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INITIAL BOUNDS FOR A CLASS OF bi−UNIVALENT FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH CHEBYSHEV POLYNOMIALS

M. K. AOUF, A. O. MOSTAFA, F. Y. AL-QUHALI

Abstract. In this paper, we obtain initial coefficient bounds for functions belong to a subclass of *bi*−univalent functions by using the Salagean differential operator and Chebyshev polynomials and also we find Fekete-Szego inequalities for functions in this class.

1. Introduction

Let S be the class of analytic and univalent functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \ , z \in \mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\} . \quad (1)$$

For f and g analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$ $(f(z) \prec g(z))$ if there exists an analytic Schwarz function $w(z)$ in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$ (see [19]).

Indeed, it is known that

$$f(z) \prec g(z) \ \ (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \ \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}),$$

and if g is univalent in $\mathbb{U}$, then

$$f(z) \prec g(z) \ \ (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \ \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

It is well known (see Duren [13]) that every function $f \in S$ has an inverse map f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_o(f); \ r_o(f) \geq \frac{1}{4}).$$

In fact, the inverse function $g = f^{-1}$ is given by

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots . \quad (2)$$

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A function $f \in S$ is said to be $bi-$univalent function in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Denote by Δ the class of $bi-$univalent functions in $\mathbb{U}$. For a history and examples of functions which are (or which are not) in the class Δ, together with various other properties one can refer to [1, 9, 11, 15, 17, 21, 22, 24, 27].

The Chebyshev polynomials of the first and second kinds are well known and defined by (see [2, 10, 12, 14, 16, 18])

$$T_k(t) = \cos k\theta \quad \text{and} \quad U_k(t) = \frac{\sin(k+1)\theta}{\sin\theta} \qquad (-1 < t < 1),$$

where the degree of the polynomial is k and $t = \cos\theta$.

Consider the function

$$H(z,t) = \frac{1}{1 - 2tz + z^2}.$$

Note that if $t = \cos\alpha,\ \alpha \in (\frac{-\pi}{3}, \frac{\pi}{3})$, then for all $z \in \mathbb{U}$

$$\begin{aligned} H(z,t) &= 1 + \sum_{k=1}^{\infty} \frac{\sin(k+1)\alpha}{\sin\alpha} z^k \\ &= 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha)z^2 + \dots . \end{aligned} \tag{3}$$

Thus, we have [26]

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \dots \qquad (z \in \mathbb{U},\ t \in (-1,1)), \tag{4}$$

where $U_{k-1} = \frac{\sin(k \arccos t)}{\sqrt{1-t^2}}$, for $k \in \mathbb{N} = \{1, 2, \dots\}$, are the second kind of the Chebyshev polynomials. Also, it is known that

$$U_k(t) = 2tU_{k-1}(t) - U_{k-2}(t), \tag{5}$$

and

$$U_1(t) = 2t,\ \ U_2(t) = 4t^2 - 1,\ \ U_3(t) = 8t^3 - 4t,\ \ U_4(t) = 16t^4 - 12t^2 + 1, \dots . \tag{6}$$

The Chebyshev polynomials $T_k(t)$, $t \in [-1,1]$, of the first kind have the generating function of the form

$$\sum_{k=0}^{\infty} T_k(t) z^k = \frac{1 - tz}{1 - 2tz + z^2} \quad (z \in \mathbb{U}). \tag{7}$$

The first kind of Chebyshev polynomial $T_k(t)$ and second kind of Chebyshev polynomial $U_k(t)$ are connected by:

$$\frac{dT_k(t)}{dt} = kU_{k-1}(t); \quad T_k(t) = U_k(t) - tU_{k-1}(t); \quad 2T_k(t) = U_k(t) - U_{k-2}(t). \tag{8}$$

For $f(z) \in S$, the Salagean operator is defined by (see [23] and [3, 4, 5, 6, 7, 8])

$$\begin{aligned} D^1 f(z) = Df(z) = zf^{'}(z), \\ \vdots \\ D^n f(z) &= D(D^{n-1}f(z)) = z(D^{n-1}f(z))^{'}, \\ &= z + \sum_{k=2}^{\infty} k^n a_k z^k \qquad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \dots\}). \end{aligned} \tag{9}$$

