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FRACTIONAL LYAPUNOV-TYPE INEQUALITIES WITH MIXED BOUNDARY CONDITIONS ON UNIVARIATE AND MULTIVARIATE DOMAINS

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1. Introduction

For the second-order linear differential equation

$$x'' + q(t)x = 0 \quad \text{on } (a,b) \tag{1}$$

with $q \in C([a,b],\mathbb{R})$, the following result is known as the Lyapunov inequality, see [4, 20].

Theorem 1.1. *Assume Eq. (1) has a solution $x(t)$ satisfying $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a,b)$. Then*

$$\int_a^b |q(t)|dt > \frac{4}{b-a}. \tag{2}$$

This inequality was improved by Wintner [27] and several others by replacing $|q(t)|$ by $q_+(t) := \max\{q(t), 0\}$, the nonnegative part of $q(t)$, to become

$$\int_a^b q_+(t)dt > \frac{4}{b-a}. \tag{3}$$

Inequality (3) was extended to a more general form of second-order linear differential equations by Hartman [12, Chapter XI], and improved by Harris and Kong [13] and Brown and Hinton [3] later on.

These Lyapunov inequalities have been used as an important tool in oscillation, disconjugacy, control theory, eigenvalue problems, and many other areas of differential equations. Due to their importance in applications, they have been extended in various directions by many authors. Actually, Lyapunov-type inequalities have been developed for higher order linear differential equations and half-linear differential equations. See [8, 23, 24, 28, 29] for the higher order linear case, [6, 7] for the half-linear case, and Pinasco [25] for an excellent survey.

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Recently, Lyapunov-type inequalities have been further established for fractional differential equations. See [5,9–11,15,16,18,22,26] and the references cited therein. Among these, Ferreira [11] studied the equation

$$\Big(D^{\alpha}_{a^+}x\Big)(t)+q(t)x=0,\quad 1<\alpha\le 2, \tag{4}$$

where $D^{\alpha}_{a^+}x$ denotes the α-th Riemann-Liouville derivative of x and $q\in C\left([a,b],\mathbb{R}\right)$.

Theorem 1.2. *Assume Eq. (4) has a solution $x(t)$ satisfying $x(a)=x(b)=0$ and $x(t)\neq 0$ for $t\in(a,b)$. Then*

$$\int_a^b |q(t)|dt>\Gamma(\alpha)\Big(\frac{4}{b-a}\Big)^{\alpha-1}. \tag{5}$$

Lyapunov-type inequalities of integer and fractional order have also been obtained on multivariate domains or for partial differential equations by many authors. To name a few, we refer the reader to [1,14,17,21]. In particular, Anastassiou [1] obtained Lyapunov-type inequalities for third-order multivariate equations on special domains in $\mathbb{R}^n$, as shown below: For $N\ge 2$ denote

$$B(0,R):=\{x\in\mathbb{R}^N:|x|<R\}\text{ for }R>0,$$

and let A be an open spherical shell in $\mathbb{R}^N$ centered at the origin, i.e., $A:=B(0,b)\setminus\overline{B(0,a)}$ for $0<a<b$. Consider the following equation

$$\frac{\partial^3 y(x)}{\partial r^3}+q(x)y(x)=0 \tag{6}$$

with the boundary condition (BC)

$$y(\partial B(0,a))=y(\partial B(0,b))=0\text{ and }\frac{\partial^2 y(\partial B(0,\xi))}{\partial r^2}=0\text{ for }\xi\in[a,b], \tag{7}$$

where $q\in C(\overline{A})$ and the derivatives with respect to r are directional derivatives in the radial direction.

Theorem 1.3. *Assume Eq. (6) has a nontrivial solution $y(x)$ satisfying (7) and $y(x)\neq 0$ on A. Then*

$$\int_A |q(x)|dx>\frac{8\pi^{N/2}a^{N-1}}{\Gamma(N/2)(b-a)^2}. \tag{8}$$

Note that inequality (8) holds only when the domain A is radially symmetric and contains a circular hole of radius $a>0$ inside. Moreover, it becomes less sharp when a is small and provides no information when $a=0$, i.e., when the hole shrinks to a point.

