z) \prec p(z)$.

For some recent investigations of subordination and superordination results and sandwich results in analytic function theory, one can find in ([6], [7], [12], [13], [15], [16], [17]).

2. Main results

First we define the following class of admissible functions that will be required in our first result.

Definition 2.1. Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}_0 \bigcap \mathcal{H}_0$. The class of admissible functions $\Phi_H(\Omega, q)$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega,$$

whenever

$$u = q(\zeta), \quad v = C_j^m(\lambda)k\zeta q'(\zeta) + (1 - C_j^m(\lambda))q(\zeta)$$

and

$$\Re\Big(\frac{1}{C_j^m(\lambda)}\Big\{\frac{w - (1 - C_j^m(\lambda))^2u}{v - (1 - C_j^m(\lambda))u} - 2\Big\} + 2\Big) \geq k\,\Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\Big),$$

for $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}\backslash E(q)$, $k \geq 1, \lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Theorem 2.1. Let $\phi \in \Phi_H(\Omega, q), \lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $f \in \mathcal{A}$ satisfies

$$\{\phi\big(D^{\alpha}_{m,\lambda}f(z), D^{\alpha+1}_{m,\lambda}f(z), D^{\alpha+2}_{m,\lambda}f(z); z\big) : z \in \mathbb{D}\} \subset \Omega, \tag{11}$$

then

$$D^{\alpha}_{m,\lambda}f(z) \prec q(z), \qquad (z \in \mathbb{D}).$$

Proof. Let us consider the analytic function $p(z)$ in $\mathbb{D}$ by

$$p(z) = D^{\alpha}_{m,\lambda}f(z).$$

By using recurrence relation (7), we get

$$D^{\alpha+1}_{m,\lambda}f(z) = C^m_j(\lambda)zp'(z) + (1 - C^m_j(\lambda))p(z)$$

and

$$D^{\alpha+2}_{m,\lambda}f(z) = (C^m_j(\lambda))^2z^2p''(z) + C^m_j(\lambda)(2 - C^m_j(\lambda))zp'(z) + (1 - C^m_j(\lambda))^2p(z).$$

Define the transformation from $\mathbb{C}^3$ *to* $\mathbb{C}$ by

$u = r$, $v = (C^m_j(\lambda))s+(1-C^m_j(\lambda))r$ and $w = (C^m_j(\lambda))^2t+C^m_j(\lambda)(2-C^m_j(\lambda))s+(1-C^m_j(\lambda))^2r.$

Let $\psi(r, s, t; z) = \phi(u, v, w; z)$

$$= \phi\Big(r, C^m_j(\lambda)s+(1-C^m_j(\lambda))r, (C^m_j(\lambda))^2t+C^m_j(\lambda)(2-C^m_j(\lambda))s+(1-C^m_j(\lambda))^2r; z\Big).$$

By using the above equations, we obtain

$$\psi\Big(p(z), zp'(z), z^2p''(z); z\Big) = \phi\left(D^{\alpha}_{m,\lambda}f(z), D^{\alpha+1}_{m,\lambda}f(z), D^{\alpha+2}_{m,\lambda}f(z); z\right),$$

and hence (11) becomes

$$\psi\Big(p(z), zp'(z), z^2p''(z); z\Big) \in \Omega.$$

To complete the proof, we need to show that the admissibility condition for $\phi \in \Phi_H(\Omega, q)$ is equivalent to the admissibility condition for ψ as given in Definition 2.1. On the other hand, we note that

$$\frac{t}{s} + 1 = \frac{1}{C^m_j(\lambda)}\Big\{\frac{w - (1 - C^m_j(\lambda))^2u}{v - (1 - C^m_j(\lambda))u} - 2\Big\} + 2,$$

this shows that $\psi \in \Psi(\Omega, q)$. Hence by Lemma 1.1, we have $p(z) \prec q(z)$. This completes the proof of Theorem 2.1.

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case, the class $\Phi_H(h(\mathbb{D}), q)$ is written as $\Phi_H(h, q)$. The following result is an immediate consequence of Theorem 2.1.

Corollary 2.1. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ *and* $\phi \in \Phi_H(h, q)$ *with* q(0)=0. If $f \in \mathcal{A}$ satisfies

$$\phi\left(D^{\alpha}_{m,\lambda}f(z), D^{\alpha+1}_{m,\lambda}f(z), D^{\alpha+2}_{m,\lambda}f(z); z\right) \prec h(z), \tag{12}$$

then

$$D^{\alpha}_{m,\lambda}f(z) \prec q(z), \qquad z \in \mathbb{D}.$$

Our next result is an extension of Theorem 2.1 to the case when the behavior of q on $\partial\mathbb{D}$ is not known.

