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# FRACTIONAL VECTOR TAYLOR AND CAUCHY MEAN VALUE FORMULAS

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Abstract. By defining fractional integrals and fractional derivatives along directed line segments that correspond to multivariable, we derive fractional vector Taylor formulas and fractional vector Cauchy mean value formulas in the sense of the Riemann-Liouville fractional derivative, the Caputo fractional derivative, and the sequential fractional derivative, respectively. These formulas can be reduced to some well-known results and classical formulas in calculus. Several examples are given to verify and illustrate the Taylor and Cauchy mean value formulas as well as test the applicability of the new directional fractional derivatives to the solution of fractional differential equations.

## 1. Introduction

Fractional vector calculus, which includes fractional differential operators (gradient, divergence, curl), fractional integral operations (flux, circulation), and fractional formulas of Taylor, Green, Gauss, Stokes, etc., as well as fractional calculus, fractionnal differential equations and fractional variational principle, has attracted great attentions (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] for example). This is because it is an important tool for describing processes in complex media, non-local materials and distributed systems in multi-dimensional space [8, 11, 12] and for solving fractional partial differential equations (see [6, 10, 13] for example). For this reason, in this study we focus on the development of the fractional Taylor formula and the fractional Cauchy mean value formula for multivariable functions in the sense of the Riemann-Liouville fractional order derivative, the Caputo fractional order derivative, and the sequential fractional order derivative, respectively, where the fractional order $\alpha$ is in $0 < \alpha \leq 1$. In particular, the Caputo fractional derivative is usually convenient for dealing with the initial conditions which are given in terms of the field variables and their integer orders in most physical processes [16, 17].

It is noted that the fractional Taylor formulas for the single variable case have been obtained by many researchers, such as [18, 19, 20, 21, 22, 23, 24, 25, 26], in

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various expressions for solving fractional order differential equations. Among them, Trujillo *et al.* [21] derived a fractional Taylor formula with a remainder in the form of fractional order derivative as

$$f(x)=\sum_{j=0}^{n}\frac{c_j}{\Gamma((j+1)\alpha)}(x-a)^{(j+1)\alpha-1}+\frac{1}{\Gamma((n+1)\alpha)}\int_a^x(x-t)^{(n+1)\alpha-1}D_a^{(n+1)\alpha}f(t)dt, \tag{1}$$

and a Taylor mean value expression

$$f(x)=\sum_{j=0}^{n}\frac{c_j}{\Gamma((j+1)\alpha)}(x-a)^{(j+1)\alpha-1}+\frac{(D_a^{(n+1)\alpha}f)(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha}, \tag{2}$$

as well as a Cauchy mean value equation

$$\frac{f(x)-\sum\limits_{j=0}^{n}\frac{c_j}{\Gamma((j+1)\alpha)}(x-a)^{(j+1)\alpha-1}}{g(x)-\sum\limits_{j=0}^{n}\frac{d_j}{\Gamma((j+1)\alpha)}(x-a)^{(j+1)\alpha-1}}=\frac{(D_a^{(n+1)\alpha}f)(\xi)}{(D_a^{(n+1)\alpha}g)(\xi)}. \tag{3}$$

Here, $a\le\xi\le x$, $D_a^{(n+1)\alpha}g(x)\neq0$, and $0<\alpha\le1$, $c_j=\Gamma(a)[(x-a)^{1-\alpha}D_a^{j\alpha}f](a+)$, $d_j=\Gamma(a)[(x-a)^{1-\alpha}D_a^{j\alpha}g](a+)$, $j=0,1,...,n$, where $a+$ denotes the limit obtained based on $x$ approaching to $a$ from the right-hand-side of $a$, and $D_a^{n\alpha}$ denotes the sequential fractional Riemann-Liouville derivative that is defined as $D_a^{n\alpha}=D_a^{\alpha}\cdots D_a^{\alpha}$ repeating $n$ times.

Odibat and Shawagfeh [22] obtained another fractional Taylor formula with a remainder in the form of fractional order derivative

$$f(x)=\sum_{j=0}^{n}\frac{(x-a)^{j\alpha}}{\Gamma(j\alpha+1)}({}^CD_a^{j\alpha}f)(a)+\frac{1}{\Gamma((n+1)\alpha)}\int_a^x(x-t)^{(n+1)\alpha-1}[{}^CD_a^{(n+1)\alpha}f](t)dt, \tag{4}$$

and

$$f(x)=\sum_{j=0}^{n}\frac{(x-a)^{j\alpha}}{\Gamma(j\alpha+1)}({}^CD_a^{j\alpha}f)(a)+\frac{({}^CD_a^{(n+1)\alpha}f)(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha}, \tag{5}$$

as well as a Cauchy mean value equation

$$\frac{f(x)-\sum\limits_{j=0}^{n}\frac{(x-a)^{j\alpha}}{\Gamma(j\alpha+1)}({}^CD_a^{j\alpha}f)(a)}{g(x)-\sum\limits_{j=0}^{n}\frac{(x-a)^{j\alpha}}{\Gamma(j\alpha+1)}({}^CD_a^{j\alpha}g)(a)}=\frac{({}^CD_a^{(n+1)\alpha}f)(\xi)}{({}^CD_a^{(n+1)\alpha}g)(\xi)}, \tag{6}$$

where $a\le\xi\le x$, $D_a^{(n+1)\alpha}g(x)\neq0$, and ${}^CD_a^{j\alpha}$ denotes the sequential fractional Caputo derivative that is defined as ${}^CD_a^{j\alpha}={}^CD_a^{\alpha}\cdots{}^CD_a^{\alpha}$ repeating $j$ times.

In vector calculus, the classical Taylor formula can be expressed as

$$f(\mathbf{x}+\mathbf{h})-f(\mathbf{x})=\sum_{k=1}^{m-1}\frac{1}{k!}(h_1\partial_1+\cdots+h_n\partial_n)^kf(\mathbf{x})+r_{m-1}(\mathbf{x},\mathbf{h}), \tag{7}$$

where $\mathbf{x}=(x_1,\ldots,x_n),\mathbf{x}+\boldsymbol{h}=(x_1+h_1,\ldots,x_n+h_n)$, and

$$r_{m-1}(\mathbf{x},\mathbf{h})=\frac{1}{(m-1)!}\int_0^1(1-t)^{m-1}(h_1\partial_1+\cdots+h_n\partial_n)^mf(\mathbf{x}+t\mathbf{h})dt. \tag{8}$$

Here, $\partial_i$ is the partial derivative operator corresponding to the $i$th variable of $f(\mathbf{x})$. Furthermore, when $\lim_{m\to\infty} r_{m-1}(\mathbf{x},\mathbf{h}) = 0$, the formula becomes

$$f(\mathbf{x}+\mathbf{h}) = \sum_{k=0}^{\infty} \frac{1}{k!}(h_1\partial_1 + \cdots + h_n\partial_n)^k f(\mathbf{x}). \tag{9}$$

To our best knowledge, only few references [27, 28, 29] have generalized the multivariate Taylor formula to the multivariable fractional case.

Jumarie [27] presented a fractional Taylor formula with two variables as

$$\begin{aligned} f(x_1+h_1, x_2+h_2) &= E_\alpha(h_1^\alpha \partial_1^\alpha) E_\alpha(h_2^\alpha \partial_2^\alpha) f(x_1,x_2) \\ &= E_\alpha({h_2}^\alpha \partial_2^\alpha) E_\alpha(h_1^\alpha \partial_1^\alpha) f(x_1,x_2) \\ &= E_\alpha[(h_1\partial_1 + h_2\partial_2)^\alpha] f(x_1,x_2), \end{aligned} \tag{10}$$

where $\partial_1^\alpha$ and $\partial_2^\alpha$ are Riemann-Liouville fractional partial derivatives of order $\alpha$ with respect to the first and second variables of function $f(x_1, x_2)$, respectively, and $E_\alpha(x)$ denotes the Mittage-Leffler function given by the expression [15, 17, 30, 31, 32, 33, 34]:

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+1)}, \tag{11}$$

implying

$$f(x_1+h_1, x_2+h_2) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} (h_1\partial_1 + h_2\partial_2)^{\alpha k} f(x_1,x_2). \tag{12}$$

However, Eq. (10) seems not quite right since $E_\alpha[(u+v)^\alpha] \neq E_\alpha(u^\alpha)E_\alpha(v^\alpha)$ except $\alpha = 1$. This may lead to Eq. (12) to be correct only when $\alpha = 1$.