In this paper, we will first establish Lyapunov-type inequalities for a univariate Riemann-Liouville fractional differential equation of order $\alpha\in(2,3]$ together with several pointwise or mixed BCs. Based on the results, we will further develop Lyapunov-type inequalities for multivariate fractional boundary value problems (BVPs) on domains in $\mathbb{R}^N$ which are not necessarily radially symmetric. We point out that when $\alpha=3$, our result improves Theorem 1.3 even for the special domain A, see Remark 4.1, Part (ii). Moreover, our result provides estimates for general simply connected regions as long as they contain the origin inside, see Remark 4.1, Part (iii).

This paper is organized as follows: After this introduction, we present results for the univariate equation in Section 2 with their proofs given in Section 3. Multivariate Lyapunov-type inequalities are given in Section 4.

2. Main Results

Recall that for any $\gamma > 0$ and $t > a$,

$$\Big(I_{a^+}^{\gamma} x\Big)(t) := \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} x(s)ds$$

denotes the γ-th order (left-sided) Riemann-Liouville fractional integral of $x(t)$ at a, and

$$\Big(D_{a^+}^{\gamma} x\Big)(t) := \frac{d^n}{dt^n}\Big(I_{a^+}^{n-\gamma} x\Big)(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\gamma-1} x(s)ds \tag{0}$$

denotes the γ-th order Riemann-Liouville fractional derivative of $x(t)$ at a, where $n = \lfloor\gamma\rfloor + 1$ with $\lfloor\gamma\rfloor$ the integer part of γ and $\Gamma(\gamma) = \int_0^\infty t^{\gamma-1}e^{-t}dt$ is the Gamma function.

In this section, we consider the α-th order fractional linear differential equation

$$\Big(D_{a^+}^{\alpha} x\Big)(t) + q(t)x = 0, \quad 2 < \alpha \le 3, \tag{1}$$

where $q \in C([a,b], \mathbb{R})$. We will derive Lyapunov-type inequalities for the BVPs consisting of Eq. (1) and one of the following BCs:

$$x(a) = 0 \quad \text{and} \quad x'(a) = x'(b) = 0; \tag{2}$$

$$x(a) = x(b) = 0 \quad \text{and} \quad x'(a) = 0; \tag{3}$$

$$x(a) = 0 \quad \text{and} \quad \Big(D_{a^+}^{\alpha-2} x\Big)(a) = \Big(D_{a^+}^{\alpha-2} x\Big)(b) = 0; \tag{4}$$

$$x(a) = x(b) = 0 \quad \text{and} \quad \Big(D_{a^+}^{\alpha-2} x\Big)(a) = 0; \tag{5}$$

$$x(a) = x(b) = 0 \quad \text{and} \quad \Big(D_{a^+}^{\alpha-1} x\Big)(\xi) = 0, \quad \xi \in [a,b]. \tag{6}$$

In the following, we say that a solution $x(t)$ of Eq. (1) does not change sign on $[a,b]$ if $x(t) \ge 0$ on $[a,b]$ or $x(t) \le 0$ on $[a,b]$. The first result is for Eq. (1) with BC (2) or BC (3).

Theorem 2.1. *Assume Eq. (1) has a nontrivial solution $x(t)$ satisfying either (2) or (3) and $x(t)$ does not change sign on $[a,b]$. Then*

$$\int_a^b q_+(t)dt > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}. \tag{7}$$

The next result is for Eq. (1) with BC (4) or BC (5).

Theorem 2.2. *Assume Eq. (1) has a nontrivial solution $x(t)$ satisfying either (4) or (5) and $x(t)$ does not change sign on $[a,b]$. Then*

$$\int_a^b q_+(t)dt > \frac{4\Gamma(\alpha-1)}{(b-a)^{\alpha-1}}. \tag{8}$$

The last result is for Eq. (1) with BC (6).