Corollary 2.2. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \Omega \subset \mathbb{C}$ and $q \in \mathcal{S}(\mathbb{D})$ with $q(0) = 0$. Let $\phi \in \Phi_H(\Omega, q_\rho)$ for some $\rho \in (0, 1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}$ satisfies

$$\phi\left(D^{\alpha}_{m,\lambda}f(z), D^{\alpha+1}_{m,\lambda}f(z), D^{\alpha+2}_{m,\lambda}f(z); z\right) \in \Omega,$$

then

$$D^{\alpha}_{m,\lambda}f(z) \prec q(z), \quad z \in \mathbb{D}.$$

Proof. Corollary 2.1. yields that under the hypothesis $D^{\alpha}_{m,\lambda}f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$.
The following main result is similar to corollary 2.1.
Theorem 2.2. Let $h, q \in \mathcal{H}(\mathbb{D})$ with $q(0) = 0$, also set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Suppose that $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfies one of the following conditions:
(i) $\phi \in \Phi_H(h, q_\rho)$ for some $\rho \in (0, 1)$, or
(ii) there exist $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_H(h_\rho, q_\rho)$ for all $\rho \in (\rho_0, 1)$.
If $f \in \mathcal{A}$ satisfies (12), then

$$D^{\alpha}_{m,\lambda}f(z) \prec q(z), \quad z \in \mathbb{D}.$$

Proof. The proof is similar to the proof of a known result ([8], Theorem 2.3d, p.30) and so it is omitted here.

The next theorem yields the best dominant of the differential subordination.
Theorem 2.3. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $h \in \mathcal{S}(\mathbb{D})$. Suppose that $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ and the differential equation

$$\phi\Big(q(z), C^m_j(\lambda)zq'(z) + (1 - (C^m_j(\lambda))q(z),$$

$$(C^m_j(\lambda))^2z^2q''(z) + C^m_j(\lambda)(2 - C^m_j(\lambda))zq'(z) + (1 - (C^m_j(\lambda))^2q(z); z\Big) = h(z)$$

has a solution q with $q(0) = 0$, which satisfies one of the following conditions:
(i) $q \in \mathcal{Q}_0$ and $\phi \in \Phi_H(h, q)$,
(ii) $q \in \mathcal{S}(\mathbb{D})$ and $\phi \in \Phi_H(h, q_\rho)$, for some $\rho \in (0, 1)$, or
(iii) $q \in \mathcal{S}(\mathbb{D})$ and there exist $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_H(h_\rho, q_\rho)$ for all $\rho \in (\rho_0, 1)$.
If $f \in \mathcal{A}$ satisfies (12) then

$$D^{\alpha}_{m,\lambda}f(z) \prec q(z), \quad z \in \mathbb{D}$$

and q is the best dominant.
Proof. By applying corollary 2.1 and Theorem 2.2, we deduce that $q(z)$ is a dominant of (12). Since q satisfies the subordination $D^{\alpha}_{m,\lambda}f(z) \prec q(z)$, it is also a solution of (12) and therefore q will be dominated by all dominants of (12). Hence q is the best dominant.
In the particular case $q(z) = Mz$, $M > 0$, and in view of Definition 2.1, the class of admissible functions $\Phi_H(\Omega, q)$ is denoted by $\Phi_H(\Omega, M)$, as described below.
Definition 2.2. Let Ω be a set in $\mathbb{C}$ and $M > 0$. Suppose $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The class of admissible function $\Phi_H(\Omega, M)$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that

$$\phi\Big(Me^{i\theta}, (1 + (k-1)C^m_j(\lambda))Me^{i\theta}, \tag{13}$$

$$(C^m_j(\lambda))^2L + \big(1 + (k-1)C^m_j(\lambda)(2 - C^m_j(\lambda))\big)Me^{i\theta}; z\Big) \notin \Omega,$$

whenever $z \in \mathbb{D}$, $\Re(Le^{-i\theta}) \geq Mk(k-1)$, $\theta \in \mathbb{R}$, and $k \geq 1$.
Corollary 2.3. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $\phi \in \Phi_H(\Omega, M)$. If $f \in \mathcal{A}$ satisfies

$$\phi\left(D^{\alpha}_{m,\lambda}f(z), D^{\alpha+1}_{m,\lambda}f(z), D^{\alpha+2}_{m,\lambda}f(z); z\right) \in \Omega,$$

then

$$\left|D^{\alpha}_{m,\lambda}f(z)\right| < M, \quad z \in \mathbb{D}.$$

In the special case $\Omega = q(\mathbb{D}) = \{\omega : |\omega| < M\}$, the class $\Phi_H(\Omega, M)$ is simply denoted by $\Phi_H(M)$ and thus Corollary 2.3 can be written in the following form.

Corollary 2.4. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $\phi \in \Phi_H(M)$. If $f \in \mathcal{A}$ satisfies

$$\left|\phi\left(D^{\alpha}_{m,\lambda}f(z), D^{\alpha+1}_{m,\lambda}f(z), D^{\alpha+2}_{m,\lambda}f(z); z\right)\right| < M,$$

then

$$\left|D^{\alpha}_{m,\lambda}f(z)\right| < M, \quad z \in \mathbb{D}.$$

Corollary 2.5. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $M > 0$. If $f(z) \in \mathcal{A}$ satisfies

$$\left|D^{\alpha+1}_{m,\lambda}f(z) - D^{\alpha}_{m,\lambda}f(z)\right| < MC^m_j(\lambda),$$

then

$$\left|D^{\alpha}_{m,\lambda}f(z)\right| < M, \quad z \in \mathbb{D}.$$

Proof. Let $\phi(u, v, w; z) = v - u$ and $\Omega = h(\mathbb{D})$, where $h(z) = MzC^m_j(\lambda)$ and $M > 0$. To use Corollary 2.3, we need to show that $\phi \in \Phi_H(\Omega, M)$. That is, the admissibility condition in definition 2.2 is satisfied. This follows since

$$\left|\phi\Big(Me^{i\theta}, (1 + (k-1)C^m_j(\lambda))Me^{i\theta}, (C^m_j(\lambda))^2L + \big(1 + (k-1)C^m_j(\lambda)(2 - C^m_j(\lambda))\big)Me^{i\theta}; z\Big)\right|$$
$$= \left|(1 + (k-1)C^m_j(\lambda))Me^{i\theta} - Me^{i\theta}\right| \geq MC^m_j(\lambda),$$

whenever $z \in \mathbb{D}$, $\theta \in \mathbb{R}$ and $k \geq 1$.The required result now follows from Corollary 2.3. Theorem 2.3, shows that the result is sharp. The differential equation $zq'(z) = Mz$ has a univalent solution $q(z) = Mz$, which in view of Theorem 2.3 is the best dominant.