Furthermore, Anastassiou [28, 29] obtained a fractional vector Taylor formula with the remainder consisting of two Riemann-Liouville fractional integrals as follows:

$$\begin{aligned} f(\mathbf{y}) = f(\mathbf{x}) &+ \sum_{i=1}^{n} [y_i - x_i] \frac{\partial f(\mathbf{x})}{\partial x_i} + \sum_{l=2}^{m-1} \frac{1}{l!} \left( \sum_{i=1}^{n} \{[y_i - x_i] \frac{\partial}{\partial x_i}\}^l f \right)(\mathbf{x}) \\ &+ \frac{1}{\Gamma(\gamma)} \int_0^1 (1-t)^{\gamma-1} D_0^{-(m-\gamma)} \left( \sum_{i=1}^{n} \{[y_i - x_i] \frac{\partial}{\partial x_i}\}^m f \right)(\mathbf{x} + t(\mathbf{y}-\mathbf{x}))dt. \end{aligned} \tag{13}$$

From our view, Eq. (13) is the first correct and significant generalization of the classical multivariate Taylor formula. The only drawback seems that the expression is not concise as desired and its proof is a little tediou, and in addition, Eq. (13) was given based on the Caputo fractional derivative, and on the other hand, Cauchy mean value formulas were not given there. Thus, simplifying the expression in Eq. (13) and further obtaining multivariate fractional Taylor formulas based on other definitions such as the Riemann-Liouville derivative and the sequential fractional derivative, as well as obtaining multivariate fractional Cauchy mean value formulas are interesting and could be useful for the reason aforementioned. For this purpose, in our manuscript, the novelty idea is that we firstly define fractional integrals and fractional derivatives along directed line segments which correspond to multivariable and then we find that it is very convenient to use them to derive fractional vector Taylor formulas and fractional vector Cauchy mean value formulas in the sense of the Riemann-Liouville fractional order derivative, the Caputo fractional order derivative, and the sequential fractional order derivative, respectively, where the fractional order $\alpha$ is in $0 < \alpha \leq 1$. Several examples are given to verify and illustrate Taylor and Cauchy mean value formulas as well as test the applicability of the new directional fractional derivatives to the solution of fractional differential equations.

## 2. Definitions and Lemmas

We start with some fundamental fractional definitions. For the detailed information on them, we refer the readers to these articles and books in the literature [3, 5, 15, 17, 21, 29, 30, 31, 32, 33, 34].

**Definition 2.1** [21, 22] A function $\varphi(s)(s \geq 0)$ is said to be in the space $C_v(\nu \in R)$ if it can be written as $\varphi(s) = s^p \varphi_1(s)$ for $p > \nu$ where $\varphi_1(s)$ is continuous in $[0, \infty)$, and it is said to be in the space $C_\nu^{(m)}$ if $\nu^{(m)} \in C_\nu, m \in N$.

**Definition 2.2** [17, 32, 33] Assume $\varphi(s) \in C_\nu(a, \infty)$. The Riemann-Liouville integral operator of order $\nu > 0$ is defined as

$$(D_a^{-\nu}\varphi)(s) = \frac{1}{\Gamma(\nu)} \int_a^s (s-t)^{\nu-1}\varphi(t)dt, \quad s > a. \tag{14}$$

**Definition 2.3** [17, 32, 33] The Riemann-Liouville fractional derivative of $\varphi(s)$ of order $\mu > 0$ is defined as

$$(D_a^\mu \varphi)(s) = \frac{d^m}{ds^m}(D_a^{\mu-m}\varphi)(s) = \frac{d^m}{ds^m}[\frac{1}{\Gamma(m-\mu)} \int_a^s \frac{\varphi(t)}{(x-t)^{\mu+1-m}} dt] \tag{15}$$

for $m - 1 < \mu \leq m, m \in N, s \geq a$.

**Definition 2.4** [17, 32, 33] The Caputo fractional derivative of $\varphi(s)$ of order $\mu > 0$ is defined as

$$({}^C D_a^\mu \varphi)(s) = (D_a^{\mu-m}\varphi^{(m)})(s) = \frac{1}{\Gamma(m-\mu)} \int_a^s \frac{\varphi^{(m)}(t)}{(x-t)^{\mu+1-m}} dt \tag{16}$$

for $m - 1 < \mu \leq m, m \in N, s \geq a$.

The fractional Taylor formulas, Taylor mean value theorems, and Cauchy mean value theorems in one variable have been developed by many researchers in various expressions. Here, we list some important results for reference and comparison later.

**Theorem 2.1** [32, 33] **(Univariate fractional Taylor theorem with Riemann-Liouville derivative).** Assume $\nu > 0$, $m$ is the smallest integer exceeding $\nu$, and $\varphi \in C^k([a, b])$. Then

$$\begin{aligned}
\varphi(b) &= \sum_{j=1}^{m} \frac{D_a^{\nu-j}\varphi(a)}{\Gamma(\nu-j+1)}(b-a)^{\nu-j} + D_a^{-\nu}[D_a^\nu \varphi(t)] \\
&= \sum_{k=0}^{m} \frac{D_a^{\nu-j}\varphi(a)}{\Gamma(\nu-j+1)}(b-a)^{\nu-j} + \frac{1}{\Gamma(\nu)} \int_a^b (b-t)^{\nu-1} D_0^\nu \varphi(t) dt. \qquad (17)
\end{aligned}$$

**Theorem 2.2** [32, 33] **(Univariate fractional Taylor theorem with Caputo derivative).** Assume $\nu > 0$, $m$ is the smallest integer exceeding $\nu$, and $\varphi \in$

$C^k([a,b])$. Then

$$\begin{aligned}\varphi(b) &= \sum_{k=0}^{m-1} \frac{\varphi^{(k)}(a)}{k!}(b-a)^k + D_a^{-\nu}[{}^C D_a^\nu \varphi(t)] \\ &= \sum_{k=0}^{m-1} \frac{\varphi^{(k)}(a)}{k!}(b-a)^k \frac{1}{\Gamma(\nu)} \int_0^s (s-t)^{\nu-1}[{}^C D_0^\nu \varphi(t)]dt. \end{aligned} \tag{18}$$

Using the classical Cauchy integral mean value theorem and the Lagrange mean value theorem together with Eq. (17), one may easily obtain the following corollary.

**Corollary 2.1** [32, 33] **(Univariate fractional Taylor and Cauchy mean formulas with Riemann-Liouville derivative).** Assume $\nu > 0$ and $m$ is the smallest integer exceeding $\nu$. Then, there exists $\xi \in (a,b)$ such that

$$\varphi(b) = \sum_{j=1}^{m} \frac{D_a^{\nu-j}\varphi(a)}{\Gamma(\nu-j+1)}(b-a)^{\nu-j} + \frac{D_0^\nu \varphi(\xi)}{\Gamma(\nu+1)}(b-a)^\nu, \tag{19}$$

and

$$\frac{\varphi(b) - \sum\limits_{j=1}^{m} \frac{D_a^{\nu-j}\varphi(a)}{\Gamma(\nu-j+1)}(b-a)^{\nu-j}}{\psi(b) - \sum\limits_{j=1}^{m} \frac{D_a^{\nu-j}\psi(a)}{\Gamma(\nu-j+1)}(b-a)^{\nu-j}} = \frac{D_0^\nu \varphi(\xi)}{D_0^\nu \psi(\xi)}, \tag{20}$$

where ${}^C D_0^\nu \psi(x) \neq 0$.

**Corollary 2.2** [32, 33] **(Univariate fractional Taylor and Cauchy mean formulas with Caputo derivative).** Assume $\nu > 0$ and $m$ is the smallest integer exceeding $\nu$. Then, there exists $\xi \in (a,b)$ such that

$$\varphi(b) = \sum_{k=0}^{m-1} \frac{\varphi^{(k)}(a)}{k!}(b-a)^k + \frac{{}^C D_0^\nu \varphi(\xi)}{\Gamma(\nu+1)}(b-a)^\nu, \tag{21}$$

and

$$\frac{\varphi(b) - \sum\limits_{k=0}^{m-1} \frac{\varphi^{(k)}(a)}{k!}(b-a)^k}{\psi(b) - \sum\limits_{k=0}^{m-1} \frac{\psi^{(k)}(a)}{k!}(b-a)^k} = \frac{{}^C D_0^\nu \varphi(\xi)}{{}^C D_0^\nu \psi(\xi)}, \tag{22}$$

where ${}^C D_0^\nu \psi(x) \neq 0$.

## 3. Fractional Vector Taylor and Cauchy formulas

In order to derive fractional vector Taylor formulas and Cauchy mean value formulas, we first give some related definitions about fractional integral and derivative corresponding to multivariable.

Consider two points $P_0 : \mathbf{x} = (x_1, ..., x_n)$ and $P_1 : \mathbf{y} = (y_1, ..., y_n)$ in $\Omega$, where $\Omega \subset R^n$ is a compact and convex domain. Let $\mathbf{h} = \mathbf{y} - \mathbf{x} = (h_1, ..., h_n)$. For any $0 \leq s \leq 1$, denote $P_s : \mathbf{z} = \mathbf{x} + s\mathbf{h}$, and assume $f(\mathbf{z}) \in C(\Omega)$. It can be seen that every point $P_s$ is in the directed line segment $\mathbf{P}_0\mathbf{P}_1$. Define

$$\varphi(s) = f(\mathbf{x} + s\mathbf{h}), \quad 0 \leq s \leq 1. \tag{23}$$

Then, for $\nu > 0$, Definition 2.2 gives

$$\begin{aligned}(D_0^{-\nu}\varphi)(s) &= \frac{1}{\Gamma(\nu)}\int_0^s (s-t)^{\nu-1}\varphi(t)dt \\ &= \frac{1}{\Gamma(\nu)}\int_0^s (s-t)^{\nu-1} f(\mathbf{x}+t\mathbf{h})dt. \qquad (24)\end{aligned}$$

From Eq. (24), we can define the Riemann-Liouville fractional integral of $f(\mathbf{z})$ of order $\nu > 0$ along the directed line segment $\mathbf{P}_0\mathbf{P}_s$.