Theorem 2.3. *Assume Eq. (1) has a nontrivial solution $x(t)$ satisfying (6) and $x(t)$ does not change sign on $[a, b]$. Then*

$$\int_a^\xi q_-(t)dt + \int_\xi^b q_+(t)dt > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}. \tag{9}$$

Consequently,

$$\int_a^b |q(t)|dt > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}. \tag{10}$$

Remark 2.1. As special cases, when $a = \xi$, then (9) becomes

$$\int_a^b q_+(t)dt > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}};$$

and when $\xi = b$, then (9) becomes

$$\int_a^b q_-(t)dt > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}.$$

In general, (9) is sharper than (10). Furthermore, (9) shows that under the assumptions of Theorem 2.3, we never expect that

$$q(t)\begin{cases}\geq 0, & t \in [a, \xi]\\ \leq 0, & t \in (\xi, b]\end{cases}$$

would happen. However, this cannot be observed from (10).

3. Proofs

In order to prove Theorems 2.1-2.3, we will use the following lemmas. The first one is from [11] which extends the one given in [2] for the case that $a = 0$ and $b = 1$.

Lemma 3.1. *Consider the BVP*

$$-\left(D_{a^+}^\beta u\right)(t) = h(t), \quad u(a) = u(b) = 0 \tag{1}$$

with $1 < \beta \leq 2$ and $h \in C([a, b], \mathbb{R})$. Then $u(t)$ is a solution of BVP (1) if and only if $u(t)$ satisfies the following integral equation

$$u(t) = \int_a^b G_\beta(t, s)h(s)ds, \tag{2}$$

where $G_\beta(t, s)$ is referred as the Green's function for BVP (1) and is given by

$$G_\beta(t, s) = \frac{1}{\Gamma(\beta)}\begin{cases}\frac{(t-a)^{\beta-1}(b-s)^{\beta-1}}{(b-a)^{\beta-1}} - (t-s)^{\beta-1}, & a \leq s \leq t \leq b\\ \frac{(t-a)^{\beta-1}(b-s)^{\beta-1}}{(b-a)^{\beta-1}}, & a \leq t \leq s \leq b\end{cases}. \tag{3}$$

Moreover, $G_\beta(t, s) \geq 0$.

The next lemma provides an estimate for the integral of $G_\beta(t, s)$ in t.

Lemma 3.2. *For $s \in [a, b]$ we have*

$$\int_a^b G_\beta(t, s)dt \leq \frac{(\beta-1)^{\beta-1}(b-a)^\beta}{\beta^\beta\Gamma(\beta+1)}. \tag{4}$$

Proof. For $s \in [a,b]$,

$$\int_a^b G_\beta(t,s)dt = \int_a^s G_\beta(t,s)dt + \int_s^b G_\beta(t,s)dt$$

$$= \frac{1}{\Gamma(\beta)}\left[\left(\frac{b-s}{b-a}\right)^{\beta-1}\int_a^s (t-a)^{\beta-1}\,dt + \left(\frac{b-s}{b-a}\right)^{\beta-1}\int_s^b (t-a)^{\beta-1}\,dt - \int_s^b (t-s)^{\beta-1}dt\right]$$

$$= \frac{1}{\Gamma(\beta)}\left[\left(\frac{b-s}{b-a}\right)^{\beta-1}\int_a^b (t-a)^{\beta-1}\,dt - \int_s^b (t-s)^{\beta-1}dt\right]$$

$$= \frac{1}{\beta\Gamma(\beta)}\left[\left(\frac{b-s}{b-a}\right)^{\beta-1}(b-a)^\beta - (b-s)^\beta\right]$$

$$= \frac{1}{\Gamma(\beta+1)}(s-a)(b-s)^{\beta-1}.$$

Let $g(s) := (s-a)(b-s)^{\beta-1}$. It is easy to see that the maximum of $g(s)$ occurs at $d = [b+(\beta-1)a]/\beta$. Hence

$$g(s) \le g(d) = \frac{(\beta-1)^{\beta-1}(b-a)^\beta}{\beta^\beta}.$$

This shows that (4) holds. □

The last lemma provides an extension of Rolle's theorem to Riemann-Liouville fractional derivatives.