By using the Salagean differential operator for g of the form (2), Vijaya et al. [25] (also see [20]) defined $D^n g(w)$ as follows:

$$D^n g(w) = w - a_2 2^n w^2 + (2a_2^2 - a_3)3^n w^3 + \dots . \tag{10}$$

Definition 1. For $\alpha \geq 0$, $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $t \in (-1, 1)$, a function $f \in \Delta$ of form (1) is said to be in the class $R_\Delta^n(b, \alpha, t)$ if the following subordinations hold:

$$1 + \frac{1}{b}\left[\frac{z(D^n f(z))' + \alpha z^2 (D^n f(z))''}{(1-\alpha)D^n f(z) + \alpha z (D^n f(z))'} - 1\right] \prec H(z,t) = \frac{1}{1 - 2tz + z^2}, \tag{11}$$

and

$$1 + \frac{1}{b}\left[\frac{z(D^n g(w))' + \alpha z^2 (D^n g(w))''}{(1-\alpha)D^n g(w) + \alpha z (D^n g(w))'} - 1\right] \prec H(w,t) = \frac{1}{1 - 2tw + w^2}, \tag{12}$$

where $z, w \in \mathbb{U}$ and g is given by (2).

For suitable choices of n, α and b, we obtain:

(i) $R_\Delta^0(b, \alpha, t) = R_\Delta(b, \alpha, t)$, where

$$1 + \frac{1}{b}\left[\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} - 1\right] \prec H(z,t) = \frac{1}{1 - 2tz + z^2},$$

and

$$1 + \frac{1}{b}\left[\frac{zg'(w) + \alpha z^2 g''(w)}{(1-\alpha)g(w) + \alpha z g'(w)} - 1\right] \prec H(w,t) = \frac{1}{1 - 2tw + w^2},$$

(ii) $R_\Delta^n(b, 0, t) = R_\Delta^{n*}(b, t)$, if

$$1 + \frac{1}{b}\left[\frac{z(D^n f(z))'}{D^n f(z)} - 1\right] \prec H(z,t) = \frac{1}{1 - 2tz + z^2},$$

and

$$1 + \frac{1}{b}\left[\frac{z(D^n g(w))'}{D^n g(w)} - 1\right] \prec H(w,t) = \frac{1}{1 - 2tw + w^2},$$

(iii) $R_\Delta^n(b, 1, t) = R_\Delta^n(b, t)$, if

$$1 + \frac{1}{b}\frac{z(D^n f(z))''}{(D^n f(z))'} \prec H(z,t) = \frac{1}{1 - 2tz + z^2},$$

and

$$1 + \frac{1}{b}\frac{z(D^n g(w))''}{(D^n g(w))'} \prec H(w,t) = \frac{1}{1 - 2tw + w^2},$$

(iv) $R_\Delta^0(1, \alpha, t) = R_\Delta(\alpha, t)$, if

$$\frac{zf'(z) + \alpha z^2 f''(z)}{(1-\alpha)f(z) + \alpha z f'(z)} \prec H(z,t) = \frac{1}{1 - 2tz + z^2},$$

and

$$\frac{zg'(w) + \alpha z^2 g''(w)}{(1-\alpha)g(w) + \alpha z g'(w)} \prec H(w,t) = \frac{1}{1 - 2tw + w^2},$$

(v) $R_\Delta^n(1, 0, t) = R_\Delta^{n*}(t)$, if

$$\frac{z(D^n f(z))'}{D^n f(z)} \prec H(z,t) = \frac{1}{1 - 2tz + z^2}, \prec H(z,t) = \frac{1}{1 - 2tz + z^2},$$

and

$$\frac{z(D^n g(w))'}{D^n g(w)} \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

(vi) $R^n_\Delta(1,1,t) = R^n_\Delta(t)$, if

$$\frac{z(D^n f(z))''}{(D^n f(z))'} \prec H(z,t) = \frac{1}{1-2tz+z^2},$$

and

$$\frac{z(D^n g(w))''}{(D^n g(w))'} \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

(vii) $R^n_\Delta((1-\lambda)e^{-i\theta}\cos\theta,\alpha,t) = R^n_\Delta(\lambda,\theta,\alpha,t)$ $\left(0 \le \lambda < 1, |\theta| < \frac{\pi}{2}\right)$, if

$$e^{i\theta}\left[\frac{z(D^n f(z))' + \alpha z^2 (D^n f(z))''}{(1-\alpha)D^n f(z) + \alpha z (D^n f(z))'}\right] \prec H(z,t)(1-\lambda)\cos\theta + \lambda\cos\theta + i\sin\theta,$$

and

$$e^{i\theta}\left[\frac{z(D^n g(w))' + \alpha z^2 (D^n g(w))''}{(1-\alpha)D^n g(w) + \alpha z (D^n g(w))'}\right] \prec H(w,t)(1-\lambda)\cos\theta + \lambda\cos\theta + i\sin\theta.$$

In this paper, we obtain the initial coefficients bounds and Fekete-Szego problem for functions in the class $R^n_\Delta(b,\alpha,t)$.