**Definition 3.1.** For fixed points $P_0 : \mathbf{x} = (x_1, ..., x_n)$, $P_1 : \mathbf{y} = (y_1, ..., y_n)$ in $\Omega$ and for any point $P_s : \mathbf{z} = \mathbf{x} + s\mathbf{h}$,assume $f(\mathbf{z}) \in C(\Omega)$. Define the Riemann-Liouville fractional integral of $f(\mathbf{z})$ of order $\nu$ along the directed line segment $\mathbf{P}_0\mathbf{P}_s$ as

$$(D_{\mathbf{x}}^{-\nu} f)(\mathbf{x}+s\mathbf{h}) = \frac{1}{\Gamma(\nu)}\int_0^s (s-t)^{\nu-1} f(\mathbf{x}+t\mathbf{h})dt, \qquad (25)$$

where $\mathbf{h} = \mathbf{y} - \mathbf{x}$ and $0 \le s \le 1$. In particular, when $s = 1$, it gives

$$(D_{\mathbf{x}}^{-\nu} f)(\mathbf{y}) = \frac{1}{\Gamma(\nu)}\int_0^1 (1-t)^{\nu-1} f(\mathbf{x}+t\mathbf{h})dt. \qquad (26)$$

**Remark 3.1.** If $h_i = y_i - x_i \ne 0, h_1 = \cdots = h_{i-1} = h_{i+1} = \cdots = h_n = 0$, then Eq. (26) can be reduced to

$$\begin{aligned}(D_{\mathbf{x}}^{-\nu} f)(\mathbf{y}) &= \frac{1}{\Gamma(\nu)}\int_0^1 (1-t)^{\nu-1} f(x_1, ..., x_{i-1}, x_i + t(y_i - x_i), x_{i+1},..., x_n)dt \\ &= (y_i - x_i)^{-\nu}\frac{1}{\Gamma(\nu)}\int_{x_i}^{y_i} (y-\tau)^{\nu-1} f(x_1, ..., x_{i-1}, \tau, x_{i+1}, ..., x_n)d\tau \\ &= (y_i - x_i)^{-\nu}[D_{x_i}^{-\nu} f](x_1, ..., x_{i-1}, y_i, x_{i+1},..., x_n).\end{aligned}$$

From Eqs. (23)-(26), one may obtain the following lemma.

**Lemma 3.1.** For fixed vectors $\mathbf{x}, \mathbf{y} \in \Omega \subset R^n$ and any vector $\mathbf{z} = \mathbf{x} + s\mathbf{h}, (0 \le s \le 1)$, assume $f(\mathbf{z}) \in C(\Omega)$. Then it holds

$$(D_{\mathbf{x}}^{-\nu} f)(\mathbf{x}+s\mathbf{h}) = (D_0^{-v}\varphi)(s), \quad 0 \le s \le 1, \qquad (27)$$

and

$$(D_{\mathbf{x}}^{-\nu} f)(\mathbf{y}) = (D_0^{-v}\varphi)(1), \quad (D_{\mathbf{x}}^{-\nu} f)(\mathbf{x}) = (D_0^{-v}\varphi)(0). \qquad (28)$$

**Lemma 3.2.** For $\alpha, \beta > 0$ and fixed vectors $\mathbf{x}, \mathbf{y} \in \Omega \subset R^n$, and for any vector $\mathbf{z} = \mathbf{x} + s\mathbf{h}, (0 \le s \le 1)$, assume $f(\mathbf{z}) \in C(\Omega)$. Then it holds

$$D_{\mathbf{x}}^{-\alpha} D_{\mathbf{x}}^{-\beta} f(\mathbf{x} + s\mathbf{h}) = D_{\mathbf{x}}^{-(\alpha+\beta)} f(\mathbf{x} + s\mathbf{h}), \qquad (29)$$

and

$$D_{\mathbf{x}}^{-\alpha} D_{\mathbf{x}}^{-\beta} f(\mathbf{y}) = D_{\mathbf{x}}^{-(\alpha+\beta)} f(\mathbf{y}). \qquad (30)$$

**Proof.** By Definition 3.1 and Lemma 3.1**,** we obtain

$$\begin{aligned} D_{\mathbf{x}}^{-\alpha} D_{\mathbf{x}}^{-\beta} f(\mathbf{x}+s\mathbf{h}) &= \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} D_{\mathbf{x}}^{-\beta} f(\mathbf{x}+t\mathbf{h}) dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} D_0^{-\beta} \varphi(t) dt \\ &= D_0^{-\alpha} D_0^{-\beta} \varphi(s) \\ &= D_0^{-(\alpha+\beta)} \varphi(s) \\ &= D_{\mathbf{x}}^{-(\alpha+\beta)} f(\mathbf{x}+s\mathbf{h}). \end{aligned}$$

When $s=1$,the above equation becomes

$$D_{\mathbf{x}}^{-\alpha} D_{\mathbf{x}}^{-\beta} f(\mathbf{y}) = D_{\mathbf{x}}^{-(\alpha+\beta)} f(\mathbf{y}),$$

which completes the proof.

Again, from $\varphi(s) = f(\mathbf{x}+s\mathbf{h}), 0 \le s \le 1$,we may obtain the derivative of $\varphi(s)$ of order $k \in N$ as

$$\varphi^{(k)}(s) = (h_1\partial_1 + \cdots + h_n\partial_n)^k f(\mathbf{x}+s\mathbf{h}), \tag{31}$$

and by Definition 2.2 and Definition 2.3, we further obtain the Riemann-Liouville fractional derivative of order $\mu \in R^+$ and the Caputo fractional derivative of order $\mu \in R^+$ for $\varphi(s)$ as

$$D_0^{\mu}\varphi(s) = D^k[D_0^{\mu-k}\varphi(s)], \tag{32}$$

$${}^C D_0^{\mu}\varphi(s) = D_0^{\mu-k}\varphi^{(k)}(s)], \tag{33}$$

where $k$ is the smallest integer exceeding $\mu$. Based on Eq. (31), we define

$$D^k f(\mathbf{x}+s\mathbf{h}) = (h_1\partial_1 + \cdots + h_n\partial_n)^k f(\mathbf{x}+s\mathbf{h})$$

and

$$D^k f(\mathbf{y}) = (h_1\partial_1 + \cdots + h_n\partial_n)^k f(\mathbf{y}),$$

when $s=1$.

We now define directional Riemann-Liouville and Caputo fractional derivatives of order $\mu$ for $f(\mathbf{z})$ along the directed line segment $\mathbf{P}_0\mathbf{P}_s$.

**Definition 3.2.** Assume $\mu > 0$ and $k$ is the smallest integer exceeding $\mu$. For fixed points $P_0 : \mathbf{x} = (x_1, ..., x_n), P_1 : \mathbf{y} = (y_1, ..., y_n) \in \Omega \subset R^n$, and any point $P_s : \mathbf{z} = \mathbf{x}+s\mathbf{h}, (0 \le s \le 1)$, define the directional Riemann-Liouville fractional derivative of order $\mu$ for $f(\mathbf{z})$ along the directed line segment $\mathbf{P}_0\mathbf{P}_s$ as follows:

$$(D_{\mathbf{x}}^{\mu} f)(\mathbf{x}+s\mathbf{h}) = (D^k D_{\mathbf{x}}^{\mu-k} f)(\mathbf{x}+s\mathbf{h}). \tag{34}$$

In particular, when $s=1$, it gives

$$(D_{\mathbf{x}}^{\mu} f)(\mathbf{y}) = (D^k D_{\mathbf{x}}^{\mu-k} f)(\mathbf{y}). \tag{35}$$

**Definition 3.3.** Assume $\mu > 0$ and $k$ is the smallest integer exceeding $\mu$. For fixed points $P_0 : \mathbf{x} = (x_1, ..., x_n), P_1 : \mathbf{y} = (y_1, ..., y_n) \in \Omega \subset R^n$ and any point $P_s : \mathbf{z} = \mathbf{x}+s\mathbf{h}$, define the directional Caputo fractional derivative of $f(\mathbf{z})$ of order $\mu$ along the directed line segment $\mathbf{P}_0\mathbf{P}_s$ as follows:

$$({}^C D_{\mathbf{x}}^{\mu} f)(\mathbf{x}+s\mathbf{h}) = (D_{\mathbf{x}}^{\mu-k} D^k f)(\mathbf{x}+s\mathbf{h}). \tag{36}$$

In particular, when $s=1$, it gives

$$({}^C D_{\mathbf{x}}^{\mu} f)(\mathbf{y}) = (D_{\mathbf{x}}^{\mu-k} D^k f)(\mathbf{y}). \tag{37}$$

**Definition 3.4.** Let $0 \le \alpha \le 1$ and $n \in N$. $D_{\mathbf{x}}^{n\alpha} f(\mathbf{x}+s\mathbf{h})$ denotes the sequential directional fractional Riemann-Liouville derivative along the directed line segment $\mathbf{P}_0\mathbf{P}_s$, which is defined as

$$D_{\mathbf{x}}^{n\alpha} f(\mathbf{x}+s\mathbf{h}) = D_0^{\alpha} \cdots D_0^{\alpha} \cdot D_{\mathbf{x}}^{\alpha} f(\mathbf{x}+s\mathbf{h}), \tag{38}$$

where $D_0^{\alpha}$ is given in Eq. (32) and $D_0^{\alpha} \cdots D_0^{\alpha}$ indicates repeating $D_0^{\alpha}$ $(n-1)$ times. In particular, when $s = 1$, it gives

$$D_{\mathbf{x}}^{n\alpha} f(\mathbf{y}) = D_0^{\alpha} \cdots D_0^{\alpha} \cdot D_{\mathbf{x}}^{\alpha} f(\mathbf{y}).$$