Lemma 3.3. *Assume a function $u(t)$ satisfies $u(a) = u(b) = 0$ and for $0 < \gamma \le 1$, $\left(D^\gamma_{a^+}u\right)(t)$ defined by (0) exists on (a,b). Then there exists a $c \in (a,b)$ such that $\left(D^\gamma_{a^+}u\right)(c) = 0$.*

Proof. The case for $\gamma = 1$ follows immediately from Rolle's theorem. Now, let $0 < \gamma < 1$. Assume the contrary and without loss of generality, let $\left(D^\gamma_{a^+}u\right)(t) > 0$ for $t \in (a,b)$. By integration we have

$$u(t) = I^\gamma_{a^+}\left(D^\gamma_{a^+}u(t)\right) + C(t-a)^{\gamma-1}$$

with $C \in \mathbb{R}$. Now $u(a) = 0$ implies $C = 0$. Hence

$$u(t) = I^\gamma_{a^+}\left(D^\gamma_{a^+}u(t)\right) = \frac{1}{\Gamma(\gamma)}\int_a^t (t-s)^{\gamma-1}\left(D^\gamma_{a^+}u\right)(s)ds.$$

Then

$$u(b) = \frac{1}{\Gamma(\gamma)}\int_a^b (b-s)^{\gamma-1}\left(D^\gamma_{a^+}u\right)(s)ds > 0.$$

We have reached a contradiction. □

Now we prove Theorems 2.1-2.2.

Proof of Theorem 2.1. (i) We assume $x(t)$ satisfies (2) and without loss of generality, let that $x(t) \ge 0$ for $t \in [a,b]$. We claim that $\left(D^\alpha_{a^+}x\right)(t) = \left(D^{\alpha-1}_{a^+}x'\right)(t)$ for $x(a) = 0$. Clearly the claim holds for $\alpha = 3$. Now let $2 < \alpha < 3$ which implies $0 < \alpha - 2 < 1$. By [19, (2.1.28)] and from the fact that $x(a) = 0$, we have

$$\left(D^{\alpha-2}_{a^+}x\right)(t) = \frac{1}{\Gamma(3-\alpha)}\left[\frac{x(a)}{(t-a)^{\alpha-2}} + \int_a^t \frac{x'(s)}{(t-s)^{\alpha-2}}ds\right] = \left(I^{3-\alpha}(x')\right)(t).$$

Differentiating both sides twice and using (0) with $\gamma = \alpha - 1$ we have

$$\left(D^{\alpha}_{a+}x\right)(t) = \frac{d^2}{dt^2}\left(D^{\alpha-2}_{a+}x\right)(t) = \frac{d^2}{dt^2}\left(I^{3-\alpha}(x')\right)(t) = (D^{\alpha-1}_{a+}x')(t).$$

Hence (1) and (2) lead to

$$-\left(D^{\alpha-1}_{a^+}x'\right)(t) = q(t)x, \quad x'(a) = x'(b) = 0.$$

Using Lemma 3.1 with $u = x'$ and $\beta = \alpha - 1$ we have

$$x'(t) = \int_a^b G_{\alpha-1}(t,s)q(s)x(s)ds,$$

Since $x(a) = 0$, it follows that

$$x(t) = \int_a^t \int_a^b G_{\alpha-1}(\tau,s)q(s)x(s)ds\,d\tau. \tag{5}$$

Denote $m = \max\{x(t) : \ t \in [a,b]\}$. Note that $G_{\alpha-1}(t,s) \geq 0$, $0 \leq x(t) \leq m$ and $x(t) \not\equiv m$, and $q(t) \leq q_+(t)$, we have

$$m < m\int_a^b \int_a^b G_{\alpha-1}(\tau,s)q_+(s)dsd\tau = m\int_a^b \Big(\int_a^b G_{\alpha-1}(\tau,s)d\tau\Big)q_+(s)ds. \tag{6}$$

canceling m from both sides and applying (4) with $\beta = \alpha - 1$, we obtain (7).

