Journal of Fractional Calculus and Applications
Vol. 11(2) July 2020, pp. 160-172.
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

STABILITY ANALYSIS OF THE FRACTIONAL DIFFERENTIAL EQUATIONS WITH THE CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE

N. SENE

Abstract. This paper aims to study the stability of the fractional differential equations without inputs. The fractional differential equations without inputs considered in this paper are defined with the Caputo-Fabrizio fractional derivative operator. We will investigate to find the stability conditions related to the fractional differential equations. We will address how to characterize the stability of fractional differential equations using the Lyapunov candidates' functions. The stability conditions of the perturbed fractional differential equations will be discussed. The Lyapunov characterization of the stability will be proposed to avoid the difficulty existing when we analyze stability using the Gronwall lemma and the trajectories analysis. Several examples to illustrate the mains results will be provided.

1. Introduction

In the last decade the fractional calculus received main attention due to its important role in modeling the anomalous dynamics of various processes related to complex systems in the most areas of science and engineering as provided by Baleanu et al. in [6], Caputo and Fabrizio in [7], Atangana et al. in [4], Podlubny in [24], Torres and Malinowska in [19], Kilbas et al. in [16], Petras in [23], Sene et al. in [32, 36, 37, 38, 39]. Many papers appeared and gave some results and role of the fractional calculus in physics, control engineering, and signal processing [8, 10, 20, 22].

In fractional calculus, there exist various types of the fractional derivatives operators as: the Riemann-Liouville fractional derivative [27, 28, 25], the generalized fractional derivative [1, 14, 15], the conformable fractional derivative [21, 29, 31], the Hadamard and Hilfer fractional derivative [14], the Caputo fractional derivative [27], the Caputo-Fabrizio fractional derivative [18], the Atangana-Baleanu fractional derivative [3] and others. The Riemann-Liouville and the Caputo-Fabrizio fractional derivatives are fractional derivatives with the singular kernel. There exist some disadvantage related to the existence of the singularity in the fractional derivatives operators, it for this reason Caputo and Fabrizio proposed in 2015 [7] a new fractional derivative without singular kernel. Another fractional derivative

---

2010 *Mathematics Subject Classification.* 26A33, 34D20, 93D05.
Key words and phrases. Caputo-Fabrizio fractional derivative, exponential stability, fractional differential equations.
Submitted Nov. 14, 2018.

without a singular kernel exists in the literature and appeared in 2016, which was the Atangana-Baleanu fractional derivative. The Caputo-Fabrizio fractional derivative and the Atantana fractional derivative are known to be very useful and helpful to study real-world problems. Many physical applications with the Caputo-Fabrizio fractional derivative and the Atantaga fractional derivative exist, see in for examples in [3, 4].

The stability problem is one of the fundamental subjects existing in the fractional calculus. Many stability analysis of the fractional differential equations using Caputo-Liouville fractional derivative, and Riemann-Liouville fractional derivative exist in the literature, see in [2, 25, 26, 27, 28]. With a new fractional derivative as the Caputo-Fabrizio fractional derivative, we will analyze the stability of a particular class of the fractional differential equations. Two methods will be used: the first will use the trajectories, and the second will use the Lyapunov direct method.

The paper is organized as follows. In Section 2, we introduce the basic definitions and provide some lemmas. In Section 3, we give the mains results of this paper. In Section 4, we present several examples and illustrate our main results. The conclusions and remarks are summarized in Section 5.

Notation. $\mathcal{PD}$ denotes the set of all continuous functions $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\chi(0) = 0$ and $\chi(s) > 0$ for all $s > 0$. A class $\mathcal{K}$ function is an increasing $\mathcal{PD}$ function. The class $\mathcal{K}_\infty$ denotes the set of all unbounded $\mathcal{K}$ function. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be class $\mathcal{KL}$ if $\beta(., t) \in \mathcal{K}$ for any $t \geq 0$ and $\beta(s, .)$ is non increasing and tends to zero as its arguments tends to infinity. Given $x \in \mathbb{R}^n$, $\|x\|$ stands for its Euclidean norm: $\|x\| := \sqrt{x_1^2 + \ldots + x_n^2}$. For a matrix A, $\lambda_{max}(A)$ and $\lambda_{min}(A)$ denote the maximal and the minimal eigenvalue of A, respectively. If the condition $Re\left(\lambda_i\right) < 0, \forall i = 1, 2, ..., n$, holds then the matrix A is said Hurwitz.

2. Background on fractional derivatives operators

In this section, we recall some preliminary definitions and fundamental lemmas. There exist in the literature various types of fractional derivatives operators with multiple types of kernels. We summarize the different kernels as follows, see in [3]:

• • • Riemann-Liouville and Caputo kernel with $\alpha \in (0, 1)$

$$K^{RL}\left(t-\tau\right) = \frac{\left(t-\tau\right)^{-\alpha}}{\Gamma\left(1-\alpha\right)} \tag{1}$$

• • • Caputo-Fabrizio kernel with $\alpha \in (0, 1)$

$$K^{CF}\left(t-\tau\right) = \frac{M(\alpha)}{1-\alpha} \exp\left(-\frac{\alpha}{1-\alpha}\left(t-\tau\right)\right) \tag{2}$$