On the other hand, ${}^{C}D_{\mathbf{x}}^{n\alpha} f(\mathbf{x}+s\mathbf{h})$ denotes the sequential directional fractional Caputo derivative along the directed line segment $\mathbf{P}_0\mathbf{P}_s$, which is defined as

$${}^{C}D_{\mathbf{x}}^{n\alpha} f(\mathbf{x}+s\mathbf{h}) = ({}^{C}D_0^{\alpha}) \cdots ({}^{C}D_0^{\alpha}) \cdot ({}^{C}D_{\mathbf{x}}^{\alpha} f)(\mathbf{x}+s\mathbf{h}), \tag{39}$$

where ${}^{C}D_0^{\alpha}$ is given in Eq. (33) and $({}^{C}D_0^{\alpha}) \cdots ({}^{C}D_0^{\alpha})$ indicates repeating ${}^{C}D_0^{\alpha}$ $(n-1)$ times. When $s = 1$, it gives

$${}^{C}D_{\mathbf{x}}^{n\alpha} f(\mathbf{y}) = ({}^{C}D_0^{\alpha}) \cdots ({}^{C}D_0^{\alpha}) \cdot ({}^{C}D_{\mathbf{x}}^{\alpha} f)(\mathbf{y}).$$

**Lemma 3.3.** Assume $\mu > 0$ and $k$ is the smallest integer exceeding $\mu$. For fixed vectors $\mathbf{x}, \mathbf{y} \in \Omega$ and any vector $\mathbf{z} = \mathbf{x}+s\mathbf{h}, (0 \le s \le 1)$, it holds

$$D^k f(\mathbf{x}+s\mathbf{h}) = \varphi^{(k)}(s), \tag{40}$$

$$D^k f(\mathbf{y}) = \varphi^{(k)}(1), \quad D^k f(\mathbf{x}) = \varphi^{(k)}(0). \tag{41}$$

For the directional Riemann-Liouville fractional derivative, it holds

$$(D_{\mathbf{x}}^{\mu} f)(\mathbf{x}+s\mathbf{h}) = (D_0^{\mu}\varphi)(s), \tag{42}$$

$$(D_{\mathbf{x}}^{\mu} f)(\mathbf{y}) = (D_0^{\mu}\varphi)(1), \qquad (D_{\mathbf{x}}^{\mu} f)(\mathbf{x}) = (D_0^{\mu}\varphi)(0). \tag{43}$$

For the directional Caputo fractional derivative, it holds that

$$({}^{C}D_{\mathbf{x}}^{\mu} f)(\mathbf{x}+s\mathbf{h}) = ({}^{C}D_0^{\mu}\varphi)(s), \tag{44}$$

$$({}^{C}D_{\mathbf{x}}^{\mu} f)(\mathbf{y}) = ({}^{C}D_0^{\mu}\varphi)(1), \qquad ({}^{C}D_{\mathbf{x}}^{\mu} f)(\mathbf{x}) = ({}^{C}D_0^{\mu}\varphi)(0). \tag{45}$$

**Proof**. Eqs. (40) and (41) can be easily obtained based on the definition of $D^k f(\mathbf{x}+s\mathbf{h})$. By Definition 3.2 and Lemma 3.1, we have

$$\begin{aligned}(D_{\mathbf{x}}^{\mu} f)(\mathbf{x}+s\mathbf{h}) &= (D^k D_{\mathbf{x}}^{\mu-k} f)(\mathbf{x}+s\mathbf{h}) \\ &= D^k[D_{\mathbf{x}}^{\mu-k} f(\mathbf{x}+s\mathbf{h})] \\ &= D^k D_0^{\mu-k}\varphi(s) \\ &= D_0^{\mu}\varphi(s),\end{aligned}$$

implying that

$$(D_{\mathbf{x}}^{\mu} f)(\mathbf{y}) = (D_0^{\mu}\varphi)(1), \qquad (D_{\mathbf{x}}^{\mu} f)(\mathbf{x}) = (D_0^{\mu}\varphi)(0).$$

On the other hand, by Definition 3.1 and Definition 3.3, we obtain

$$\begin{aligned}({}^C D_{\mathbf{x}}^{\mu} f)(\mathbf{x}+s\mathbf{h}) &= (D_{\mathbf{x}}^{\mu-k} D^k f)(\mathbf{x}+s\mathbf{h}) \\ &= D_{\mathbf{x}}^{\mu-k}[D^k f(\mathbf{x}+s\mathbf{h})] \\ &= \frac{1}{\Gamma(k-\mu)}\int_0^s (s-t)^{k-\mu-1} D^k f(\mathbf{x}+t\mathbf{h})dt \\ &= \frac{1}{\Gamma(k-\mu)}\int_0^s (s-t)^{k-\mu-1} \varphi^{(k)}(t)dt \\ &= ({}^C D^{\mu}\varphi)(s),\end{aligned}$$

implying that

$$({}^C D_{\mathbf{x}}^{\mu} f)(\mathbf{y}) = ({}^C D_0^{\mu}\varphi)(1), \quad ({}^C D_{\mathbf{x}}^{\mu} f)(\mathbf{x}) = ({}^C D_0^{\mu}\varphi)(0).$$

By Lemma 3.3, the following lemma can be obtained.

**Lemma 3.4.** For $\alpha > 0$ and fixed vectors $\mathbf{x}, \mathbf{y} \in \Omega \subset R^n$, denote $\mathbf{h} = \mathbf{y} - \mathbf{x}$. For any vector $\mathbf{z} = \mathbf{x} + s\mathbf{h}, (0 \le s \le 1)$,it holds

$$D_{\mathbf{x}}^{n\alpha} f(\mathbf{x}+s\mathbf{h}) = D_0^{n\alpha}\varphi(s), \quad {}^C D_{\mathbf{x}}^{n\alpha} f(\mathbf{x}+s\mathbf{h}) = ({}^C D_0^{n\alpha}\varphi)(s), \quad 0 \le s \le 1,$$

and

$$\begin{aligned}D_{\mathbf{x}}^{n\alpha} f(\mathbf{y}) = D_0^{n\alpha}\varphi(1), &\quad D_{\mathbf{x}}^{n\alpha} f(\mathbf{x}) = D_0^{n\alpha}\varphi(0), \\ {}^C D_{\mathbf{x}}^{n\alpha} f(\mathbf{y}) = ({}^C D_0^{n\alpha}\varphi)(1), &\quad {}^C D_{\mathbf{x}}^{n\alpha} f(\mathbf{x}) = ({}^C D_0^{n\alpha}\varphi)(0),\end{aligned}$$

where $D_0^{n\alpha}\varphi(s)$ and $({}^C D_0^{n\alpha}\varphi)(s)$ denote the sequential fractional Riemann-Liouville and Caputo derivatives, respectively.

**Remark 3.2.** If $h_i = y_i - x_i \neq 0, h_1 = \cdots = h_{i-1} = h_{i+1} = \cdots = h_n = 0$, then

$$\varphi(s) = f(x_1, ..., x_{i-1}, x_i + sh, x_{i+1}, ..., x_n), \quad 0 \le s \le 1,$$

and

$$\begin{aligned}\varphi^{(n)}(s) &= h^n \partial_i^n f(x_1, ..., x_{i-1}, x_i + sh, x_{i+1}, ..., x_n) \\ &= (y_i - x_i)^n \partial_i^n f(x_1, ..., x_{i-1}, x_i + sh, x_{i+1}, ..., x_n).\end{aligned}$$

Thus, along this special directed line segment, the Riemann-Liouville fractional derivative in Eq. (35) becomes

$$\begin{aligned}(D_{\mathbf{x}}^{\mu} f)(\mathbf{y}) &= (h_i\partial_i)^n[\frac{1}{\Gamma(n-\mu)}\int_0^1 (1-t)^{n-\mu-1}\varphi(t)dt] \\ &= \frac{(y_i-x_i)^n}{\Gamma(n-\mu)}[\partial_i{}^n \int_0^1 (1-t)^{n-\mu-1} f(x_1, ..., x_{i-1}, x_i + th, x_{i+1}, ..., x_n)dt] \\ &= \frac{(y_i-x_i)^n}{\Gamma(n-\mu)}(y_i-x_i)^{\mu-n}[\partial_i{}^{\mu}\int_{x_i}^{y_i} (y-\tau)^{n-\mu-1} f(x_1, ..., x_{i-1}, \tau, x_{i+1}, ..., x_n)d\tau] \\ &= (y_i-x_i)^{\mu}[D_{x_i}^{\mu} f](x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n),\end{aligned}$$

and the Caputo fractional derivative in Eq. (37) becomes

$$\begin{aligned}({}^C D^{\mu}_{\mathbf{x}} f)(\mathbf{y}) &= \frac{1}{\Gamma(n-\mu)}\int_0^1 (1-t)^{n-\mu-1}\varphi^{(n)}(t)dt \\ &= \frac{(y_i-x_i)^n}{\Gamma(n-\mu)}\int_0^1 (1-t)^{n-\mu-1}\partial_i^n f(x_1,...,x_{i-1},x_i+th,x_{i+1},...,x_n)dt \\ &= (y_i-x_i)^{\mu-n}\frac{(y_i-x_i)^n}{\Gamma(n-\mu)}\int_{x_i}^{y_i} (y-\tau)^{n-\mu-1}\partial_i^n f(x_1,...,x_{i-1},\tau,x_{i+1},...,x_n)d\tau \\ &= (y_i-x_i)^{\mu}[{}^C D^{\mu}_{x_i} f](x_1,...,x_{i-1},y_i,x_{i+1},...,x_n).\end{aligned}$$

Using a similar argument, we can obtain the sequential directional fractional derivatives along the directed line segment as

$$\begin{aligned}D^{n\alpha}_{\mathbf{x}} f(\mathbf{y}) &= (y_i-x_i)^{n\alpha}[D^{n\alpha}_{x_i} f](x_1,...,x_{i-1},y_i,x_{i+1},...,x_n), \\ {}^C D^{n\alpha}_{\mathbf{x}} f(\mathbf{y}) &= (y_i-x_i)^{n\alpha}[{}^C D^{n\alpha}_{x_i} f](x_1,...,x_{i-1},y_i,x_{i+1},...,x_n).\end{aligned}$$

Based on the above definitions and lemmas, we now are in the position to state the main results of this article.