(ii) We assume $x(t)$ satisfies (3). By Rolle's theorem, there exists a $c \in (a,b)$ such that $x'(c) = 0$. Hence $x(t)$ satisfies BC (2) with b replaced by c. Now applying Part (i) we obtain

$$\int_a^b q_+(t)dt \geq \int_a^c q_+(t)dt > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(c-a)^{\alpha-1}} > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}.$$

□

Proof of Theorem 2.2. (i) We assume $x(t)$ satisfies (3) and without loss of generality, let that $x(t) \geq 0$ for $t \in [a,b]$. We note that $\left(D^{\alpha}_{a^+}x\right)(t) = \left(D^{\alpha-2}_{a^+}x\right)''(t)$. Hence (1) and (4) lead to

$$-\left(D^{\alpha-2}_{a^+}x\right)'' = q(t)x \quad \text{with} \quad \left(D^{\alpha-2}_{a^+}x\right)(a) = \left(D^{\alpha-2}_{a^+}x\right)(b) = 0. \tag{7}$$

Using Lemma 3.1 with $u = D^{\alpha-2}_{a^+}x$ and $\beta = 2$ we have

$$\left(D^{\alpha-2}_{a^+}x\right)(t) = \int_a^b G_2(t,s)q(s)x(s)ds, \tag{8}$$

where

$$G_2(t,s) = \frac{1}{b-a}\begin{cases}(s-a)(b-t), & a \leq s \leq t \leq b\\ (t-a)(b-s), & a \leq t \leq s \leq b\end{cases}$$

satisfying

$$0 \leq G_2(t,s) \leq G_2(s,s) = (s-a)(b-s)/(b-a). \tag{9}$$

It follows that

$$x(t) = \left(I^{\alpha-2}_{a^+}\left(D^{\alpha-2}_{a^+}x\right)\right)(t) + C(t-a)^{\alpha-3}$$

with $C \in \mathbb{R}$. Now $x(a) = 0$ implies $C = 0$ and thus

$$x(t) = \left(I^{\alpha-2}_{a^+}\left(D^{\alpha-2}_{a^+}x\right)\right)(t) = \frac{1}{\Gamma(\alpha-2)}\int_a^t (t-s)^{\alpha-3}\left(D^{\alpha-2}_{a^+}x\right)(s)ds.$$

From (8) it follows that

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha-2)}\int_a^t (t-s)^{\alpha-3}\int_a^b G_2(s,\tau)q(\tau)x(\tau)d\tau ds \\ &= \frac{1}{\Gamma(\alpha-2)}\int_a^t\int_a^b (t-s)^{\alpha-3}G_2(s,\tau)q(\tau)x(\tau)d\tau ds \\ &= \frac{1}{\Gamma(\alpha-2)}\int_a^b \left(\int_a^t (t-s)^{\alpha-3}G_2(s,\tau)ds\right)q(\tau)x(\tau)d\tau. \qquad (10) \end{aligned}$$

Denote $m = \max\{x(t):\ t\in[a,b]\}$. Note that $0 \le G_2(s,\tau)\le G_2(\tau,\tau)$, $0\le x(t)\le m$ and $x(t)\not\equiv m$, and $q(t)\le q_+(t)$, from (9) we have that for $t\in[a,b]$

$$m < \frac{m}{(b-a)\Gamma(\alpha-2)}\int_a^b\left(\int_a^t (t-s)^{\alpha-3}ds\right)(\tau-a)(b-\tau)q_+(\tau)d\tau. \qquad (11)$$

Note that

$$\int_a^t (t-s)^{\alpha-3}ds = \frac{(t-a)^{\alpha-2}}{\alpha-2}\le\frac{(b-a)^{\alpha-2}}{\alpha-2},\ t\in[a,b]$$

and

$$(\tau-a)(b-\tau)\le\frac{(b-a)^2}{4},\ \tau\in[a,b].$$

Then (8) follows from (11).