• • • Atangana-Baleanu kernel with $\alpha \in (0, 1)$

$$K^{AB}\left(t-\tau\right) = \frac{AB(\alpha)}{1-\alpha} E_\alpha\left(-\frac{\alpha}{1-\alpha}\left(t-\tau\right)^\alpha\right) \tag{3}$$

$M(\alpha)$ and $AB(\alpha)$ have the same properties as in Caputo-Fabrizio fractional derivative, see in [18]. The Riemann-Liouville and Caputo Kernel is a singular kernel because we can observe the function K^{RL} is not well defined when $t = \tau$. In other words, the function K^{RL} admit singularity at the point $t=\tau$. The fractional derivatives operators associated to the function K^{RL} kernel are the Riemann-Liouville fractional derivative and the Caputo-Liouville fractional derivative.

Definition 1. [25, 27] *Let's a function* $f:[a,+\infty[\longrightarrow\mathbb{R}$. *Then the Riemann-Liouville fractional derivative of* f *of order* α *is defined as*

$$D_{\alpha}^{RL}f(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t(t-s)^{-\alpha}f(s)ds \tag{4}$$

for all $t>a,\ \alpha\in(0,1)$, *where* $\Gamma(...)$ *is the Gamma function.*

Definition 2. [25, 27] *Let's a function* $f:[a,+\infty[\longrightarrow\mathbb{R}$. *Then the Caputo-Liouville fractional derivative of* f *of order* α *is defined by*

$$D_{\alpha}^{c}f(t)=\frac{1}{\Gamma(1-\alpha)}\int_a^t\frac{f'(s)}{(t-s)^{\alpha}}ds \tag{5}$$

for all $t>a,\ \alpha\in(0,1)$, *where* $\Gamma(...)$ *is Gamma function.*

Definition 3. [25, 27] *Let's a function* $f:[a,+\infty[\longrightarrow\mathbb{R}$. *Then the Riemann-Liouville integral of* f *of order* α *is defined by*

$$I_{\alpha}^{RL}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s)ds \tag{6}$$

all $t>a,\ \alpha\in(0,1)$, *where* $\Gamma(...)$ *is Gamma function.*

In some cases, using the fractional derivative with a singular kernel has many inconveniences. In 2015, Caputo and Fabrizio proposed a new fractional derivative without a singular kernel. The new definition of the fractional derivative introduced by Caputo and Fabrizio consists of replacing the singular kernel $K^{RL}(t-\tau)$ by the non-singular kernel

$$K^{CF}=\frac{M(\alpha)}{1-\alpha}\exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right). \tag{7}$$

The associated fractional derivative is called the Caputo-Fabrizio fractional derivative. It was provided in the literature the Caputo-Fabrizio fractional derivative operator is very useful to describe real-world problems. It is applied to the Cattaneo-Hristov diffusion model in [12, 13], see also Koca and Atangana investigations in [17]. Escamilla et al. in [9] investigate to find analytic solutions of some electrical circuits as RL, LC and RLC described by Caputo- Fabrizio fractional derivative, and others. Let's recall the definition of the Caputo-Fabrizio fractional derivative.

Definition 4. [18] *Let's* $a>0$, $f\in C(0,a)$ *and* $\alpha\in(0,1)$, *the Caputo-Fabrizio fractional derivative of the function* f *of order* α *is given by*

$$D_{\alpha}^{CF}f(t)=\frac{(2-\alpha)M(\alpha)}{2(1-\alpha)}\int_0^t\exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right)f'(s)ds \tag{8}$$

for all $t>0$, *and where* $M(\alpha)$ *is a normalization constant depending on* α.

It is provided by the Caputo and Fabrizio in [7], if $\alpha\to1$, we recover the classical derivative. The normalization constant has the following property $M(0)=M(1)=1$ and is given by [18]

$$M(\alpha)=\frac{2}{2-\alpha},$$

for all $\alpha \in (0,1)$. And the associated integral is given in the following definition.

Definition 5. [18] *Let's $a > 0$, $f \in C(0,a)$ and $\alpha \in (0,1)$, the Caputo-Fabrizio fractional integral of order α of the function f is given by*

$$I_{\alpha}^{CF} f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t f(s)ds, \tag{9}$$

for all $t > 0$, and where $M(\alpha)$ is the normalization constant depending on α.

In 2016, Atangana and Baleanu proposed another type of fractional derivative because Caputo-Fabrizio fractional derivative is the average of a function and its integral. Furthermore, the solution of the Cauchy problem is an exponential form, not a non-local function, see in [5]. They propose a new kernel called Atangana-Baleanu kernel defined as

$$K^{AB}(t-\tau) = \frac{AB(\alpha)}{1-\alpha} E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right). \tag{10}$$

The fractional derivative associated with Atangana-Baluanu fractional derivative kernel is defined in the following definition.