**Theorem 3.1 (Fractional vector Taylor formula with Riemann-Liouville derivative).** Assume that $\Omega \subset R^n$ is a compact and convex domain and $D^{\nu}_{\mathbf{x}} f(\mathbf{y}) \in C(\Omega)$. Let $\nu > 0$ and $m$ be the smallest integer exceeding $\nu$. Then, for any vectors $\mathbf{x}, \mathbf{y} \in \Omega$, it holds

$$f(\mathbf{y}) = \sum_{j=1}^{m} \frac{D^{\nu-j}_{\mathbf{x}} f(\mathbf{x})}{\Gamma(\nu-j+1)} + D^{-\nu}_{\mathbf{x}}[D^{\nu}_{\mathbf{x}} f(\mathbf{y})]. \tag{46}$$

**Proof.** Let $\varphi(t) = f(\mathbf{x}+t(\mathbf{y}-\mathbf{x})) = f(\mathbf{x}+t\mathbf{h})$ and $a=0, b=1$. By Theorem 2.1, we have

$$\varphi(1) = \sum_{j=1}^{m} \frac{D^{\nu-j}_0 \varphi(0)}{\Gamma(\nu-j+1)} + \frac{1}{\Gamma(\nu)}\int_0^1 (1-t)[D^{\nu}_0 \varphi(t)]dt. \tag{47}$$

From Eq. (43) and by Lemma 3.3, we further obtain

$$\varphi(1) = f(\mathbf{y}), \quad D^{\nu-j}_0 \varphi(0) = \ D^{\nu-j}_{\mathbf{x}} f(\mathbf{x}), \tag{48a}$$

$$\begin{aligned}\frac{1}{\Gamma(\nu)}\int_0^1 (1-t)^{\nu-1} D^{\nu}_0 \varphi(t)dt &= \frac{1}{\Gamma(\nu)}\int_0^1 (1-t)^{\nu-1} D^{\nu}_{\mathbf{x}} f(\mathbf{x}+t\mathbf{h})dt \\ &= D^{-\nu}_{\mathbf{x}}[D^{\nu}_{\mathbf{x}} f(\mathbf{y})].\end{aligned} \tag{48b}$$

Thus, substituting them into Eq. (47) gives Eq. (46).

**Theorem 3.2 (Fractional vector Taylor formula with Caputo derivative).** Assume that $\Omega \subset R^n$ is a compact and convex domain and ${}^C D^{\nu}_{\mathbf{x}} f(\mathbf{y}) \in C(\Omega)$. Let $\nu > 0$ and $m$ being the smallest integer exceeding $\nu$. Then, for any vectors $\mathbf{x}, \mathbf{y} \in \Omega$, it holds

$$f(\mathbf{y}) = \sum_{k=0}^{m-1} \frac{D^{k}_{\mathbf{x}} f(\mathbf{x})}{k!} + D^{-\nu}_{\mathbf{x}}[{}^C D^{\nu}_{\mathbf{x}} f(\mathbf{y})]. \tag{49}$$

**Proof.** Let $\varphi(t) = f(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) = f(\mathbf{x} + t\mathbf{h})$ and denote $a = 0, b = 1$. By Theorem 2.2, we obtain

$$\varphi(1) = \sum_{k=0}^{m-1} \frac{\varphi^{(k)}(0)}{k!} + \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} [{}^C D_0^\nu \varphi(t)] dt. \tag{50}$$

From Eq. (45) and by Lemma 3.3, we further obtain

$$\varphi(1) = f(\mathbf{y}), \quad \varphi^{(k)}(0) = D_{\mathbf{x}}^k f(\mathbf{x}), \tag{51a}$$

$$\begin{aligned} \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} [{}^C D_0^\nu \varphi(t)] dt &= \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} [{}^C D_{\mathbf{x}}^\nu f(\mathbf{x}+t\mathbf{h})] dt \\ &= D_{\mathbf{x}}^{-\nu} [{}^C D_{\mathbf{x}}^\nu f(\mathbf{y})]. \end{aligned} \tag{51b}$$

Thus, substituting them into Eq. (50) gives Eq. (49).

**Remark 3.3.** It should be pointed out that Eq. (49) is essentially the same as Eq. (13), and however, the expression in Eq. (49) looks much more concise than that in Eq. (13). Also, the proof for Eq. (49) is much simpler than that for Eq. (13).

**Theorem 3.3 (Fractional vector Taylor formula with sequential Caputo derivative).** Assume that $\Omega \subset R^n$ is a compact and convex domain and the sequential Caputo derivative ${}^C D_{\mathbf{x}}^{k\alpha} f(\mathbf{y}) \in C(\Omega), k = 0, 1, ..., m+1,\ 0 < \alpha \le 1$. Then, for any vectors $\mathbf{x}, \mathbf{y} \in \Omega$ and $\mathbf{h} = \mathbf{y} - \mathbf{x}$, it holds

$$f(\mathbf{y}) = \sum_{k=0}^{m} \frac{{}^C D_{\mathbf{x}}^{k\alpha} f(\mathbf{x})}{\Gamma(k\alpha+1)} + \frac{1}{\Gamma((m+1)\alpha)} \int_0^1 (1-t)^{(m+1)\alpha-1}\ {}^C D_{\mathbf{x}}^{(m+1)\alpha} f(\mathbf{x}+t\mathbf{h}) dt. \tag{52}$$

**Proof.** Let $\varphi(t) = f(\mathbf{x} + t\mathbf{h})$. Replacing $f$ by $\varphi$ with $a = 0, b = 1$ in Eq. (4), we obtain

$$\varphi(1) = \sum_{k=0}^{m} \frac{{}^C D_0^{k\alpha} \varphi(0)}{\Gamma(k\alpha+1)} + \frac{1}{\Gamma((m+1)\alpha)} \int_0^1 (1-t)^{(m+1)\alpha-1} ({}^C D_0^{(m+1)\alpha} \varphi)(t) dt. \tag{53}$$

From Eqs. (44)-(45), we further obtain

$$({}^C D_{\mathbf{x}}^\mu f)(\mathbf{x} + s\mathbf{h}) = ({}^C D_0^\mu \varphi)(s), \tag{54}$$

$$({}^C D_{\mathbf{x}}^\mu f)(\mathbf{y}) = ({}^C D_0^\mu \varphi)(1), \qquad ({}^C D_{\mathbf{x}}^\mu f)(\boldsymbol{x}) = ({}^C D_0^\mu \varphi)(0). \tag{55}$$

Thus, substituting them into Eq. (53) gives Eq. (52).

**Theorem 3.4 (Fractional vector Taylor mean value theorem with sequential Caputo derivative).** Assume that $\Omega \subset R^n$ is a compact and convex domain and the sequential Caputo derivative ${}^C D_{\mathbf{x}}^{k\alpha} f(\mathbf{y}) \in C(\Omega), k = 0, 1, ..., m+1$, $0 < \alpha \le 1$. Then, for any points $\mathbf{x}, \mathbf{y} \in \Omega$, it holds

$$f(\mathbf{y}) = \sum_{k=0}^{m} \frac{{}^C D_{\mathbf{x}}^{k\alpha} f(\mathbf{x})}{\Gamma(k\alpha+1)} + \frac{{}^C D_{\mathbf{x}}^{(m+1)\alpha} f(\xi)}{\Gamma((m+1)\alpha+1)}, \tag{56}$$

where $\xi = \mathbf{x} + \theta(\mathbf{y} - \mathbf{x})$ for some $\theta$ in $0 < \theta < 1$.

**Proof.** Let $\varphi(t) = f(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))$ and denote $a = 0, b = 1$. From Eq. (4), we obtain

$$\varphi(1) = \sum_{k=0}^{m} \frac{({}^C D_0^{k\alpha} \varphi)(0)}{\Gamma(k\alpha+1)} + \frac{{}^C D_0^{(m+1)\alpha} \varphi(\theta)}{\Gamma((m+1)\alpha+1)}, \tag{57}$$

for some $\theta$ in $0<\theta<1$. From Eq. (45), we further obtain

$$\varphi(1)=f(x,y),\quad ({}^CD_0^{k\alpha}\varphi)(0)={}^CD_{\mathbf{x}}^{k\alpha}f(\mathbf{x}),\quad {}^CD_0^{(m+1)\alpha}\varphi(\theta)={}^CD_{\mathbf{x}}^{(m+1)\alpha}f(\xi). \tag{58}$$

Thus, substituting them into Eq. (57) gives Eq. (56).