(ii) We assume $x(t)$ satisfies (5). Note that $0<\alpha-2\le 1$, by Lemma 3.3 with $\gamma=\alpha-2$, there exists a $c\in(a,b)$ such that $\left(D_{a^+}^{\alpha-2}x\right)(c)=0$. Hence $x(t)$ satisfies BC (4) with b replaced by c. Now applying Part (i) we obtain

$$\int_a^b q_+(t)dt \ge \int_a^c q_+(t)dt > \frac{4\Gamma(\alpha-1)}{(c-a)^{\alpha-1}} > \frac{4\Gamma(\alpha-1)}{(b-a)^{\alpha-1}}.$$

□

Proof of Theorem 2.3. Without loss of generality we assume that $x(t)\ge 0$ for $t\in[a,b]$. Rewrite Eq. (1) as

$$\left(D_{a^+}^{\alpha-1}x\right)'(t)+q(t)x=0.$$

Integrating both sides from ξ to t and using the fact that $\left(D_{a^+}^{\alpha-1}x\right)(\xi)=0$ we have

$$-\left(D_{a^+}^{\alpha-1}x\right)=\int_\xi^t q(\tau)x(\tau)d\tau.$$

By Lemma 3.1 with $u=x$ and $\beta=\alpha-1$ we see that

$$x(t)=\int_a^b G_{\alpha-1}(t,s)\int_\xi^s q(\tau)x(\tau)d\tau ds=\int_a^b\int_\xi^s G_{\alpha-1}(t,s)q(\tau)x(\tau)d\tau ds.$$

It follows that

$$\begin{aligned} x(t) &= \int_a^\xi\int_\xi^s G_{\alpha-1}(t,s)q(\tau)x(\tau)d\tau ds+\int_\xi^b\int_\xi^s G_{\alpha-1}(t,s)q(\tau)x(\tau)d\tau ds \\ &= \int_a^\xi\int_s^\xi G_{\alpha-1}(t,s)(-q(\tau))x(\tau)d\tau ds+\int_\xi^b\int_\xi^s G_{\alpha-1}(t,s)q(\tau)x(\tau)d\tau ds \\ &= \int_a^\xi\left(\int_a^\tau G_{\alpha-1}(t,s)ds\right)(-q(\tau))x(\tau)d\tau+\int_\xi^b\left(\int_\tau^b G_{\alpha-1}(t,s)ds\right)q(\tau)x(\tau)d\tau. \end{aligned}$$

Denote $m = \max\{x(t) : t \in [a,b]\}$. Note that $G_{\alpha-1}(t,s) \geq 0$, $0 \leq x(t) \leq m$ and $x(t) \not\equiv m$ on $[a,b]$, $-q(t) \leq q_-(t)$, and $q(t) \leq q_+(t)$. Hence, we have that for $t \in [a,b]$

$$m < m \int_a^b G_{\alpha-1}(t,s)ds \left(\int_a^\xi q_-(\tau)d\tau + \int_\xi^b q_+(\tau)d\tau \right). \tag{12}$$

Lemma 3.2 with $\beta = \alpha - 1$ lead to

$$\int_a^b G_{\alpha-1}(t,s)ds = \frac{1}{\Gamma(\alpha)} \max_{t\in[a,b]} (t-a)^{\alpha-2}(b-t) = \frac{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}.$$

Then (9) follows from (12). (10) follows from (9) immediately. □

4. Multivariate Lyapunov-type Inequalities

In the last section, we show how the Lyapunov-type inequalities in Section 2 can be extended to fractional multivariate equations. To avoid redundancy, we only give the extension of Theorem 2.3. To present our results, we introduce some notations.

For $N \geq 2$, we denote

$$S^{N-1} := \{u \in \mathbb{R}^N : |u| = 1\}$$

as the unit sphere in $\mathbb{R}^N$. It is well known that the surface area of S^{N-1} is

$$\int_{S^{N-1}} d\omega = \frac{2\pi^{N/2}}{\Gamma(N/2)},$$

where Γ stands for the Gamma function as given in Section 2. Note that every $u \in \mathbb{R}^N \setminus \{0\}$ has a unique representation of the form $u = r\omega$ with $|u| = r$ for some $r > 0$ and $\omega \in S^{N-1}$.