Definition 6. [4] *Let's the function $f \in H^1(a,b)$, $b > a$ and $\alpha \in [0,1]$, then the Atangana-Baleanu fractional derivative in Caputo sense is given as:*

$$D_{\alpha}^{ABC} f(t) = \frac{AB(\alpha)}{1-\alpha} \int_a^t f'(s) E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-s)^{\alpha}\right) ds \tag{11}$$

It is now provided in the literature the Caputo-Fabrizio and the Atangana-Baleanu fractional derivative are useful and helpful to study the real-world problems. The Caputo-Fabrizio and the Atangana-Baleanu, fractional derivatives operators, can be applied and used in many fields in science and engineering: as linear viscoelasticity, as in diffusion [32, 33, 34, 35], as Navier Stokes problems [30], as Rayleigh-Stokes problems [30, 41], and many others. In this paper, we work with Caputo-Fabrizio fractional derivative. Let us recall the Laplace transform of Caputo-Fabrizio fractional derivative [18], defined as:

$$\mathcal{L}\left\{{}_0^{CF}D_t^{\alpha} f(t)\right\} = \frac{s\mathcal{L}\{f(t)\}(s) - f(0)}{s + \alpha(1-s)}. \tag{12}$$

The Laplace transform will be used to solve the Cauchy problem described by the Caputo-Fabrizio fractional derivative. There exist some modifications in the structure of the solution of the Cauchy problem when the Caputo-Fabrizio derivative is used. Let's recall the first fundamental lemma of this paper.

Lemma 1. *Let $x(.) \in \mathbb{R}^n$, the solution of the Cauchy problem $D_{\alpha}^{CF} x(t) = \lambda x(t)$ with initial boundary condition defined by $x(t_0) = \eta$, is given by*

$$x(t) = \frac{\eta}{1 + \lambda(\alpha - 1)} \exp\left(\frac{\lambda\alpha}{1 + \lambda(\alpha-1)} t\right). \tag{13}$$

Proof: We first apply the Laplace transform, thus it follows that

$$\begin{aligned} \mathcal{L}\left(D_c^{CF} x(t)\right) &= \lambda \mathcal{L}(x(t)) \\ \frac{s\mathcal{L}\{x(t)\}(s) - \eta}{s + \alpha(1-s)} &= \lambda \mathcal{L}(x(t)) \\ \mathcal{L}_{\rho}(x(t)) &= \frac{\eta}{1 + \lambda(\alpha-1)} \frac{\eta}{s - \frac{\lambda\alpha}{1+\lambda(\alpha-1)}} \end{aligned} \tag{14}$$

Applying the inverse of Laplace transform to both sides of Eq. (14), we obtain that

$$x(t) = \frac{\eta}{1+\lambda(\alpha-1)} \exp\left(\frac{\lambda\alpha}{1+\lambda(\alpha-1)} t\right)$$

We can observe if the constant $\frac{\lambda\alpha}{1+\lambda(\alpha-1)}$ is negative then all the solution converge to zero. The classical solution of the equation $x'(t) = \lambda x(t)$ with boundary condition defined by $x(t_0) = \eta$ can be obtained with Eq. (13), when the order of the fractional derivative converge to 1, and we have $x(t) = \eta \exp(t)$. We finish this section by providing a fundamental lemma, which we will use in the Lyapunov direct method. This lemma will simplify the calculation when we use the quadratic Lyapunov function. We have the following lemma.

Lemma 2. *Let $x(.) \in \mathbb{R}^n$ be a vector of differentiable functions and there exists a positive definite, symmetric and constant matrix $P \in \mathbb{R}^{n\times n}$. Then, the following relationship is hold*

$$D_\alpha^{CF}\left(x(t)^T P x(t)\right) \le 2x(t)^T P D_\alpha^{CF} x(t) \qquad \alpha \in [0,1) \qquad \textit{for all } t \ge 0.$$

Proof: We define an intermediary function

$$f(t) = D_\alpha^{CF}\left(x(t)^T P x(t)\right) - 2x(t)^T P D_\alpha^{CF} x(t), \tag{15}$$

which we will study. The objective of the proof is to prove the function f is negative definite. Using the Caputo- Fabrizio fractional derivative, the function f can be expressed as follows:

$$\begin{aligned}
f(t) &= \frac{M(\alpha)}{\Gamma(1-\alpha)} \int_{t_0}^t \left(\dot{x}^T(s)Px(s) - x^T(s)P\dot{x}(s)\right) \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) ds \\
&\qquad - \frac{2M(\alpha)x^T(t)P}{\Gamma(1-\alpha)} \int_{t_0}^t \dot{x}(s) \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) ds \\
&= \frac{M(\alpha)}{\Gamma(1-\alpha)} \int_{t_0}^t \left(\dot{x}^T(s)Px(s) - x^T(s)P\dot{x}(s) - 2x^T(t)P\dot{x}(s)\right) \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) ds \\
&= \frac{M(\alpha)}{\Gamma(1-\alpha)} \int_{t_0}^t \left(2x^T(s)P\dot{x}(s) - 2x^T(t)P\dot{x}(s)\right) \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) ds
\end{aligned}$$

when we change the variable in the integration by letting that $y(s) = x^T(s) - x^T(t)$, then the function f can be rewritten as follows

$$\begin{aligned}
f(t) &= \frac{M(\alpha)}{\Gamma(1-\alpha)} \int_{t_0}^t 2y^T(s)P\dot{y}(s) \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) ds \\
&= \frac{M(\alpha)}{\Gamma(1-\alpha)} \int_{t_0}^t d\left(y^T(s)Py(s)\right) \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) ds
\end{aligned}$$