To verify the above Theorems 3.3 and 3.4, we consider the following simple example.

**Example 3.1.** Consider $f(\mathbf{y})=(y_1^{2\alpha}+\cdots+y_n^{2\alpha})/\Gamma(2\alpha+1)$, where $0<\alpha\le 1$, and $\Omega=\{y_1^2+\cdots+y_n^2\leqslant 1\}$. Here, we choose $\mathbf{x}=\mathbf{0}=(0,\cdots,0)$ and obtain $f(\mathbf{0})=0$. For any vector $\mathbf{y}\in\Omega$, we denote $\varphi(t)\equiv f(t\mathbf{y})=t^{2\alpha}(y_1^{2\alpha}+\cdots+y_n^{2\alpha})/\Gamma(2\alpha+1)$. This gives ${}^CD_0^{\alpha}\varphi(t)=t^{\alpha}(y_1^{2\alpha}+\cdots+y_n^{2\alpha})/\Gamma(\alpha+1)$ and ${}^CD_0^{2\alpha}\varphi(t)=y_1^{2\alpha}+\cdots+y_n^{2\alpha}$, implying that $f(\mathbf{0})=\varphi(0)=0, {}^CD_{\mathbf{0}}^{\alpha}f(\mathbf{0})={}^CD_0^{\alpha}\varphi(0)=0$, and ${}^CD_{\mathbf{0}}^{2\alpha}f(\boldsymbol{z})={}^CD_0^{2\alpha}\varphi(s)=y_1^{2\alpha}+\cdots+y_n^{2\alpha}$. Note that ${}^CD_{\mathbf{0}}^{k\alpha}f(\mathbf{z})\in C(\Omega)$, $k=0,1,2$. Thus, from Eq. (53) with $m=1$, we obtain

$$\begin{aligned}\varphi(1)&=\varphi(0)+\frac{{}^CD_0^{\alpha}\varphi(0)}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(2\alpha)}\int_0^1(1-t)^{2\alpha-1}\;[{}^CD_0^{2\alpha}\varphi(t)]dt\\&=0+0+\frac{y_1^{2\alpha}+\cdots+y_n^{2\alpha}}{\Gamma(2\alpha+1)},\end{aligned} \tag{59}$$

implying that

$$\begin{aligned}f(\mathbf{y})&=\frac{y_1^{2\alpha}+\cdots+y_n^{2\alpha}}{\Gamma(2\alpha+1)}=0+0+\frac{y_1^{2\alpha}+\cdots+y_n^{2\alpha}}{\Gamma(2\alpha+1)}\\&=f(\mathbf{0})+\frac{{}^CD_{\mathbf{0}}^{\alpha}f(\mathbf{0})}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(2\alpha)}\int_0^1(1-t)^{2\alpha-1}[{}^CD_{\mathbf{x}}^{(m+1)\alpha}f](\mathbf{x}+t\mathbf{h})dt.\end{aligned} \tag{60}$$

Hence, we have verified Eq. (52) to be true when $m=1$.

Furthermore, from ${}^CD_0^{\alpha}\varphi(t)=t^{\alpha}(y_1^{2\alpha}+\cdots+y_n^{2\alpha})/\Gamma(\alpha+1)$ and ${}^CD_0^{2\alpha}\varphi(t)=y_1^{2\alpha}+\cdots+y_n^{2\alpha}$, we see ${}^CD_{\mathbf{0}}^{\alpha}f(\mathbf{y})={}^CD_0^{\alpha}\varphi(1)=(y_1^{2\alpha}+\cdots+y_n^{2\alpha})/\Gamma(\alpha+1)$, ${}^CD_{\mathbf{0}}^{\alpha}f(\mathbf{0})={}^CD_0^{\alpha}\varphi(0)=0$, ${}^CD_{\mathbf{0}}^{2\alpha}f(\boldsymbol{y})={}^CD_0^{2\alpha}\varphi(1)=y_1^{2\alpha}+\cdots+y_n^{2\alpha}$, and

$${}^CD_{\mathbf{0}}^{2\alpha}f(\xi)={}^CD_{\mathbf{0}}^{2\alpha}f(\theta\mathbf{y})={}^CD_0^{2\alpha}\varphi(\theta)=y_1^{2\alpha}+\cdots+y_n^{2\alpha}.$$

Note that ${}^CD_{\mathbf{0}}^{k\alpha}f(\mathbf{y})\in C(\Omega)$, $k=0,1,2$. Thus, from Eq. (56) with $m=1$, we obtain

$$\varphi(1)=\varphi(0)+\frac{{}^CD_0^{\alpha}\varphi(0)}{\Gamma(\alpha+1)}+\frac{{}^CD_0^{2\alpha}\varphi(\theta)}{\Gamma(2\alpha+1)},$$

implying that

$$\begin{aligned}f(\mathbf{y})&=\frac{y_1^{2\alpha}+\cdots+y_n^{2\alpha}}{\Gamma(2\alpha+1)}=0+0+\frac{y_1^{2\alpha}+\cdots+y_n^{2\alpha}}{\Gamma(2\alpha+1)}\\&=f(\mathbf{0})+\frac{{}^CD_{\mathbf{0}}^{\alpha}f(\mathbf{0})}{\Gamma(\alpha+1)}+\frac{{}^CD_{\mathbf{0}}^{2\alpha}f(\xi)}{\Gamma(2\alpha+1)},\end{aligned} \tag{61}$$

where $\xi=\theta\mathbf{y}$ for $\theta$ in $0<\theta<1$. Hence, we have verified Eq. (56) to be true when $m=1$.

To illustrate the applicability of the fractional vector Taylor formula, we consider the following simple example.

**Example 3.2.** Let $0<\alpha\le 1$. Given points $P_0:\mathbf{x}=(x_1,...,x_n), P_1:\mathbf{y}=(y_1,...,y_n)\in\Omega\subset R^n$, and given $f(\mathbf{x})=C_0$ and the directional fractional derivative at $P_s$ as ${}^CD_{\mathbf{x}}^{\alpha}f(\mathbf{z})=s^{2\alpha}[(y_1-x_1)^{2\alpha}+\cdots+(y_n-x_n)^{2\alpha}]/\Gamma(2\alpha+1)$, where $\mathbf{z}=\mathbf{x}+s\mathbf{h},\mathbf{h}=$

$\mathbf{y}-\mathbf{x}=(h_1,...,h_n)$, and $0\leq s\leq 1$. We will use the multivariate fractional Taylor formula in Eq. (56) to evaluate $f(\mathbf{y})$.

To this end, we denote ${}^{C}D^{\alpha}_{\mathbf{x}}f(\mathbf{x}+s\mathbf{h})=s^{2\alpha}(h_1^{2\alpha}+\cdots+h_n^{2\alpha})/\Gamma(2\alpha+1)$ as $g(s)$. It can be seen that ${}^{C}D^{\alpha}_{\mathbf{x}}f(\mathbf{x})=\lim\limits_{s\to 0^+}g(s)=\lim\limits_{s\to 0^+}s^{2\alpha}(h_1^{2\alpha}+\cdots+h_n^{2\alpha})/\Gamma(2\alpha+1)=0$, ${}^{C}D^{2\alpha}_{\mathbf{x}}f(\mathbf{x})=\lim\limits_{s\to 0^+}{}^{C}D^{\alpha}_{0}g(s)=\lim\limits_{s\to 0^+}s^{\alpha}(h_1^{2\alpha}+\cdots+h_n^{2\alpha})/\Gamma(\alpha+1)=0$, and ${}^{C}D^{3\alpha}_{\mathbf{x}}f(\mathbf{x}+s\mathbf{h})={}^{C}D^{2\alpha}_{0}g(s)=h_1^{2\alpha}+\cdots+h_n^{2\alpha}$. By Theorem 3.4, we obtain

$$\begin{aligned}f(\mathbf{y})&=f(\mathbf{x}+\mathbf{h})\\&=f(\mathbf{x})+\frac{{}^{C}D^{\alpha}_{\mathbf{x}}f(\mathbf{x})}{\Gamma(\alpha+1)}+\frac{{}^{C}D^{2\alpha}_{\mathbf{x}}f(\mathbf{x})}{\Gamma(2\alpha+1)}+\frac{{}^{C}D^{3\alpha}_{\mathbf{x}}f(\xi)}{\Gamma(3\alpha+1)}\\&=C_0+\frac{h_1^{2\alpha}+\cdots+h_n^{2\alpha}}{\Gamma(3\alpha+1)}.\end{aligned}\tag{62}$$

**Theorem 3.5 (Fractional vector Cauchy mean value theorem with Caputo derivative).** Assume that $\Omega\subset R^n$ is a compact and convex domain and the sequential Caputo derivatives ${}^{C}D^{k\alpha}_{\mathbf{x}}f(\mathbf{y}),{}^{C}D^{k\alpha}_{\mathbf{x}}g(\mathbf{y})\in C(\Omega), k=0,1,...,m+1$, $0<\alpha\leq 1$. Then, for any vectors $\mathbf{x},\mathbf{y}\in\Omega$, it holds

$$\frac{f(\mathbf{y})-\sum\limits_{k=0}^{m}\frac{{}^{C}D^{k\alpha}_{\mathbf{x}}f(\mathbf{x})}{\Gamma(k\alpha+1)}}{g(\mathbf{y})-\sum\limits_{k=0}^{m}\frac{{}^{C}D^{k\alpha}_{\mathbf{x}}g(\mathbf{x})}{\Gamma(k\alpha+1)}}=\frac{{}^{C}D^{(m+1)\alpha}_{\mathbf{x}}f(\xi)}{{}^{C}D^{(m+1)\alpha}_{\mathbf{x}}g(\xi)},\tag{63}$$

where $\xi=\mathbf{x}+\theta(\mathbf{y}-\mathbf{x})\equiv\mathbf{x}+\theta\mathbf{h}$ for some $\theta$ in $0<\theta<1$. Here, it is assumed that ${}^{C}D^{k\alpha}_{\mathbf{x}}g(\mathbf{x})\neq 0$ in $\Omega\backslash\{\mathbf{x}\}$.