Assume that $a, b, \xi \in C(S^{N-1}, \mathbb{R})$ and $0 < a(\omega) \leq \xi(\omega) \leq b(\omega)$ for all $\omega \in S^{N-1}$. We define a doubly connected region A in $\mathbb{R}^N$ as

$$A := \{u = r\omega : r \in (a(\omega), b(\omega)), \omega \in S^{N-1}\},$$

together with its subregions

$$A_1 := \{u = r\omega : r \in (a(\omega), \xi(\omega)), \omega \in S^{N-1}\}$$

and

$$A_2 := \{u = r\omega : r \in (\xi(\omega), b(\omega)), \omega \in S^{N-1}\}.$$

Clearly, $A = A_1 \cup A_2$. Let the corresponding boundaries be denoted by

$$B_a = \{u = r\omega : r = a(\omega), \omega \in S^{N-1}\},$$
$$B_b = \{u = r\omega : r = b(\omega), \omega \in S^{N-1}\},$$

and

$$B_\xi = \{u = r\omega : r = \xi(\omega), \omega \in S^{N-1}\}.$$

The following gives a graphical interpretation for the region A.

Let $\omega \in S^{N-1}$ be fixed. For any $\gamma > 0$ and $u = r\omega$ with $r > a(\omega)$, we denote by $(\mathcal{D}_r^\gamma y)(u)$ the γ-th order Riemann-Liouville directional derivative of $y(u)$ in the radial direction at $a(\omega)$, i.e.,

$$\left(\mathcal{D}_r^\gamma y\right)(u) := \frac{1}{\Gamma(n-\gamma)} \frac{\partial^n}{\partial r^n} \int_{a(\omega)}^r (r-s)^{n-\gamma-1} y(s\omega)ds, \tag{0}$$

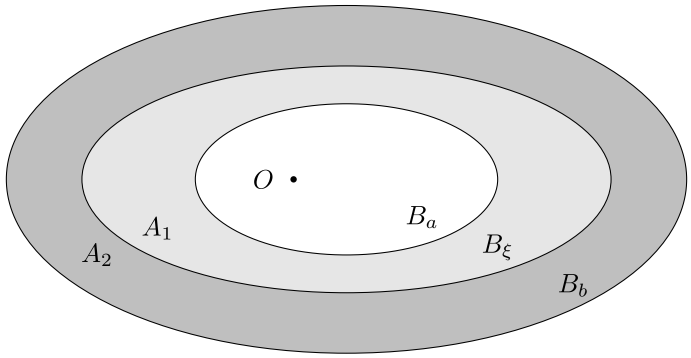

FIGURE 1. Region A

where n and Γ are given in Section 1. Now, on the region A, we consider the equation

$$\left(\mathcal{D}_r^\alpha y\right)(u) + q(u)y = 0, \quad 2 < \alpha \le 3, \tag{1}$$

where $q \in C(\overline{A})$, together with the BC

$$y(B_a) = y(B_b) = 0 \quad \text{and} \quad \left(\mathcal{D}_r^{\alpha-1} y\right)(B_\xi) = 0. \tag{2}$$

The following provides a Lyapunov-type inequality for BVP (1), (2).

Theorem 4.1. *Assume Eq. (1) has a nontrivial solution $y(u)$ satisfying (2) and $y(u)$ does not change sign on A . Then*

$$\int_{A_1} |u|^{1-N} q_-(u)du + \int_{A_2} |u|^{1-N} q_+(u)du > \frac{2\pi^{N/2}\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{\Gamma(N/2)(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}, \tag{3}$$

where $a = \min\{a(\omega) : \omega \in S^{N-1}\}$ and $b = \max\{b(\omega) : \omega \in S^{N-1}\}$. Consequently,

$$\int_A |u|^{1-N} |q(u)| du > \frac{2\pi^{N/2}\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{\Gamma(N/2)(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}. \tag{4}$$

Proof. For a fixed $\omega \in S^{N-1}$, we denote $z(r) := y(r\omega)$. By comparing (0) with (0) we see that $\left(\mathcal{D}_r^\gamma y\right)(u) = \left(D_{a(\omega)^+}^\gamma z\right)(r)$ for $\gamma > 0$. Since $y(u)$ is a nontrivial solution of Eq. (1) satisfying BC (2) and $y(u)$ does not change sign on A, we have that for $\omega \in S^{N-1}$, $z(r)$ is a nontrivial solution of the equation

$$\left(D_{a(\omega)^+}^\alpha z\right)(r) + q(r\omega)z = 0, \quad 2 < \alpha \le 3 \tag{5}$$