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $R^n_\Delta(b,\alpha,t)$

Unless indicated, we assume that $\alpha \ge 0$, $t \in (-1,1)$, f given by (1.1) and $b \in \mathbb{C}^*$.

Theorem 1. Let $f \in R^n_\Delta(b,\alpha,t)$. Then

$$|a_2| \le \frac{|b|2t\sqrt{2t}}{\sqrt{\left|\left\{4t^2\left\{b\left[(2+4\alpha)3^n-(1+\alpha)^2 2^{2n}\right]-(1+\alpha)^2 2^{2n}\right\}+(1+\alpha)^2 2^{2n}\right\}\right|}}, \tag{13}$$

and

$$|a_3| \le \frac{4t^2|b|^2}{(1+\alpha)^2 2^{2n}} + \frac{2t|b|}{(2+4\alpha)3^n}. \tag{14}$$

Proof. Let $f \in R^n_\Delta(b,\alpha,t)$ and $g = f^{-1}$. From (11) and (12) , we have

$$1+\frac{1}{b}\left[\frac{z(D^n f(z))' + \alpha z^2 (D^n f(z))''}{(1-\alpha)D^n f(z) + \alpha z (D^n f(z))'} - 1\right] = 1 + U_1(t)p(z) + U_2(t)p^2(z) + ... \tag{15}$$

and

$$1+\frac{1}{b}\left[\frac{z(D^n g(w))' + \alpha z^2 (D^n g(w))''}{(1-\alpha)D^n g(w) + \alpha z (D^n g(w))'} - 1\right] = 1 + U_1(t)q(w) + U_2(t)q^2(w) + ... \tag{16}$$

for some analytic functions

$$p(z) = c_1 z + c_2 z^2 + c_3 z^3 + ... \quad (z \in \mathbb{U}), \tag{17}$$

$$q(w) = d_1 w + d_2 w^2 + d_3 w^3 + ... \quad (w \in \mathbb{U}), \tag{18}$$

such that $p(0) = q(0) = 0$, $|p(z)| < 1 \quad (z \in \mathbb{U})$ and $|q(w)| < 1 \quad (w \in \mathbb{U})$. It is well known that if $|p(z)| < 1$ and $|q(w)| < 1$, then

$$|c_j| \le 1 \quad \text{and} \quad |d_j| \le 1 \quad \text{for all } j \in \mathbb{N}. \tag{19}$$

From (15) — (18), we have

$$\begin{aligned}
&\frac{1}{b}\left[\frac{z(D^n f(z))' + \alpha z^2 (D^n f(z))''}{(1-\alpha)D^n f(z) + \alpha z (D^n f(z))'} - 1\right] \\
= &\frac{1}{b}\left\{(1+\alpha)\,2^n a_2 z + \left[(2+4\alpha)\,3^n a_3 - (1+\alpha)^2\, 2^{2n} a_2^2\right] z^2 + \ldots\right\} \\
= &U_1(t)c_1 z + \left[U_1(t)c_2 + U_2(t)c_1^2\right] z^2 + \ldots \qquad (20)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{b}\left[\frac{z(D^n g(w))' + \alpha z^2 (D^n g(w))''}{(1-\alpha)D^n g(w) + \alpha z (D^n g(w))'} - 1\right] \\
= &\frac{1}{b}\{-(1+\alpha)\,2^n a_2 w + \\
&\left\{\left[(4+8\alpha)\,3^n - (1+\alpha)^2\, 2^{2n}\right] a_2^2 - (2+4\alpha)\,3^n a_3\right\} w^2 + \ldots\} \\
= &U_1(t)d_1 w + \left[U_1(t)d_2 + U_2(t)d_1^2\right] w^2 + \ldots\,. \qquad (21)
\end{aligned}$$