Definition 2.3. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}_1$. The class of admissible functions $\Phi_{H,1}(\Omega, q)$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega,$$

whenever $q(\zeta) \neq 0$, we have $u = q(\zeta)$,

$$v = C^m_j(\lambda)\frac{k\zeta q'(\zeta)}{q(\zeta)} + q(\zeta) \text{ and}$$

$$\Re\left(\frac{1}{C^m_j(\lambda)}\Big(\frac{wv + 2u^2 - 3uv}{v - u}\Big)\right) \geq k\,\Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\Big),$$

where $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{D},\ \zeta \in \partial\mathbb{D}\backslash E(q),$ and $k \geq 1$.

Theorem 2.4. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $\phi \in \Phi_{H,1}(\Omega, q)$. If $f(z) \in \mathcal{A}$ satisfies

$$\left\{\phi\left(\frac{D^{\alpha+1}_{m,\lambda}f(z)}{D^{\alpha}_{m,\lambda}f(z)}, \frac{D^{\alpha+2}_{m,\lambda}f(z)}{D^{\alpha+1}_{m,\lambda}f(z)}, \frac{D^{\alpha+3}_{m,\lambda}f(z)}{D^{\alpha+2}_{m,\lambda}f(z)}, z\right); z \in \mathbb{D}\right\} \subset \Omega,$$

then

$$\frac{D^{\alpha+1}_{m,\lambda}f(z)}{D^{\alpha}_{m,\lambda}f(z)} \prec q(z), \quad z \in \mathbb{D}.$$

Proof. Define the analytic function

$$p(z) = \frac{D^{\alpha+1}_{m,\lambda}f(z)}{D^{\alpha}_{m,\lambda}f(z)}, \quad z \in \mathbb{D}.$$

Logarithmic differentiation gives

$$\frac{D^{\alpha+2}_{m,\lambda}f(z)}{D^{\alpha+1}_{m,\lambda}f(z)} = C^m_j(\lambda)\frac{zp'(z)}{p(z)} + p(z)$$

and $\frac{D^{\alpha+3}_{m,\lambda}f(z)}{D^{\alpha+2}_{m,\lambda}f(z)} = C^m_j(\lambda)\frac{zp'(z)}{p(z)} + p(z)$

$$+\frac{C^m_j(\lambda)zp'(z) + (C^m_j(\lambda))^2\Big(\frac{z^2p''(z)}{p(z)} + \frac{zp'(z)}{p(z)} - (\frac{zp'(z)}{p(z)})^2\Big)}{C^m_j(\lambda)\frac{zp'(z)}{p(z)} + p(z)}.$$

Now, we define the transformation from $\mathbb{C}^3$ *to* $\mathbb{C}$ by:

$$u = r, \quad v = C^m_j(\lambda)\frac{s}{r} + r,$$

$$w = C^m_j(\lambda)\frac{s}{r} + r + \frac{C^m_j(\lambda)s + (C^m_j(\lambda))^2\Big(\frac{t}{r} + \frac{s}{r} - (\frac{s}{r})^2\Big)}{C^m_j(\lambda)\frac{s}{r} + r}$$

Let

$$\begin{aligned}\psi(r,s,t;z) &= \phi(u,v,w;z)\\ &= \phi\Big(r,\ C^m_j(\lambda)\frac{s}{r} + r, \qquad (14)\\ C^m_j(\lambda)\frac{s}{r} + r &+ \frac{C^m_j(\lambda)s + (C^m_j(\lambda))^2\Big(\frac{t}{r} + \frac{s}{r} - (\frac{s}{r})^2\Big)}{C^m_j(\lambda)\frac{s}{r} + r}; z\Big).\end{aligned}$$

Using the above equations, it follows that

$$\psi\Big(p(z), zp'(z), z^2p''(z); z\Big) = \phi\left(\frac{D^{\alpha+1}_{m,\lambda}f(z)}{D^{\alpha}_{m,\lambda}f(z)}, \frac{D^{\alpha+2}_{m,\lambda}f(z)}{D^{\alpha+1}_{m,\lambda}f(z)}, \frac{D^{\alpha+3}_{m,\lambda}f(z)}{D^{\alpha+2}_{m,\lambda}f(z)}; z\right). \qquad (15)$$

Hence this implies that $\psi\Big(p(z), zp'(z), z^2p''(z); z\Big) \in \Omega$.
Hence proof is completed, if it can be shown that the admissibility condition $\phi \in \Phi_{H,1}(\Omega, q)$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = \frac{1}{C^m_j(\lambda)}\Big(\frac{wv + 2u^2 - 3uv}{v-u}\Big).$$