After integration by parts, we obtain the following expression

$$\begin{aligned}
f(t) &= \left[y^T(s)Py(s)\exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right)\right]_{t_0}^{t} \\
&\quad -\frac{\alpha}{1-\alpha}\int_{t_0}^{t} y^T(s)Py(s)\exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right)ds \\
&= \lim_{s\to t} y^T(s)Py(s)\exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) \\
&\quad -y^T(t_0)Py(t_0)\exp\left(-\frac{\alpha}{1-\alpha}(t-t_0)\right) \\
&\quad -\frac{\alpha}{1-\alpha}\int_{t_0}^{t} y^T(s)Py(s)\exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right)ds
\end{aligned}$$

The value of the limit is straightforward to get, by calculation we find that

$$\lim_{s\to t} y^T(s)Py(s)\exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) = 0\times 1 = 0$$

Finally, the function f is given by

$$\begin{aligned}
f(t) &= -y^T(t_0)Py(t_0)\exp\left(-\frac{\alpha}{1-\alpha}(t-t_0)\right) \\
&\quad -\frac{\alpha}{1-\alpha}\int_{t_0}^{t} y^T(s)Py(s)\exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right)ds
\end{aligned}$$

We observe that the function $f \leq 0$ from which it holds that

$$D_{\alpha}^{CF}\left(x^T Px\right) \leq 2x^T PD_{\alpha}^{CF}x$$

for all $\alpha \in [0,1)$.

3. Stability Analysis of the Fractional Differential Equation in Caputo-Fabrizio Sense

In this section, we investigate to find some conditions of the stability of the fractional differential equations defined with Caputo-Fabrizio fractional derivative. The fractional differential equation treated in this section is in general expressed as

$$D_{\alpha}^{CF}x = f(t,x) \tag{16}$$

where $x \in \mathbb{R}^n$ is the state variable, and the function $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous locally Lipschitz function and satisfies the condition $f(t,0) = 0$. Given the initial condition $x_0 \in \mathbb{R}^n$, the solution of Eq. (16) starting initially at x_0 at time $t = t_0$ is denoted by $x(.) = x(., x_0)$. We recall some stability notions used in fractional calculus.

Definition 7. [27, 40] *The trivial solution $x = 0$ of the fractional differential equation in Caputo-Fabrizio sense defined by Eq. (16) is said to be stable if, for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that for any initial condition satisfying $\|x_0\| < \delta$, the solution $x(t)$ of Eq. (16) satisfies the inequality $\|x(t)\| < \epsilon$ for all $t > t_0$.*

The trivial solution $x = 0$ of the fractional differential equation (20) is said to be asymptotically stable if it is stable and furthermore $\lim_{t\to+\infty} x(t) = 0$.

In terms of comparison function, we have the following definition of global asymptotic stability.

Definition 8. [27, 40] *The origin $x = 0$ of the fractional differential equation in Caputo-Fabrizio sense defined by Eq. (16) is said to be globally uniformly asymptotically stable if there exists a class $\mathcal{KL}$ function β such that, for any bounded initial condition x_0, its solution satisfies*

$$\|x(t, x_0)\| \leq \beta(\|x_0\|, t - t_0). \tag{17}$$

There exist two methods to analyze the stability analysis of the fractional differential equations: by trajectory method and the Lyapunov direct method.

3.1. **Stability Analysis linear fractional differential equation.** In this section, we consider the linear fractional differential equations defined by the following relation

$$D_{\alpha}^{CF} x = Ax \tag{18}$$

where $x \in \mathbb{R}^n$ is the state variable and $A \in \mathbb{R}^{n\times n}$. The equation (18) can be expressed using the Caputo fractional derivative or the Riemann-Liouville fractional derivative. The study related to these categories of fractional differential equations is already discussed in the literature, see in [26]. It was provided if the fractional linear differential equation described by Caputo fractional derivative and the Riemann-Liouville fractional derivative satisfies the condition that $|\arg(\lambda(A)| > \frac{\alpha\pi}{2}$ then its origin is asymptotically stable. The main interest in this section is to see what is the modification obtained when we study the fractional linear differential equation with Caputo-Fabrizio fractional derivative Eq. (18), using the trajectory method. Does this above condition change? We will investigate to bring some precision to this question. Let's give the first main result of this paper. We establish the asymptotic stability of the fractional linear differential equation (18) using the trajectory method. Note that the trajectory method means that we get the solution of the fractional differential equation; we analyze it respecting the stability notions.

Theorem 1. *Let $x = 0$ be an equiolibrium point for the fractional differential equation (18). If the matrix $I_n + (\alpha - 1) A$ is invertible and the condition*

$$\left|\arg(\lambda(\alpha A\left(I_n + (\alpha - 1) A\right)^{-1})\right| > \frac{\alpha\pi}{2} \tag{19}$$

is hold, then the trivial solution of the fractional linear differential equation defined by Eq. (18) is asymptotically stable.