satisfying the BC

$$z(a(\omega)) = z(b(\omega)) = 0 \quad \text{and} \quad \left(D_{a(\omega)^+}^{\alpha-1} z\right)(\xi(\omega)) = 0, \tag{6}$$

and $z(r)$ does not change sign for $r \in [a(\omega), b(\omega)]$. Thus by Theorem 2.3,

$$\int_{a(\omega)}^{\xi(\omega)} q_-(r\omega)dr + \int_{\xi(\omega)}^{b(\omega)} q_+(r\omega)dr > \frac{(\alpha-1)^{\alpha-1}\Gamma(\alpha)}{(\alpha-2)^{\alpha-2}(b(\omega)-a(\omega))^{\alpha-1}}. \tag{7}$$

Recall that $r = |u|$ and for any $\Omega \subset \mathbb{R}^N$ and $f \in C(\Omega)$,

$$\int_\Omega f(u)du = \int_\Omega r^{N-1} f(r\omega)dr\, d\omega.$$

Hence

$$\int_{A_1} |u|^{1-N} q_-(u)du = \int_{S^{N-1}} \left(\int_{a(\omega)}^{\xi(\omega)} q_-(r\omega)dr \right) d\omega$$

and

$$\int_{A_2} |u|^{1-N} q_+(u)du = \int_{S^{N-1}} \left(\int_{\xi(\omega)}^{b(\omega)} q_+(r\omega)dr \right) d\omega.$$

Integrating both sides of (7) with respect to ω on S^{N-1} and noting that $0 < a \leq a(\omega) < b(\omega) \leq b$, we obtain

$$\begin{aligned}
& \int_{A_1} |u|^{1-N} q_-(u)du + \int_{A_2} |u|^{1-N} q_+(u)du \\
= & \int_{S^{N-1}} \left(\int_{a(\omega)}^{\xi(\omega)} q_-(r\omega)dr + \int_{\xi(\omega)}^{b(\omega)} q_+(r\omega)dr \right) d\omega \\
> & \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(\alpha-2)^{\alpha-2}} \int_{S^{N-1}} \frac{d\omega}{(b(\omega)-a(\omega))^{\alpha-1}} \geq \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}} \int_{S^{N-1}} d\omega \\
= & \frac{2\pi^{N/2}\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{\Gamma(N/2)(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}},
\end{aligned}$$

i.e., (3) holds. (4) follows from (3) immediately. □

Remark 4.1. (i) A comment similar to that in Remark 2.1 can be made to inequalities (3) and (4). We omit the detail.

(ii) Let $\alpha = 3$ and the region A be given as in Theorem 1.3 with $A_1 = B(0,\xi) \setminus \overline{B(0,a)}$ and $A_2 = B(0,b) \setminus \overline{B(0,\xi)}$. Then BVP (1), (2) becomes BVP (6), (7), and inequality (3) becomes

$$\int_{A_1} |u|^{1-N} q_-(u)du + \int_{A_2} |u|^{1-N} q_+(u)du > \frac{16\pi^{N/2}}{\Gamma(N/2)(b-a)^2}. \tag{8}$$

Since $|u| = r \geq a > 0$, it follows that

$$\int_{A_1} q_-(u)du + \int_{A_2} q_+(u)du > \frac{16\pi^{N/2}a^{N-1}}{\Gamma(N/2)(b-a)^2}. \tag{9}$$

We observe that even the weakened inequality (9) is sharper than (8) in Theorem 1.3. In fact, not only the $|q|$ on the left-hand side of (8) is replaced by q_- or q_+, but also the constant 8 on the right-hand side is strengthened to 16.

(iii) In addition to the assumptions in Part (ii), we let $a(\omega) = 0$ for $\omega \in S^{N-1}$, i.e., the region A is simply connected. In this case, (8) becomes

$$\int_{A_1} |u|^{1-N} q_-(u)du + \int_{A_2} |u|^{1-N} q_+(u)du > \frac{16\pi^{N/2}}{\Gamma(N/2)b^2},$$

but (8) provides no useful information.

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