**Proof.** Denoting $\varphi(t)=f(\mathbf{x}+t\mathbf{h})$ and $\psi(t)=g(\mathbf{x}+t\mathbf{h})$, and replacing function $f$ by $\varphi$, $g$ by $\psi$ with $a=0,b=1$ in Eq. (6), we obtain

$$\frac{\varphi(1)-\sum\limits_{k=0}^{m}\frac{{}^{C}D^{\alpha}_{0}\varphi(0)}{\Gamma(k\alpha+1)}}{\psi(1)-\sum\limits_{k=0}^{m}\frac{{}^{C}D^{\alpha}_{0}\psi(0)}{\Gamma(k\alpha+1)}}=\frac{{}^{C}D^{(m+1)\alpha}_{0}\varphi(\theta)}{{}^{C}D^{(m+1)\alpha}_{0}\psi(\theta)},\tag{64}$$

for some $\theta$ in $0<\theta<1$. From Eq. (45), we further obtain

$$\varphi(1)=f(\mathbf{x}),\quad({}^{C}D^{k\alpha}_{0}\varphi)(0)={}^{C}D^{k\alpha}_{\mathbf{x}}f(\mathbf{x}),\quad{}^{C}D^{(m+1)\alpha}_{0}\varphi(\theta)={}^{C}D^{(m+1)\alpha}_{\mathbf{x}}f(\xi).\tag{65a}$$

$$\psi(1)=g(\boldsymbol{x}),\quad({}^{C}D^{k\alpha}_{0}\psi)(0)={}^{C}D^{k\alpha}_{\mathbf{x}}g(\mathbf{x}),\quad{}^{C}D^{(m+1)\alpha}_{0}\psi(\theta)={}^{C}D^{(m+1)\alpha}_{\mathbf{x}}g(\xi).\tag{65b}$$

Thus, substituting them into Eq. (64) gives Eq. (63).

To verify Theorem 3.5, we consider the following simple example.

**Example 3.3.** Consider $f(\mathbf{y})=(y_1^{2\alpha}+\cdots+y_n^{2\alpha})/\Gamma(2\alpha+1)$ and $g(\mathbf{y})=(y_1^{4\alpha}+...+y_n^{4\alpha})/\Gamma(4\alpha+1)$, where $0\leq\alpha\leq 1$ and $\Omega=\{y_1^2+\cdots+y_n^2\leqslant 1\}$. Here, we choose $\mathbf{x}=\mathbf{0}=(0,\cdots,0)$ and obtain $f(\mathbf{0})=0$, $g(\mathbf{0})=0$. For any vector $\mathbf{y}\in\Omega$, we denote $\varphi(t)\equiv f(t\mathbf{y})=t^{2\alpha}(y_1^{2\alpha}+\cdots+y_n^{2\alpha})/\Gamma(2\alpha+1)$, $\psi(t)\equiv g(t\mathbf{y})=t^{4\alpha}(y_1^{4\alpha}+\cdots+y_n^{4\alpha})/\Gamma(4\alpha+1)$. Taking the sequential Caputo fractional derivatives, we obtain ${}^{C}D^{\alpha}_{0}\varphi(t)=t^{\alpha}(y_1^{2\alpha}+\cdots+y_n^{2\alpha})/\Gamma(\alpha+1)$, ${}^{C}D^{2\alpha}_{0}\varphi(t)=y_1^{2\alpha}+\cdots+y_n^{2\alpha}$, and ${}^{C}D^{\alpha}_{0}\psi(t)=t^{3\alpha}(y_1^{4\alpha}+\cdots+y_n^{4\alpha})/\Gamma(3\alpha+1)$, and ${}^{C}D^{2\alpha}_{0}\psi(t)=t^{2\alpha}(y_1^{4\alpha}+\cdots+y_n^{4\alpha})/\Gamma(2\alpha+1)$. These indicate that ${}^{C}D^{\alpha}_{\mathbf{0}}f(\boldsymbol{y})={}^{C}D^{\alpha}_{0}\varphi(1)=(y_1^{2\alpha}+\cdots+y_n^{2\alpha})/\Gamma(\alpha+1)$, ${}^{C}D^{\alpha}_{\mathbf{0}}f(\mathbf{0})={}^{C}D^{\alpha}_{0}\varphi(0)=0$, ${}^{C}D^{2\alpha}_{\mathbf{0}}f(\mathbf{y})={}^{C}D^{2\alpha}_{0}\varphi(1)=y_1^{2\alpha}+\cdots+$

$y_n^{2\alpha}$, ${}^C D_{\mathbf{0}}^{2\alpha} f(\xi) = {}^C D_{\mathbf{0}}^{2\alpha} f(\theta \mathbf{y}) = {}^C D_0^{2\alpha} \varphi(\theta) = y_1^{2\alpha} + \cdots + y_n^{2\alpha}$, and ${}^C D_{\mathbf{0}}^{\alpha} g(\mathbf{y}) = {}^C D_0^{\alpha} \psi(1) = (y_1^{4\alpha} + \cdots + y_n^{4\alpha})/\Gamma(3\alpha+1)$, ${}^C D_{\mathbf{0}}^{\alpha} g(\mathbf{0}) = {}^C D_0^{\alpha} \psi(0) = 0$, ${}^C D_{\mathbf{0}}^{2\alpha} g(\mathbf{y}) = {}^C D_0^{2\alpha} \psi(1) = (y_1^{4\alpha} + \cdots + y_n^{4\alpha})/\Gamma(2\alpha+1)$, ${}^C D_{\mathbf{0}}^{2\alpha} g(\xi) = {}^C D_{\mathbf{0}}^{2\alpha} g(\theta \mathbf{y}) = {}^C D_0^{2\alpha} \psi(\theta) = \theta^{2\alpha}(y_1^{4\alpha} + \cdots + y_n^{4\alpha})/\Gamma(2\alpha+1)$. Note that ${}^C D_{\mathbf{0}}^{k\alpha} f(\mathbf{y})$, ${}^C D_{\mathbf{0}}^{k\alpha} g(\mathbf{y}) \in C(\Omega)$, $k = 0, 1, 2$, and ${}^C D_{\mathbf{0}}^{2\alpha} g(\mathbf{y}) = (y_1^{4\alpha} + \cdots + y_n^{4\alpha})/\Gamma(2\alpha+1) \neq 0$ in $\{0 < y_1^2 + \cdots + y_n^2 \leqslant 1\}$. Thus, from Eq. (64) with $m = 1$, we obtain

$$\frac{\varphi(1) - \varphi(0) - \frac{{}^C D_0^{\alpha} \varphi(0)}{\Gamma(\alpha+1)}}{\psi(1) - \psi(0) - \frac{{}^C D_0^{\alpha} \psi(0)}{\Gamma(\alpha+1)}} = \frac{{}^C D_0^{2\alpha} \varphi(\theta)}{{}^C D_0^{2\alpha} \psi(\theta)}, \tag{66}$$

implying that

$$\begin{aligned} \frac{f(\boldsymbol{y}) - f(\mathbf{0}) - \frac{{}^C D_{\mathbf{0}}^{\alpha} f(\mathbf{0})}{\Gamma(\alpha+1)}}{g(\boldsymbol{y}) - g(\mathbf{0}) - \frac{{}^C D_{\mathbf{0}}^{\alpha} g(\mathbf{0})}{\Gamma(\alpha+1)}} &= \frac{\frac{y_1^{2\alpha} + \ldots + y_n^{2\alpha}}{\Gamma(2\alpha+1)} - 0 - 0}{\frac{y_1^{4\alpha} + \ldots + y_n^{4\alpha}}{\Gamma(4\alpha+1)} - 0 - 0} \\ &= \frac{y_1^{2\alpha} + \ldots + y_n^{2\alpha}}{\frac{\theta^{2\alpha}(y_1^{4\alpha} + \ldots + y_n^{4\alpha})}{\Gamma(2\alpha+1)}} \\ &= \frac{{}^C D_{\mathbf{0}}^{2\alpha} f(\xi)}{{}^C D_{\mathbf{0}}^{2\alpha} g(\xi)}, \end{aligned} \tag{67}$$

where the value of $\theta$ is calculated to be $\theta = [\Gamma^2(2\alpha+1)/\Gamma(4\alpha+1)]^{\frac{1}{2\alpha}} < 1$ for $0 < \alpha \leq 1$ and $\xi = \theta \mathbf{y}$. Hence, we have verified Eq. (63) to be true for $m = 1$.