Hence $\psi \in \Psi(\Omega, q)$ and by Lemma 1.1, we get $p(z) \prec q(z)$, which completes the proof.
In the case when $\Omega \neq \mathbb{C}$ is a simply connected domain with $\Omega = h(\mathbb{D})$ for some conformal mapping $h : \mathbb{D} \to \Omega$, the class $\Phi_{H,1}(h(\mathbb{D}), q)$ is written as $\Phi_{H,1}(h, q)$. The following result is an immediate consequence of above theorem.
Corollary 2.6. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $\phi \in \Phi_{H,1}(h, q)$ with $q(0) = 1$. If $f \in \mathcal{A}$ satisfies

$$\phi\left(\frac{D^{\alpha+1}_{m,\lambda}f(z)}{D^{\alpha}_{m,\lambda}f(z)}, \frac{D^{\alpha+2}_{m,\lambda}f(z)}{D^{\alpha+1}_{m,\lambda}f(z)}, \frac{D^{\alpha+3}_{m,\lambda}f(z)}{D^{\alpha+2}_{m,\lambda}f(z)}; z\right) \prec h(z), \quad z \in \mathbb{D}$$

then

$$\frac{D^{\alpha+1}_{m,\lambda}f(z)}{D^{\alpha}_{m,\lambda}f(z)} \prec q(z), \quad z \in \mathbb{D}.$$

In the particular case $q(z) = 1 + Mz$, $M > 0$. The class of admissible function $\Phi_{H,1}(\Omega, q)$ is simply denoted by $\Phi_{H,1}(\Omega, M)$.

Definition 2.4. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,1}(\Omega, M)$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that

$$\phi\Big(1 + Me^{i\theta}, 1 + Me^{i\theta}\big(1 + \frac{kC_j^m(\lambda)}{1 + Me^{i\theta}}\big), \tag{16}$$
$$\frac{(C_j^m(\lambda))^2(L + kMe^{i\theta}) + (1 + Me^{i\theta})\big((1 + Me^{i\theta})^2 + 3kC_j^m(\lambda)Me^{i\theta}\big)}{(1 + Me^{i\theta})^2 + C_j^m(\lambda)(kMe^{i\theta})}; z\Big) \notin \Omega$$

whenever $z \in \mathbb{D}$, $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\Re(Le^{-i\theta}) \geq (k-1)kM$, for all real θ and $k \geq 1$.

Corollary 2.7. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $\phi \in \Phi_{H,1}(\Omega, M)$. If $f \in \mathcal{A}$ satisfies

$$\phi\left(\frac{D_{m,\lambda}^{\alpha+1}f(z)}{D_{m,\lambda}^{\alpha}f(z)}, \frac{D_{m,\lambda}^{\alpha+2}f(z)}{D_{m,\lambda}^{\alpha+1}f(z)}, \frac{D_{m,\lambda}^{\alpha+3}f(z)}{D_{m,\lambda}^{\alpha+2}f(z)}; z\right) \in \Omega$$

then

$$\Big|\frac{D_{m,\lambda}^{\alpha+1}f(z)}{D_{m,\lambda}^{\alpha}f(z)} - 1\Big| < M, \quad z \in \mathbb{D}.$$

In the special case $\Omega = q(\mathbb{D}) = \{\omega : |\omega - 1| < M\}$, the class $\Phi_{H,1}(\Omega, M)$ is denoted by $\Phi_{H,1}(M)$ and Corollary 2.7 takes the following form.

Corollary 2.8. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $\phi \in \Phi_{H,1}(M)$. If $f(z) \in \mathcal{A}$ satisfies

$$\Big|\phi\left(\frac{D_{m,\lambda}^{\alpha+1}f(z)}{D_{m,\lambda}^{\alpha}f(z)}, \frac{D_{m,\lambda}^{\alpha+2}f(z)}{D_{m,\lambda}^{\alpha+1}f(z)}, \frac{D_{m,\lambda}^{\alpha+3}f(z)}{D_{m,\lambda}^{\alpha+2}f(z)}; z\right) - 1\Big| < M,$$

then

$$\Big|\frac{D_{m,\lambda}^{\alpha+1}f(z)}{D_{m,\lambda}^{\alpha}f(z)} - 1\Big| < M, \quad z \in \mathbb{D}.$$

Crollary 2.9. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $M > 0$. If $f \in \mathcal{A}$ satisfies

$$\Big|\Big(\frac{D_{m,\lambda}^{\alpha+2}f(z)}{D_{m,\lambda}^{\alpha+1}f(z)} - \frac{D_{m,\lambda}^{\alpha+1}f(z)}{D_{m,\lambda}^{\alpha}f(z)}\Big)\Big| < \frac{MC_j^m(\lambda)}{M+1},$$

then

$$\Big|\frac{D_{m,\lambda}^{\alpha+1}f(z)}{D_{m,\lambda}^{\alpha}f(z)} - 1\Big| < M, \quad z \in \mathbb{D}.$$

Proof. This follows from Corollary 2.7 by taking $\phi(u, v, w; z) = v - u$ and $\Omega = h(\mathbb{D})$, where $h(z) = C_j^m(\lambda)\dfrac{Mz}{M+1}$, $M > 0$. To use Corollary 2.7, we need to show that $\phi \in \Phi_{H,1}(M)$, that is the admissibility condition (16) is satisfied. This follows since,

$$|\phi(u, v, w; z)| = \left|1 + Me^{i\theta}\big(1 + \frac{kC_j^m(\lambda)}{1 + Me^{i\theta}}\big) - (1 + Me^{i,\theta})\right|$$
$$= \frac{Mk}{|1 + Me^{i\theta}|} \geq C_j^m(\lambda)\frac{M}{1+M}$$

for $z \in \mathbb{D}$, $\theta \in \mathbb{R}$ and $k \geq 1$. Hence the result is easily deduced from Corollary 2.7. We define the following class of admissible functions that will be required in our first result.