Equating the coefficients in (20) and (21) we get

$$\frac{1}{b}(1+\alpha)\,2^n a_2 = U_1(t)c_1, \qquad (22)$$

$$\frac{1}{b}\left[(2+4\alpha)\,3^n a_3 - (1+\alpha)^2\, 2^{2n} a_2^2\right] = U_1(t)c_2 + U_2(t)c_1^2, \qquad (23)$$

$$-\frac{1}{b}(1+\alpha)\,2^n a_2 = U_1(t)d_1, \qquad (24)$$

and

$$\frac{1}{b}\left\{\left[(4+8\alpha)3^n - (1+\alpha)^2\, 2^{2n}\right] a_2^2 - (2+4\alpha)\,3^n a_3\right\} = U_1(t)d_2 + U_2(t)d_1^2. \qquad (25)$$

From (22) and (24) we obtain

$$c_1 = -d_1 \qquad (26)$$

and

$$\frac{1}{b^2}(1+\alpha)^2\, 2^{2n+1} a_2^2 = U_1^2(t)\left(c_1^2 + d_1^2\right). \qquad (27)$$

Also, (23) and (25) yield

$$\frac{1}{b}\left[(4+8\alpha)\,3^n - (1+\alpha)^2\, 2^{2n+1}\right] a_2^2 = U_1(t)\,(c_2 + d_2) + U_2(t)\left(c_1^2 + d_1^2\right), \qquad (28)$$

which by (**??**), leads to

$$\left[(4+8\alpha)\,3^n - (1+\alpha)^2\, 2^{2n+1} - \frac{U_2(t)2^{2n+1}}{bU_1^2(t)}(1+\alpha)^2\right] a_2^2 = bU_1(t)\,(c_2 + d_2)\,. \qquad (29)$$

From (6), (19) and (29), we have (13).

Next, by subtracting (**??**) from (23), we have

$$\frac{2}{b}(2+4\alpha)\,3^n (a_3 - a_2^2) = U_1(t)\,(c_2 - d_2) + U_2(t)\left(c_1^2 - d_1^2\right).$$

Further, in view of (26), we obtain

$$a_3 = a_2^2 + \frac{bU_1(t)}{2\,(2+4\alpha)\,3^n}\,(c_2 - d_2)\,. \qquad (30)$$

Hence using (27) and applying (6), we get (14).

This completes the proof of Theorem 1. □

Taking $n=0$ in Theorem 1, we get the following consequence.

Corollary 1. Let $f\in\Delta$ be in the class $R_\Delta(b,\alpha,t)$. Then

$$|a_2|\le \frac{|b|2t\sqrt{2t}}{\sqrt{\left|\left\{4t^2\left\{b\left[(2+4\alpha)-(1+\alpha)^2\right]-(1+\alpha)^2\right\}+(1+\alpha)^2\right\}\right|}},$$

and

$$|a_3|\le\frac{4t^2|b|^2}{(1+\alpha)^2}+\frac{2t|b|}{(2+4\alpha)}.$$

Taking $\alpha=1$ in Corollary 1, we get the following consequence.

Corollary 2. Let $f\in\Delta$ be in the class $R_\Delta(b,t)$. Then

$$|a_2|\le\frac{|b|t\sqrt{2t}}{\sqrt{|t^2[2b-4]+1|}},$$

and

$$|a_3|\le t^2|b|^2+\frac{t|b|}{3}.$$

Taking $b=e^{-i\theta}(1-\lambda)\cos\theta$ $(0\le\lambda<1,\ |\theta|<\frac{\pi}{2})$ in Corollary 2, we get the following consequence.

Corollary 3. Let $f\in\Delta$ be in the class $R_\Delta(\lambda,\theta,t)$. Then

$$|a_2|\le\frac{t\sqrt{2t}(1-\lambda)\cos\theta}{\sqrt{|t^2[2(1-\lambda)\cos\theta-4]+1|}},$$

and

$$|a_3|\le t^2(1-\lambda)^2\cos^2\theta+\frac{t(1-\lambda)\cos\theta}{3}.$$

Taking $\lambda=0$ in Corollary 3, we get the following consequence.