**Remark 3.4.** Here, we would like to discuss some special cases of Theorem 3.3.

*Case 1.* When $n = 1$ (*i.e.*, one-dimensional space $R$) and $0 < \alpha \leq 1$, Eq. (52) gives

$$f(y) = \sum_{k=0}^{m} \frac{{}^C D_x^{k\alpha} f(x)}{\Gamma(k\alpha+1)} + \frac{1}{\Gamma((m+1)\alpha)} \int_0^1 (1-t)^{(m+1)\alpha - 1}\, {}^C D_x^{(m+1)\alpha} f(x + t\,(y-x))dt. \tag{68}$$

Based on Eq. (26) in one-dimensional space $R$, we obtain

$$\begin{aligned} (D_0^{-\nu} f)(y) &= \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} f(x + t(y-x))dt \\ &= (y-x)^{-\nu} \frac{1}{\Gamma(\nu)} \int_x^y (y-\tau)^{\nu-1} f(\tau)d\tau \\ &= (y-x)^{-\nu} [D_x^{-\nu} f](y), \end{aligned} \tag{69}$$

where $[D_x^{-\nu} f](y)$ is the Riemann-Liouville integral and $\nu > 0$. Similarly, by Definition 3.3 and Lemma 3.3 in one dimension, we obtain

$$({}^C D_0^{\mu} f)(y) = (y-x)^{\mu} [{}^C D_x^{\mu} f](y). \tag{70}$$

where $[{}^C D_x^{\mu} f](y)$ is the Caputo fractional derivative. Substituting Eq. (70) into Eq. (68), we obtain a fractional Taylor formula with an integral remainder via the sequential fractional Caputo derivative as

$$\begin{aligned} f(y) = &\sum_{k=0}^{m} \frac{(y-x)^{k\alpha}}{\Gamma(k\alpha+1)} [{}^C D_x^{k\alpha} f](x) \\ &+ \frac{1}{\Gamma((m+1)\alpha)} \int_x^y (y-t)^{(m+1)\alpha - 1}\, {}^C D_x^{(m+1)\alpha} f(t)dt, \end{aligned} \tag{71}$$

which is the same as Eq. (4) obtained by Odibat and Shawagfeh [22].

In particular, when $\alpha = 1$, Eq. (71) becomes

$$f(y) = \sum_{k=0}^{m} \frac{(y-x)^k}{k!} f^k(x) + \frac{1}{m!} \int_x^y (y-t)^m f^{(m+1)}(t)dt, \tag{72}$$

which is the same as classical Taylor formula in calculus.

Especially, when $m = 0$, Eq. (72) further simplifies to the well-known Newton-Leibnitz fundamental theorem in calculus as

$$f(y) = f(x) + \int_x^y f'(t)dt.$$

*Case 2***.** When $n > 1$ (*i.e.*, multi-dimensional space $R^n$) and $\alpha = 1$, by Definition 3.2, Eq. (52) becomes

$$\begin{aligned} f(\mathbf{y}) = &\sum_{k=0}^{m} \frac{1}{k!} (h_1 \frac{\partial}{\partial x_1} + h_2 \frac{\partial}{\partial x_2} + \cdots + h_n \frac{\partial}{\partial x_n})^k f(\mathbf{x}) \\ &+ \frac{1}{m!} \int_0^1 (1-t)^m D^{m+1} f(\mathbf{x} + t\mathbf{h})dt, \end{aligned} \tag{73}$$

which is the classical multivariate Taylor formula in Eqs. (7) and (8).

Especially, when $m = 0$, Eq. (73) further simplifies to

$$f(\mathbf{y}) = f(\mathbf{x}) + \int_0^1 (h_1 \frac{\partial}{\partial x_1} + \cdots + h_n \frac{\partial}{\partial x_n}) f(\mathbf{x} + t\mathbf{h})dt, \tag{74}$$

which is the famous Hadamard formula.

*Case 3.* Based on the classical Cauchy mean value theorem of integral, the integral term in Eq. (52) can be re-written as

$$\begin{aligned} &\int_0^1 (1-t)^{(m+1)\alpha-1} \, {}^C D_{\mathbf{x}}^{(m+1)\alpha} f(\mathbf{x} + s\mathbf{h})ds \\ = &\ {}^C D_{\mathbf{x}}^{(m+1)\alpha} f(\mathbf{x} + \theta\mathbf{h}) \int_0^1 (1-t)^{(m+1)\alpha-1} ds \\ = &\ \frac{({}^C D_{\mathbf{x}}^{(m+1)\alpha} f)(\xi)}{(m+1)\alpha}, \end{aligned} \tag{75}$$

where $\xi = \mathbf{x} + \theta\mathbf{h}$ for some $\theta$ in $0 < \theta < 1$. This indicates that the multivariate fractional Taylor formula in Eq. (56) can also be obtained from Eq. (52).

It should be pointed out that by using the definitions of fractional integrals and fractional derivatives along directed line segments corresponding to multivariable, we have derived fractional vector Taylor formulas and Cauchy mean value formulas. Here, we further give couple of examples for solving directional fractional differential equations along directed line segments.

**Example 3.4.** Given two vectors $\mathbf{x}, \mathbf{y} \in \Omega \subset R^n$, any vector $\mathbf{z} = \mathbf{x} + s\mathbf{h}$, where $0 \leq s \leq 1$ and $\mathbf{h} = \mathbf{y} - \mathbf{x}$. Assume $0 < \alpha \leq 1$ and $f(\mathbf{z}) \in C(\Omega)$. Consider the Cauchy initial value problem with the directional Caputo derivative as

$$\begin{cases} {}^C D_{\mathbf{x}}^{\alpha} f(\mathbf{z}) = \lambda f(\mathbf{z}), \\ \qquad f(\mathbf{x}) = 1. \end{cases} \tag{76}$$

To find its solution, we use those fractional integrals and fractional derivatives along directed line segments defined in the previous text and obtain that Eq. (76) is equivalent to

$$\begin{cases} {}^{C}D_{0}^{\alpha}\varphi(s) = \lambda\varphi(s), \\ \quad\varphi(0) = 1, \end{cases} \tag{77}$$

where the solution is given as

$$\varphi(s) = E_{\alpha}(\lambda s^{\alpha}) = \sum_{k=0}^{\infty} \frac{\lambda^{k} s^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{78}$$

Hence, the solution of Eq. (76) is

$$f(\mathbf{z}) = E_{\alpha}(\lambda s^{\alpha}) = \sum_{k=0}^{\infty} \frac{\lambda^{k} s^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{79}$$

**Example 3.5.** Given two vectors $\mathbf{x}, \mathbf{y} \in \Omega \sqsubset R^{n}$, any $\mathbf{z} = \mathbf{x} + s\mathbf{h}$, where $0 \le s \le 1$ and $\mathbf{h} = \mathbf{y} - \mathbf{x}$. Assume $0 < \alpha \le 1$ and $f(\mathbf{z}) \in C(\Omega)$. Consider the Cauchy initial value problem with the directional Riemann-Liouville derivative as

$$\begin{cases} D_{\mathbf{x}}^{\alpha} f(\mathbf{z}) = \lambda f(\mathbf{z}), \\ D_{\mathbf{x}}^{\alpha-1} f(\mathbf{x}) = 1. \end{cases} \tag{80}$$

To find its solution, again, we use those fractional integrals and fractional derivatives along directed line segments defined in the previous text and obtain that Eq. (80) is equivalent to

$$\begin{cases} D_{0}^{\alpha}\varphi(s) = \lambda\varphi(s), \\ D_{0}^{\alpha-1}\varphi(0) = 1, \end{cases} \tag{81}$$

where the solution of Eq. (81) is given as

$$\varphi(s) = s^{\alpha-1}E_{\alpha,\alpha}(\lambda s^{\alpha}) = s^{\alpha-1}\sum_{k=0}^{\infty} \frac{\lambda^{k} s^{k\alpha}}{\Gamma(k\alpha+\alpha)}. \tag{82}$$

Hence, the solution of Eq. (80) is

$$f(\mathbf{z}) = s^{\alpha-1}E_{\alpha,\alpha}(\lambda s^{\alpha}) = s^{\alpha-1}\sum_{k=0}^{\infty} \frac{\lambda^{k} s^{k\alpha}}{\Gamma(k\alpha+\alpha)}, \tag{83}$$

where $E_{\alpha,\beta}(t)$ is defined by

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(k\alpha+\beta)}. \tag{84}$$

## 4. Conclusion

By defining fractional integrals and fractional derivatives along directed line segments corresponding to multivariable, we derive fractional vector Taylor formulas and fractional vector Cauchy mean value formulas in the sense of the Riemann-Liouville fractional order derivative, the Caputo fractional order derivative, and the sequential fractional order derivative, respectively, where the fractional order $\alpha$ is in $0 < \alpha \le 1$. When $\alpha = 1$, these formulas can be reduced to the corresponding classical Taylor formula and Cauchy mean value formula. These new formulas and directional fractional derivatives are verified and illustrated by several examples. The obtained formulas may be useful in fractional vector calculus which is an important tool for describing processes in complex media, non-local materials and distributed systems in multi-dimensional space and for solving fractional partial differential equations.

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