Definition 2.5. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \bigcap \mathcal{H}_1$. The class of admissible functions $\Phi_{H,2}(\Omega, q)$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; z) \notin \Omega,$$

whenever

$$u = q(\zeta), \quad v = q(\zeta) + k\zeta q'(\zeta) C_j^m(\lambda)$$

and

$$\Re\Big(\frac{1}{C_j^m(\lambda)}\Big\{\frac{w-u}{v-u} - 2\Big\}\Big) \geq k\,\Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\Big),$$

for $z \in \mathbb{D}$, $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\zeta \in \partial\mathbb{D} \backslash E(q)$, $k \geq 1$.

Theorem 2.5. Let $\phi \in \Phi_{H,2}(\Omega, q)$. If $f \in \mathcal{A}$ satisfies

$$\left\{\phi\Big(\frac{D^{\alpha}_{m,\lambda} f(z)}{z}, \frac{D^{\alpha+1}_{m,\lambda} f(z)}{z}, \frac{D^{\alpha+2}_{m,\lambda} f(z)}{z}; z\Big) : z \in \mathbb{D}\right\} \subset \Omega,$$

then

$$\frac{D^{\alpha}_{m,\lambda} f(z)}{z} \prec q(z), \qquad z \in \mathbb{D}.$$

Proof. Let us consider the analytic function $p(z)$ in $\mathbb{D}$ by

$$p(z) = \frac{D^{\alpha}_{m,\lambda} f(z)}{z}.$$

Differentiating logarithmically with respect to z and using (7), we obtain

$$\frac{D^{\alpha+1}_{m,\lambda} f(z)}{z} = p(z) + C_j^m(\lambda) z p'(z)$$

and

$$\frac{D^{\alpha+2}_{m,\lambda} f(z)}{z} = (C_j^m(\lambda))^2 z^2 p''(z) + C_j^m(\lambda)(C_j^m(\lambda) + 2) z p'(z) + p(z).$$

Now, define the transformation from $\mathbb{C}^3$ *to* $\mathbb{C}$ by

$$u = r, \quad v = r + (C_j^m(\lambda))s \ \text{ and } \ w = r + C_j^m(\lambda)(C_j^m(\lambda) + 2)s + (C_j^m(\lambda))^2 t.$$

Let $\psi(r, s, t; z) = \phi(u, v, w; z)$

$$= \phi\Big(r, r + (C_j^m(\lambda))s, r + C_j^m(\lambda)(C_j^m(\lambda) + 2)s + (C_j^m(\lambda))^2 t; z\Big).$$

By using the above equations it follows that

$$\psi\Big(p(z), zp'(z), z^2 p''(z); z\Big) = \phi\left(\frac{D^{\alpha}_{m,\lambda} f(z)}{z}, \frac{D^{\alpha+1}_{m,\lambda} f(z)}{z}, \frac{D^{\alpha+2}_{m,\lambda} f(z)}{z}; z\right)$$

and consequently we have that

$$\psi\Big(p(z), zp'(z), z^2 p''(z); z\Big) \in \Omega.$$

Thus, the proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{H,2}(\Omega, q)$ is equivalent to the admissibility condition for ψ as given in Definition 1.1. Note that

$$\frac{t}{s} + 1 = \frac{1}{C_j^m(\lambda)}\Big(\frac{w-u}{v-u} - 2\Big),$$

this shows that $\psi \in \Psi(\Omega, q)$. Hence by Lemma 1.1, we have $p(z) \prec q(z)$.
If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case, the class $\Phi_{H,2}(h(\mathbb{D}), q)$ is written as $\Phi_{H,2}(h, q)$. The following result is an immediate consequence of the above theorem.

Corollary 2.10. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \phi \in \Phi_{H,2}(h, q), with$ q(0)=1. If $f \in \mathcal{A}$ satisfies

$$\phi\left(\frac{D^{\alpha}_{m,\lambda}f(z)}{z}, \frac{D^{\alpha+1}_{m,\lambda}f(z)}{z}, \frac{D^{\alpha+2}_{m,\lambda}f(z)}{z}; z\right) \prec h(z),$$

then

$$\frac{D^{\alpha}_{m,\lambda}f(z)}{z} \prec q(z), \qquad z \in \mathbb{D}.$$

In the particular case $q(z) = 1 + Mz$, $M > 0$, and in view of Definition 2.5, the class of admissible functions $\Phi_{H,2}(\Omega, q)$ is denoted by $\Phi_{H,2}(\Omega, M)$ as described below.

Definition 2.6. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible function $\Phi_{H,2}(\Omega, M)$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that

$$\phi\Big(1+Me^{i\theta}, 1+Me^{i\theta}(1+kC^{m}_{j}(\lambda)), 1+(C^{m}_{j}(\lambda))^2L+Me^{i\theta}(1+kC^{m}_{j}(\lambda)(C^{m}_{j}(\lambda)+2)); z\Big) \notin \Omega,$$

whenever $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{D}$, $\Re(Le^{-i\theta}) \geq Mk(k-1)$, $\theta \in \mathbb{R}$, and $k \geq 1$.