Proof: We first apply the Laplace transform, it follows that

$$\begin{aligned}
\mathcal{L}\left(D_c^{CF} x(t)\right) &= A\mathcal{L}\left(x(t)\right)\\
\frac{s\mathcal{L}\left\{x(t)\right\}(s) - \eta}{s + \alpha\left(1 - s\right)} &= A\mathcal{L}\left(x(t)\right)\\
s\mathcal{L}\left\{x(t)\right\}(s) - \eta &= sA\mathcal{L}\left(x(t)\right) + \alpha\left(1 - s\right)A\mathcal{L}\left(x(t)\right)\\
\left[s\left(I_n + (\alpha - 1) A\right) - \alpha A\right]\mathcal{L}\left\{x(t)\right\}(s) &= \eta\\
\left[sI_n - \alpha A\left(I_n + (\alpha - 1) A\right)^{-1}\right]\mathcal{L}\left\{x(t)\right\}(s) &= \eta\left(I_n + (\alpha - 1) A\right)^{-1}\\
\mathcal{L}_{\rho}\left(x(t)\right) &= \frac{\eta\left(I_n + (\alpha - 1) A\right)^{-1}}{sI_n - \alpha A\left(I_n + (\alpha - 1) A\right)^{-1}}
\end{aligned}$$

Applying the inverse of Laplace transform on previous equation, we obtain that

$$\begin{aligned} x(t) &= \eta \left(I_n + (\alpha-1)\,A\right)^{-1} \exp\left(\alpha A \left(I_n + (\alpha-1)\,A\right)^{-1} t\right) \\ &= \tilde{\eta} \exp\left(\alpha A \left(I_n + (\alpha-1)\,A\right)^{-1} t\right) \end{aligned}$$

where $\tilde{\eta} = \eta \left(I_n + (\alpha-1)\,A\right)^{-1}$ is a vector. Doing same reasoning as in [26], we obtain that

$$\lim_{t\to+\infty} x(t) = 0$$

when the condition $\left|\arg(\lambda(\alpha A \left(I_n + (\alpha-1)\,A\right)^{-1})\right| > \frac{\alpha\pi}{2}$ is hold.

The condition in Theorems 1 can become more simple by using the Lyapunov direct method, as we will see in the next section. Note that using trajectories, it is not trivial to notice the condition $|\arg(\lambda(A)| > \frac{\alpha\pi}{2}$ is necessary and sufficient for the asymptotic stability of the fractional linear differential equations described by the Caputo-Fabrizio fractional derivative.

3.2. **Stability Analysis with Lyapunov direct method.** The use of the trajectories to prove the asymptotic stability is not all time possible due to the complexity of some fractional differential equations. It is for that in this section; we investigate to find the Lyapunov characterization of the asymptotic stability of the fractional differential equations defined by the Caputo-Fabrizio fractional derivative. We first begin to characterize the exponential stability.

Theorem 2. *Let $x = 0$ be an equilibrium point for the fractional differential equation (16) in the Caputo-Fabrizio sense. Let's there exists a positive function $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous and differentiable, and satisfies the conditions enumerated as follows*

(1) $\|x(t)\|^a \leq V(t,x)$
(2) $D_\alpha^{CF} V(t,x) \leq -kV(t,x)$

where k is non-negative constant, then the origin of the fractional differential equation (16) is exponential stable.

Note that in the previous theorem, we suppose the Caputo-Fabrizio fractional derivative of the function V exists.

Proof: It follows from assumption (2) the following relationship

$$D_\alpha^{CF} V(t,x) \leq -kV(t,x)$$

To saturate the above constraint, we have known that, there exist a continuous positive function m such that the following equation is hold

$$D_\alpha^{CF} V(t,x) = -kV(t,x) - m(t)$$

From which there exist κ such that we obtain the following relationship

$$V(t) \leq \frac{V(t_0)}{1+k(1-\alpha)} \exp\left(-\frac{k\alpha}{1+k(1-\alpha)} t\right) - \kappa \int_0^t \exp\left(-\frac{k\alpha}{1+k(1-\alpha)}(t-s)\right) m(s)ds.$$

Due to the negativity of the second term, we have the following relation

$$V(t) \leq \frac{V(t_0)}{1+k(1-\alpha)} \exp\left(-\frac{k\alpha}{1+k(1-\alpha)} t\right)$$

From the first assumption (1), we have the following relationship

$$\|x(t)\|^a \leq V(t) \leq \frac{V(t_0)}{1+k\,(1-\alpha)} \exp\left(-\frac{k\alpha}{1+k\,(1-\alpha)}t\right)$$

Which in turn implies that

$$\|x(t)\| \leq \left\{\frac{V(t_0)}{1+k\,(1-\alpha)}\right\}^{1/a} \exp\left(-\frac{k\alpha}{a+ka\,(1-\alpha)}t\right) \tag{20}$$

From which it follows that the origin of the fractional differential equation in Caputo-Fabrizio sense (16) is exponentially stable.

We note with Theorem 2 that we haven't obtained the Mittag-Leffler stability or the fractional exponential stability; we recover the exponential stability. This conclusion can surprise well, but that is because the used fractional derivative has an exponential kernel.

Theorem 3. *Let's $x = 0$ be an equilibrium point for the fractional differential equation (16) in the Caputo-Fabrizio sense. Let's there exists a positive Lyapunov candidate function $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying the conditions enumerated as follows*

(1) *$V(t,x)$ has Caputo-Fabrizio fractional derivative of order α for all $t > t_0 \geq 0$*
(2) $D^{CF}_{\alpha} V(t,x) \leq -\chi\left(\|x\|\right)$

Then the origin of the fractional differential equation (16) is globally uniformly asymptotically stable.