Corollary 4. Let $f\in\Delta$ be in the class $R_\Delta(\theta,t)$. Then

$$|a_2|\le\frac{t\sqrt{2t}\cos\theta}{\sqrt{|t^2[2\cos\theta-4]+1|}},$$

and

$$|a_3|\le t^2\cos^2\theta+\frac{t\cos\theta}{3}.$$

3. FEKETE- SZEGO INEQUALITIES FOR THE CLASS $R^n_\Delta(b,\alpha,t)$

Theorem 2. If $f\in R^n_\Delta(b,\alpha,t)$ and $\xi\in\mathbb{R}$, then

$$\left|a_3-\xi a_2^2\right|\le\begin{cases}\frac{2|b|t}{(2+4\alpha)3^n}, & |\xi-1|\le k\\ \frac{8|b|^2|\xi-1|t^3}{\left|\left\{4t^2\left\{b\left[(2+4\alpha)3^n-(1+\alpha)^2 2^{2n}\right]-(1+\alpha)^2 2^{2n}\right\}+(1+\alpha)^2 2^{2n}\right\}\right|}, & |\xi-1|\ge k,\end{cases}\tag{31}$$

where $k=\frac{\left|\left\{4t^2\left\{b\left[(2+4\alpha)3^n-(1+\alpha)^2 2^{2n}\right]-(1+\alpha)^2 2^{2n}\right\}+(1+\alpha)^2 2^{2n}\right\}\right|}{4t^2|b|(2+4\alpha)3^n}$.

Proof. From (29) and (30)

$$\begin{aligned}\left(a_3-\xi a_2^2\right) &= (1-\xi)\left[\frac{b^2U_1^3(t)(c_2+d_2)}{bU_1^2(t)\left[(4+8\alpha)3^n-(1+\alpha)^2 2^{2n+1}\right]-U_2(t)(1+\alpha)^2 2^{2n+1}}\right]+\frac{bU_1(t)}{2(2+4\alpha)3^n}(c_2-d_2)\\ &= bU_1(t)\left[\left(h(\xi)+\frac{1}{2(2+4\alpha)3^n}\right)c_2+\left(h(\xi)-\frac{1}{2(2+4\alpha)3^n}\right)d_2\right],\end{aligned}\tag{32}$$

where

$$h(\xi)=\frac{b(1-\xi)U_1^2(t)}{bU_1^2(t)\left[(4+8\alpha)3^n-(1+\alpha)^2 2^{2n+1}\right]-U_2(t)(1+\alpha)^2 2^{2n+1}}.$$

Then, by taking the modulus of (32) and considering (6), we have

$$\left|a_3-\xi a_2^2\right| \leq \begin{cases} \frac{2|b|t}{(2+4\alpha)3^n}, & 0 \leq |h(\xi)| \leq \frac{1}{2(2+4\alpha)3^n} \\ 4\,|b|\,|h(\xi)|\,t, & |h(\xi)| \geq \frac{1}{2(2+4\alpha)3^n}. \end{cases}$$

This completes the proof of Theorem 2. □

Taking $\xi = 1$ in Theorem 2, we get the following consequence.

Corollary 5. Let the function $f \in \Delta$ given by (1) be in the class $R_{\Delta}^{n}(b, \alpha, t)$. Then

$$\left|a_3-a_2^2\right| \leq \frac{2\,|b|\,t}{(2+4\alpha)\,3^n}.$$

Taking $\alpha = 1$ and $n = 0$ in Corollary 5, we get the following consequence.

Corollary 6. For $t \in (-1, 1)$, let the function $f \in \Delta$ given by (1) be in the class $R_{\Delta}(b, t)$. Then

$$\left|a_3-a_2^2\right| \leq \frac{t\,|b|}{3}.$$

Taking $b = e^{-i\theta}(1-\lambda)\cos\theta$ $(0 \leq \lambda < 1,\ |\theta| < \frac{\pi}{2})$ in Corollary 6, we get the following consequence.

Corollary 7. For $t \in (-1, 1)$, let the function $f \in \Delta$ given by (1) be in the class $R_{\Delta}(\lambda, \theta, t)$. Then

$$\left|a_3-a_2^2\right| \leq \frac{t(1-\lambda)\cos\theta}{3}.$$

Taking $\lambda = 0$ in Corollary 7, we get the following consequence.

Corollary 8. For $t \in (-1, 1)$, let the function $f \in \Delta$ given by (1) be in the class $R_{\Delta}(\theta, t)$. Then

$$\left|a_3-a_2^2\right| \leq \frac{t\cos\theta}{3}.$$

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M. K. Aouf, Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.
E-mail address: `mkaouf127@yahoo.com`

A. O. Mostafa, Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.
E-mail address: `adelaeg254@yahoo.com`

F. Y. Al-Quhali, Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.
E-mail address: `fyalquhali89@gmail.com`