Corollary 2.11. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, and\ \phi \in \Phi_{H,2}(\Omega, M)$. If $f \in \mathcal{A}$ satisfies

$$\phi\left(\frac{D^{\alpha}_{m,\lambda}f(z)}{z}, \frac{D^{\alpha+1}_{m,\lambda}f(z)}{z}, \frac{D^{\alpha+2}_{m,\lambda}f(z)}{z}; z\right) \in \Omega,$$

then

$$\left|\frac{D^{\alpha}_{m,\lambda}f(z)}{z} - 1\right| < M, \qquad z \in \mathbb{D}.$$

In the special case $\Omega = q(\mathbb{D}) = \{\omega : |\omega - 1| < M\}$, the class $\Phi_{H,2}(\Omega, M)$ is denoted by $\Phi_{H,2}(M)$ and the above Corollary takes the following form.

Corollary 2.12. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, and\ \phi \in \Phi_{H,2}(M)$. If $f \in \mathcal{A}$ satisfies

$$\left|\phi\left(\frac{D^{\alpha}_{m,\lambda}f(z)}{z}, \frac{D^{\alpha+1}_{m,\lambda}f(z)}{z}, \frac{D^{\alpha+2}_{m,\lambda}f(z)}{z}; z\right) - 1\right| < M,$$

then

$$\left|\frac{D^{\alpha}_{m,\lambda}f(z)}{z} - 1\right| < M, \quad z \in \mathbb{D}.$$

Corollary 2.13. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, and\ M > 0$. If $f(z) \in \mathcal{A}$ satisfies

$$\left|\frac{D^{\alpha+1}_{m,\lambda}f(z)}{z} - \frac{D^{\alpha}_{m,\lambda}f(z)}{z}\right| < C^{m}_{j}(\lambda)M,$$

then

$$\left|\frac{D^{\alpha}_{m,\lambda}f(z)}{z} - 1\right| < M, \quad z \in \mathbb{D}.$$

Proof. This follows from corollary 2.10 by taking $\phi(u, v, w; z) = v - u$ and

$\Omega = h(\mathbb{D})$, where $h(z) = C_j^m(\lambda)Mz$.

3. Superordination results for the operator $D_{m,\lambda}^{\alpha} f(z)$

The differential superordination results of the operator $D_{m,\lambda}^{\alpha} f(z)$ defined by (6) is investigated in this section. For this purpose, the class of admissible function is given as follows:

Definition 3.1. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}_0$ with $zq'(z) \neq 0$ The class of admissible functions $\Phi'_H(\Omega, q)$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfy the admissibility condition

$$\phi(u, v, w; \zeta) \in \Omega,$$

whenever $u = q(z)$,

$$v = \frac{C_j^m(\lambda)zq'(z) + (1 - C_j^m)mq(z)}{m} \quad \text{and}$$

$$\Re\Big(\frac{1}{C_j^m(\lambda)}\Big\{\frac{w - (1 - C_j^m(\lambda))^2 u}{v - (1 - C_j^m(\lambda))u} - 2\Big\} + 2\Big) \leq \frac{1}{m}\Re\Big(\frac{zq''(z)}{q'(z)} + 1\Big),$$

for $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$, $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $m \geq 1$.

Theorem 3.1. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $\phi \in \Phi'_H(\Omega, q)$. If $f \in \mathcal{A}$, $D_{m,\lambda}^{\alpha} f(z) \in \mathcal{Q}_0$ and $\Big(D_{m,\lambda}^{\alpha} f(z), D_{m,\lambda}^{\alpha+1} f(z), D_{m,\lambda}^{\alpha+2} f(z); z\Big) \in \mathcal{S}(\mathbb{D})$,
then

$$\Omega \subset \Big\{D_{m,\lambda}^{\alpha} f(z), D_{m,\lambda}^{\alpha+1} f(z), D_{m,\lambda}^{\alpha+2} f(z);\ \ z \in \mathbb{D}\Big\},$$

implies that $q(z) \prec D_{m,\lambda}^{\alpha} f(z), \quad z \in \mathbb{D}$.

Proof. With $p(z) = D_{m,\lambda}^{\alpha} f(z)$ and

$\psi(r, s, t; z) = \phi(u, v, w; z)$

$= \phi\Big(r, (C_j^m(\lambda))s + (1 - C_j^m(\lambda))r, (C_j^m(\lambda))^2 t + C_j^m(\lambda)(2 - C_j^m(\lambda))s + (1 - C_j^m(\lambda))^2 r; z\Big)$

we have

$$\Omega \subset \Big\{\psi\Big(p(z), zp'(z), z^2 p''(z); z\Big) : z \in \mathbb{D}\Big\}.$$

Since

$$\frac{t}{s} + 1 = \frac{1}{C_j^m(\lambda)}\Big\{\frac{w - (1 - C_j^m(\lambda))^2 u}{v - (1 - C_j^m(\lambda))u} - 2\Big\} + 2,$$

the admissibility condition for $\phi \in \Phi'_H(\Omega, q)$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'_H(\Omega, q)$ and by Lemma 1.2, $q(z) \prec p(z)$.