Proof: From the fact that V is a Lyapunov candidate function, then there exists a class $\mathcal{K}_\infty$ functions χ_1, χ_2 such that the following relation is holds $\chi_1\left(\|x\|\right) \leq V(t,x) \leq \chi_2\left(\|x\|\right)$. Combining it with assumption (2), we get that

$$D^{CF}_{\alpha} V(t,x) \leq -\chi\left(\chi_2^{-1}(V(t,x)\right).$$

There exist a class $\mathcal{KL}$ function μ [27] such that we have the following relation

$$V(t,x) \leq \mu\left(\chi_2(\|x_0\|), t - t_0\right).$$

Using the fact that V is a Lyapunov candidate function, we obtain the following relationship

$$\chi_1\left(\|x\|\right) \leq \mu\left(\chi_3(\|x_0\|), t - t_0\right).$$

Which in turn under assumption that $\chi_1^{-1}(a+b) \leq \chi_1^{-1}(2a) + \chi_1^{-1}(2b)$ as $\chi_1 \in \mathcal{K}_\infty$ and $a, b \in \mathbb{R}$, implies the following relation

$$\|x(t)\| \leq \chi_1^{-1}(2\mu\left(\chi_3(\|x_0\|), t - t_0\right))$$

We suppose that the function $\beta(\|x_0\|, t - t_0)) = \chi_1^{-1}(2\mu\left(\chi_2(\|x_0\|), t - t_0\right))$, which is clearly a class $\mathcal{KL}$ function. We finally obtain that

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0))$$

That corresponds to the global uniform asymptotic stability of the origin of the fractional differential equation in Caputo-Fabrizio sense defined by Eq. (16).

We finish this section by a particular Lyapunov characterization of the exponential stability of the fractional differential equation in Caputo-Fabrizio sense.

Theorem 4. *Let's $x = 0$ be an equilibrium point for the fractional differential equation (16) in Caputo-Fabrizio sense. Let's there exists a positive function $V : \mathbb{R}^+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be continuous and differentiable, and satisfies the conditions enumerated as follows*

(1) $a\|x\|^2 \leq V(t,x) \leq b\|x\|^2$
(2) $D_\alpha^{CF} V(t,x) \leq -c\|x\|^2$

Then the origin of the fractional differential equation (16) is exponentially stable.

Note that in the previous theorem, we suppose the Caputo-Fabrizio fractional derivative of the function V exists.

Proof: It comes from the assumption (2) the following relation

$$D_\alpha^{CF} V(t,x) \leq -c\|x\|^2 \leq -\frac{c}{b} V(t,x)$$

Using the same procedure as in Theorem 2, we get that

$$V(t) \leq \frac{V(t_0)}{1+\frac{c}{b}(1-\alpha)} \exp\left(-\frac{\frac{c}{b}\alpha}{1+\frac{c}{b}(1-\alpha)} t\right)$$

Under first assumption, we have the following relationship

$$a\|x(t)\|^2 \leq \frac{V(t_0)}{1+\frac{c}{b}(1-\alpha)} \exp\left(-\frac{\frac{c}{b}\alpha}{1+\frac{c}{b}(1-\alpha)} t\right)$$

From which we obtain that

$$\|x(t)\| \leq \left\{\frac{V(t_0)}{a+\frac{ca}{b}(1-\alpha)}\right\}^{1/2} \exp\left(-\frac{\frac{c}{b}\alpha}{2+\frac{2c}{b}(1-\alpha)} t\right)$$

That corresponds to the exponential stability of the origin of the fractional differential equation in Caputo-Fabrizio sense defined by equation (16).

4. Illustrative examples of the mains results

In this section, we give some examples to illustrate the main results of this paper. The examples given in this section show the Lyapunov characterization of the stability of the fractional differential equations in Caputo-Fabrizio sense, particularly. We first consider the fractional differential equation defined by

$$D_\alpha^{CF} x = Ax \tag{21}$$

where $x \in \mathbb{R}^n$ is a state variable and $A \in \mathbb{R}^{n\times n}$. The objective with this class of fractional differential equation is to prove that the condition $|\arg(\lambda(A)| > \frac{\alpha\pi}{2}$ is necessary and sufficient for the asymptotic stability. To this end, we use the Lyapunov direct method to illustrate it. Let's the Lyapunov function defined by $V(t,x) = x^T P x$ where the matrix $P = I_n$ is a square, constant, positive, and symmetric matrix. Calculating the α-derivative of the Lyapunov function V along the trajectories of the fractional linear differential equation (21), we get under Lemma 2, that

$$\begin{aligned} D_\alpha^{CF} V(t,x) \leq 2x^T P D_\alpha^{CF} x &= [Ax]^T Px + x^T P[Ax] \\ &= x^T A^T P x + x^T P A x \\ &= x^T\left(A^T P + PA\right)x \\ &\leq -\lambda_{min}(Q)\|x\|^2 \end{aligned}$$

We known that it derives from the condition $|\arg(\lambda(A)| > \frac{\alpha\pi}{2}$ the existence of a positive definite matrix Q satisfying the following condition $A^TP + PA = -Q$, thus we have

$$\begin{aligned} D_\alpha^{CF}V(t,x) \leq 2x^TPD_\alpha^{CF}x &= x^T\left(A^TP+PA\right)x \\ &= -x^TQx \\ &\leq -\lambda_{min}(Q)\left\|x\right\|^2 \end{aligned}$$

where $\lambda_{min}(Q)$ is the minimum eigenvalue of the matrix Q. It follows from Theorem 4 that the fractional linear differential equation in Caputo-Fabrizio sense (21) is exponentially stable, when the condition $|\arg(\lambda(A)| > \frac{\alpha\pi}{2}$ is held.