If $\Omega \neq \mathbb{C}$ is simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping $h(z)$ of $\mathbb{D}$ onto Ω, and then the class $\Phi'_H(h(\mathbb{D}), q)$ is written as $\Phi'_H(h, q)$. Proceeding as in the previous section, the following result is an immediate consequence of Theorem 3.1.

Corollary 3.1. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $q \in \mathcal{H}_0$, $h \in \mathcal{S}(\mathbb{D})$ and $\phi \in \Phi'_H(h, q)$. If $f(z) \in \mathcal{A}$,

$$D_{m,\lambda}^{\alpha} f(z) \in \mathcal{Q}_0 \quad \text{and}$$

$$\phi\Big(D_{m,\lambda}^{\alpha} f(z),\ D_{m,\lambda}^{\alpha+1} f(z), D_{m,\lambda}^{\alpha+2} f(z); z\Big) \in \mathcal{S}(\mathbb{D}),$$

then

$$h(z) \prec \phi\Big(D^{\alpha}_{m,\lambda}f(z), D^{\alpha+1}_{m,\lambda}f(z), D^{\alpha+2}_{m,\lambda}f(z); z\Big) \tag{17}$$

implies that $q(z) \prec D^{\alpha}_{m,\lambda}f(z)$.

The following theorem proves the existence of the best subordination for an appropriate ϕ.

Corollary 3.2. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $h \in \mathcal{H}(\mathbb{D})$ and $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$. Suppose that the differential equation

$$\phi\Big(q(z), C^m_j(\lambda)zq'(z) + (1-(C^m_j(\lambda))q(z),$$

$$(C^m_j(\lambda))^2z^2q''(z) + C^m_j(\lambda)(2 - C^m_j(\lambda))zq'(z) + (1-(C^m_j(\lambda))^2q(z); z\Big) = h(z)$$

has a solution $q(z) \in \mathcal{Q}_0$. If $\phi \in \Phi'_H(h, q)$, $f \in \mathcal{A}$,

$$D^{\alpha}_{m,\lambda}f(z) \in \mathcal{Q}_0 \quad \text{and}$$

$$\phi\Big(D^{\alpha}_{m,\lambda}f(z), D^{\alpha+1}_{m,\lambda}f(z), D^{\alpha+2}_{m,\lambda}f(z); z\Big) \in \mathcal{S}(\mathbb{D}),$$

then

$$h(z) \prec \phi\Big(D^{\alpha}_{m,\lambda}f(z), D^{\alpha+1}_{m,\lambda}f(z), D^{\alpha+2}_{m,\lambda}f(z); z\Big),$$

implies

$$q(z) \prec D^{\alpha}_{m,\lambda}f(z), \quad z \in \mathbb{D}$$

and q is the best subordinant.

Proof. The proof is similar to the proof of Theorem 2.3 and is therefore omitted. By combining corollary 2.1 and corollary 3.1, we obtain the following sandwich-type result.

Corollary 3.3. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $h_1, q_1 \in \mathcal{H}(\mathbb{D})$ *and* $h_2 \in \mathcal{S}(\mathbb{D})$. Suppose also that $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_H(h_2, q_2) \cap \Phi'_H(h_1, q_1)$. If $f(z) \in \mathcal{A}$,

$$D^{\alpha}_{m,\lambda}f(z) \in \mathcal{H}_0 \cap \mathcal{Q}_0 \quad \text{and}$$

$$\phi\left(D^{\alpha}_{m,\lambda}f(z), D^{\alpha+1}_{m,\lambda}f(z), D^{\alpha+2}_{m,\lambda}f(z); z\right) \in \mathcal{S}(\mathbb{D}),$$

then

$$h_1(z) \prec \phi\Big(D^{\alpha}_{m,\lambda}f(z), D^{\alpha+1}_{m,\lambda}f(z), D^{\alpha+2}_{m,\lambda}f(z); z\Big) \prec h_2(z),$$

implies

$$q_1(z) \prec D^{\alpha}_{m,\lambda}f(z) \prec q_2(z), \quad z \in \mathbb{D}.$$

Definition 3.2. Let Ω be a set in $\mathbb{C}$ and $q(z) \in \mathcal{H}_1$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{H,1}(\Omega, q)$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfy the admissibility condition:

$$\phi(u, v, w; \zeta) \in \Omega,$$

whenever $q(z) \neq 0, u = q(z)$,

$$v = q(\zeta) + C^m_j(\lambda)\frac{k\zeta q'(\zeta)}{q(\zeta)} \text{ and}$$

$$\Re\left(\frac{1}{C^m_j(\lambda)}\Big(\frac{wv + 2u^2 - 3uv}{v-u}\Big)\right) \leq \frac{1}{m}\,\Re\Big(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\Big),$$

where $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$ and $m \geq 1$.
Now we will establish the dual result of differential superordination, concerning Theorem 2.4.