For the second example let's analyze the stability of the fractional differential equation defined by

$$D_\alpha^{CF}x = Ax + Bx \tag{22}$$

where $x \in \mathbb{R}^n$ is the state variable, A is an Hurwitz matrix in $\mathbb{R}^{n\times n}$, B is an matrix in $\mathbb{R}^{n\times n}$. We choose the same Lyapunov function as in the previous section defined by $V(t,x) = x^TPx$ where the matrix P satisfies the condition that $A^TP+PA = -Q$ and $P = I_n$. The α-derivative of V along the trajectories of the perturbed fractional linear differential equation in Caputo-Fabrizio sense (22) give the following relation

$$\begin{aligned} D_\alpha^{CF}V(t,x) \leq 2x^TPD_\alpha^{CF}x &= \left[Ax+Bx\right]^TPx + x^TP\left[Ax+Bx\right] \\ &= x^TA^TPx + (Bx)^TPx + x^TPAx + x^TP(Bx) \\ &= x^T\left(A^TP+PA\right)x + (Bx)^TPx + x^TP(Bx) \\ &\leq -\lambda_{min}(Q)\left\|x\right\|^2 + 2\lambda_{max}(P)\left\|B\right\|\left\|x\right\|^2 \\ &= -\left[\lambda_{min}(Q) - 2\lambda_{max}(P)\left\|B\right\|\right]\left\|x\right\|^2 \end{aligned}$$

where $\lambda_{min}(Q)$ is the minimum eigenvalue of the matrix Q and $\lambda_{max}(P)$ the maximum eigenvalue of the matrix P. It follows from Theorem 4, the exponential stability of the perturbed fractional linear differential equation in Caputo-Fabrizio sense (22), when the condition $\lambda_{min}(Q) - 2\lambda_{max}(P)\left\|B\right\| > 0$ is held.

5. Conclusion

We have discussed in this paper the stability of the fractional differential equation defined with the Caputo-Fabrizio fractional derivative. We have noticed with Caputo-Fabrizio fractional derivative, we obtain exponential stability. We also see the solution to the Cauchy problem is also exponential. Lyapunov characterization of the asymptotic stability and exponential stability were proposed, and examples to illustrate theses characterizations were given.

References

[1] Y. Adjabi, F. Jarad, and T. Abdeljawad, *On generalized fractional operators and a gronwall type inequality with applications*, Filomat, **31(17)**, (2017).

[2] J. Alidousti, R. Khoshsia, and A. Bayati, *Stability analysis of nonlinear fractional differential order systems with caputo and Riemann-Liouville derivatives*, Turk. J. of Math., **41(5)**, 1260-1278, (2017).

[3] A. Atangana, *Non validity of index law in fractional calculus: A fractional differential operator with markovian and non-markovian properties*, Phys. A: Stat. Mech. and Appl., **505**, 688-706, 2018.

[4] A. Atangana, and D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model*, arXiv preprint arXiv:1602.03408, (2016).

[5] A. Atangana, and I. Koca, *Chaos in a simple nonlinear system with AtanganaBaleanu derivatives with fractional order*, Chaos, Soli. and Fract., **89**, 447-454, (2016).

[6] D. Baleanu, Z. B. Gvenc, J. A. T. Machado, et al., *New trends in nanotechnology and fractional calculus applications*, Springer, (2010).

[7] M. Caputo, and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progr. Fract. Differ. Appl, **1(2)**, 1-15, (2015).

[8] W. S. Chung, *Fractional newton mechanics with conformable fractional derivative*, J. of Comput. and Appl. Math., **290**, 150-158, 2015.

[9] A. Coronel-Escamilla, J. F Gmez-Aguilar, L. Torres, and R. F. Escobar- Jimnez, A numerical solution for a variable-order reactiondiffusion model by using fractional derivatives with non-local and non-singular kernel. Phys. A: Stat. Mech. and Appl., **491**, 406-424, 2018.

[10] M. Eslami, Exact traveling wave solutions to the fractional coupled nonlinear schrodinger equations. Appl. Math. and Comput., **285**, 141-148, 2016.

[11] J. Hristov, On the atangana-baleanu derivative and its relation to the fading memory concept: The diffusion equation formulation.

[12] J. Hristov, *Approximate solutions to fractional subdiffusion equations*, The Euro. Phys. J. Spec. Topi., **193(1)**, 229-243, (2011).

[13] J. Hristov, *Steady-state heat conduction in a medium with spatial non-singular fading memory; derivation of caputo-fabrizio space fractional derivative from cattaneo concept with jeffrey's kernel and analytical solutions*, Ther. Scie., **21(2)**, 827-839, (2017).