Theorem 3.2. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ *and* $\phi \in \Phi'_{H,1}(\Omega, q)$. If $f \in \mathcal{A}$,

$$\frac{D^{\alpha+1}_{m,\lambda} f(z)}{D^{\alpha}_{m,\lambda} f(z)} \in \mathcal{Q}_1 \text{ and } \phi\left(\frac{D^{\alpha+1}_{m,\lambda} f(z)}{D^{\alpha}_{m,\lambda} f(z)}, \frac{D^{\alpha+2}_{m,\lambda} f(z)}{D^{\alpha+1}_{m,\lambda} f(z)}, \frac{D^{\alpha+3}_{m,\lambda} f(z)}{D^{\alpha+2}_{m,\lambda} f(z)}; z\right) \in \mathcal{S}(\mathbb{D}),$$

then

$$\Omega \subset \left\{\phi\left(\frac{D^{\alpha+1}_{m,\lambda} f(z)}{D^{\alpha}_{m,\lambda} f(z)}, \frac{D^{\alpha+2}_{m,\lambda} f(z)}{D^{\alpha+1}_{m,\lambda} f(z)}, \frac{D^{\alpha+3}_{m,\lambda} f(z)}{D^{\alpha+2}_{m,\lambda} f(z)}; z\right) : z \in \mathbb{U}\right\},$$

implies

$$q(z) \prec \frac{D^{\alpha+1}_{m,\lambda} f(z)}{D^{\alpha}_{m,\lambda} f(z)}, \quad z \in \mathbb{D}.$$

Proof. In view of the proof of Theorem 2.4, we have

$$\Omega \subset \left\{\psi\Big(p(z), zp'(z), z^2p''(z); z\Big); z \in \mathbb{D}\right\},$$

and, we can see that the admissibility condition for $\phi \in \Phi'_{H,1}(\Omega, q)$ is equivalent to the admissibility condition for ψ as given in Definition 1.2. Hence $\psi \in \Psi'_{H,1}(\Omega, q)$ and by Lemma 1.2, we get $q(z) \prec p(z)$.
If $\Omega \neq \mathbb{C}$ is a simply connected domain and $\Omega = h(\mathbb{D})$ for some conformal mapping $h(z)$ of $\mathbb{D}$ onto Ω, then the class $\Phi'_{H,1}(h(\mathbb{D}), q)$ is written as $\Phi'_{H,1}(h, q)$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.2.

Corollary 3.4. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $q(z) \in \mathcal{H}$, $h \in \mathcal{H}(\mathbb{D})$ and $\phi \in \Phi'_{H,1}(h, q)$. If $f(z) \in \mathcal{A}$,

$$\frac{D^{\alpha+1}_{m,\lambda} f(z)}{D^{\alpha}_{m,\lambda} f(z)} \in \mathcal{Q}_1 \quad \text{and} \quad \phi\left(\frac{D^{\alpha+1}_{m,\lambda} f(z)}{D^{\alpha}_{m,\lambda} f(z)}, \frac{D^{\alpha+2}_{m,\lambda} f(z)}{D^{\alpha+1}_{m,\lambda} f(z)}, \frac{D^{\alpha+3}_{m,\lambda} f(z)}{D^{\alpha+2}_{m,\lambda} f(z)}; z\right) \in \mathcal{S}(\mathbb{D}),$$

then

$$h(z) \prec \phi\left(\frac{D^{\alpha+1}_{m,\lambda} f(z)}{D^{\alpha}_{m,\lambda} f(z)}, \frac{D^{\alpha+2}_{m,\lambda} f(z)}{D^{\alpha+1}_{m,\lambda} f(z)}, \frac{D^{\alpha+3}_{m,\lambda} f(z)}{D^{\alpha+2}_{m,\lambda} f(z)}; z\right),$$

implies

$$q(z) \prec \frac{D^{\alpha+1}_{m,\lambda} f(z)}{D^{\alpha}_{m,\lambda} f(z)}, \quad z \in \mathbb{D}.$$

Combining corollary 2.4 and corollary 3.4, we obtain the following sandwich-type result.

Corollary 3.5. Let $\lambda > 0, m \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, h_1, q_1 \in \mathcal{H}(\mathbb{D}), h_2 \in \mathcal{S}(\mathbb{D}), q_2(z) \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{H,1}(h_2, q_2) \cap \Phi'_{H,1}(h_1, q_1)$. If $f(z) \in \mathcal{A}$,

$$\frac{D^{\alpha+1}_{m,\lambda} f(z)}{D^{\alpha}_{m,\lambda} f(z)} \in \mathcal{H}_1 \cap \mathcal{Q}_1 \quad \text{and} \quad \phi\left(\frac{D^{\alpha+1}_{m,\lambda} f(z)}{D^{\alpha}_{m,\lambda} f(z)}, \frac{D^{\alpha+2}_{m,\lambda} f(z)}{D^{\alpha+1}_{m,\lambda} f(z)}, \frac{D^{\alpha+3}_{m,\lambda} f(z)}{D^{\alpha+2}_{m,\lambda} f(z)}; z\right) \in \mathcal{S}(\mathbb{D})$$

then

$$h_1(z) \prec \phi\left(\frac{D^{\alpha+1}_{m,\lambda} f(z)}{D^{\alpha}_{m,\lambda} f(z)}, \frac{D^{\alpha+2}_{m,\lambda} f(z)}{D^{\alpha+1}_{m,\lambda} f(z)}, \frac{D^{\alpha+3}_{m,\lambda} f(z)}{D^{\alpha+2}_{m,\lambda} f(z)}; z\right) \prec h_2(z),$$

implies

$$q_1(z) \prec \frac{D_{m,\lambda}^{\alpha+1} f(z)}{D_{m,\lambda}^{\alpha} f(z)} \prec q_2(z). \quad z \in \mathbb{D}.$$

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