[14] J. Fahd, and T. Abdeljawad, *A modified Laplace transform for certain generalized fractional operators*, Res. in Nonl. Anal., **2**, 88-98 (2018).

[15] J. Fahd, and T. Abdeljawad, and D. Baleanu, *On the generalized fractional derivatives and their caputo modification*, J. Nonlinear Sci. Appl, **10(5)**, 2607-2619, (2017).

[16] A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, (2006).

[17] I. Koca and A. Atangana, *Solutions of cattaneo-hristov model of elastic heat diffusion with caputo-fabrizio and atangana-baleanu fractional derivatives*, Ther. Scie., **00**, 103-103, (2016).

[18] J. Losada and J. J. Nieto, *Properties of a new fractional derivative without singular kernel*, Progr. Fract. Differ. Appl, **1(2)**, 87-92, (2015).

[19] A. B. Malinowska and D. FM. Torres, *Introduction to the fractional calculus of variations*, World Scie. Publis. Comp., (2012).

[20] K. S. Miller, and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, **1993**.

[21] V. F. Morales-Delgado, J. F. Gmez-Aguilar, and M. A. Taneco-Hernndez, *Analytical solutions of electrical circuits described by fractional conformable derivatives in liouville-caputo sense*, AEU-Int. J. Elec. Comm., **85**, 108-117, (2018).

[22] K. Oldham, and J. Spanier, *The fractional calculus theory and applications of differentiation and integration to arbitrary order*, Else., **111**, 65-67, (1974).

[23] I. Petras, *Fractional-order nonlinear systems: modeling, analysis and simulation*, Spring. Sci. Busi. Medi., (2011).

[24] I. Podlubny, *Matrix approach to discrete fractional calculus ii: Partial fractional differential equations*, 2009.

[25] S. Priyadharsinid, *Stability of fractional neutral and integrodifferential systems*, J. Fract. Calc. Appli., **7(1)**, 87-102, (2016).

[26] D. Qian, C. Li, R. P. Agarwal, and P. J. Y. Wong, *Stability analysis of fractional differential system with Riemann-Liouville derivative*, Math. Comput. Model., **52(5-6)**, 862-874, (2010).

[27] N. Sene, *Exponential form for Lyapunov function and stability analysis of the fractional differential equations*, J. Math. Comp. Scien., **18** (2018), :388–397.

[28] N. Sene, *Lyapunov characterization of the fractional nonlinear systems with exogenous input*, Frac. Fract., **2(2)**, (2018).

[29] N. Sene, *Solutions for some conformable differential equations*, Progr. Fract. Differ. Appl, **4(4)**, 493-501, (2018).

[30] N. Sene, *Stokes first problem for heated flat plate with AtanganaBaleanu fractional derivative*, Chaos, Soli. Fract., **117**, 68-75, (2018).

[31] N. Sene, *On stability analysis of the fractional nonlinear systems with hurwitz state matrix*, J. Frac. Calc. Appl., **10(1)**, 1-9, (2019).
[32] Sene, N., *Analytical solutions of Hristov diffusion equations with non-singular fractional derivatives*, choas, **29**, 023112, (2019).
[33] Sene, N., *Solutions of fractional diffusion equations and Cattaneo-Hristov diffusion models*, Int. J. Appli. Anal., **17(2)**, 191-207, 2019.
[34] Sene, N., *Homotopy Perturbation ρ-Laplace Transform Method and Its Application to the Fractional Diffusion Equation and the Fractional Diffusion-Reaction Equation*, Frac. Fract., **3**,14 (2019)
[35] A. N. Fall, S. N. Ndiaye, N. Sene, *Black-Scholes option pricing equations describ e d by the Caputo generalized fractional derivative*, Chaos, Sol. Fract., **125**, 108-118, (2019).
[36] N. Sene, *Analytical solutions and numerical schemes of certain generalized fractional diffusion models*, Eur. Phys. J. Plus, **134**, 199, (2019).
[37] N. Sene, J.F. Gomez-Aguilar, *Analytical solutions of electrical circuits considering certain generalized fractional derivatives*, Eur. Phys. J. Plus, **134**, 260, (2019).
[38] N. Sene, J.F. Gomez-Aguilar, *Fractional Mass-Spring-Damper System Described by Generalized Fractional Order Derivatives*, Fractal Fract., **3**, 39, (2019).
[39] N. Sene, K. Abdelmalek, *Analysis of the fractional diffusion equations describ e d by Atangana-Baleanu-Caputo fractional derivative*, Chaos, Sol. Fract., **127**, 158-164, (2019).
[40] N. Sene, G. Srivastava, *Generalized Mittag-Leffler Input Stability of the Fractional Differential Equations*, Sym., **11**, 608, (2019).
[41] N. Sene, *Integral Balance Methods for Stokes First, Equation Described by the Left Generalized Fractional Derivative*, Phys., **1**, 154-166, (2019).

NDOLANE SENE
Laboratoire Lmdan, Département de Mathématiques de la Décision, Université Cheikh Anta Diop de Dakar, Faculté des Sciences Economiques et Gestion, BP 5683 Dakar Fann, Senegal.
E-mail address: ndolanesene@yahoo.fr