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ON THE BOUNDEDNESS AND OSCILLATION OF NON-CONFORMABLE LIÉNARD EQUATION

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ABSTRACT. In this work we study the boundary and oscillation of the solutions of the non conformable equation (1), without making use of the second method of Lyapunov.

1. INTRODUCTION.

Various questions on the stability, oscillation and periodicity of solutions of classical Liénard equation $x'' + f(x)x' + a(t)g(x) = 0$ have received a considerable amount of attention in the last years (see [6], [9], [10], [11], [15], [16], [19], [25], [27], [28], [33], [34] and the references cited there) under condition $f(x)>0$ for all $x \in \mathbb{R}$. The fractional case is very different, although there are works in this direction, the results are scarce, first, because in the case of classical fractional derivatives there is no chain rule, which prevents the direct use of the second method of Lyapunov. In [14] we studied the stability of the fractional Liénard equation with derivative Caputo. As we said, since the chain rule was not valid, then the difficulties that we had to overcome were several. In [3] the results obtained with Caputo fractional derivatives and Caputo fractional Dini derivatives of Lyapunov functions are illustrated by the examples. It is emphasized that in some cases these techniques cannot be used. In this regard, it can also be consulted [31], see [13]) and in the case of local fractional derivatives, development time is very little (see [14]). Other results of various qualitative properties in the fractional case are [4, 8, 5, 1, 24, 30, 20] in the global case and [2, 23, 32, 35] in the local case.

In this paper, we study the asymptotic behaviors of solutions of (2) without considering the positivity of the function f and using a new method in which the usual Lyapunov function is not used. As we will see later, the method used allows the construction of a certain boundary region, where by imposing natural conditions, the oscillation of the solutions can be guaranteed.

To apply the direct method of Lyapunov to the classic Liénard equation, we usually define a Lyapunov function $V(t, x, y)$ given by

$$V(t, x, y) = b(t)W(t, x, y),$$

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where

$$W(t,x,y) = G(x) + \frac{y^2}{2\,[a(t)]}$$

with $G(x) = \int_0^x g(t)dt$, $b(t) = \exp\left(-\int_0^t \frac{a'(s)^-}{a(s)}ds\right)$ and $a'(t)_- = \max(-a'(t), 0)$. Let $V'(t,x,y)$ be the total derivative along the solutions of Liénard equation, which is given in the abstract of the this paper. If $V'(t,x,y)$ is non-positive in a suitable neighborhood of the $(0,0)$, then the stability of the zero solution of Liénard equation is followed. For the non-positivity of $V'(t,x,y)$ we need that $F(x)$ satisfies the following condition:

$$F(-x) \leq 0 \leq F(x) \text{ somewhere in } x \geq 0, \tag{1}$$

since $V'(t,x,y) = -\frac{b(t)}{a(t)}\left[a'(t)^-G(x) + \frac{y^2a'(t)^+}{2a^2(t)} + a(t)g(x)F(x)\right]$. In other point of view, the non-positivity of derivative of V implies that every solutions of Liénard equation departing from a bounded region by a closed curve, remains in this region as t increases. This fact play an essential role in our work where the assumptions (1) is not used. So, we need alternative assumptions on $F(x)$ and $g(x)$ under which the last remark is still valid.

We consider the following equation

$$N_1^{\alpha}\left(N_1^{\alpha}x\right) + f(x)N_1^{\alpha}x + a(t)g(x) = 0, \tag{2}$$

where a, f and g are continuous functions satisfying the following conditions:

a) $xg(x) > 0$ for $x \neq 0$, $g \in C^1(\mathbb{R})$.
b) ${}_{N_1}J_0^{\alpha}g(+\infty) = +\infty$.
c) $0 < a \leq a(t) \leq A < +\infty$ for $t \in [0, +\infty)$.

It is necessary to present some necessary definitions for our work. Let $\alpha \in (0,1]$ and we consider a continuous function $f : [t_0, +\infty) \to \mathbb{R}$.

First, let us remember the definition of $N_1^{\alpha}f(t)$, a non conformable fractional derivative of a function in a point t defined in [12] and that is the basis of our results, that are close resemblance of those found in classical qualitative theory.

Definition 1 Given a function $f : [t_0, +\infty) \to \mathbb{R}$, $t_0 > 0$. Then the N-derivative of f of order α is defined by $N_1^{\alpha}f(t) = \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon e^{t^{-\alpha}})-f(t)}{\varepsilon}$ for all $t > 0$, $\alpha \in (0,1)$. If f is $\alpha-$differentiable in some $(0,a)$, and $\lim_{t\to 0^+} N_1^{(\alpha)}f(t)$ exists, then define $N_1^{(\alpha)}f(0) = \lim_{t\to 0^+} N_1^{(\alpha)}f(t)$.

If the N-derivative of the function $x(t)$ of order α exists and is finite in (t_0, ∞), we will say that $x(t)$ is N-differentiable in $I = (t_0, \infty)$.

Remark 1 The N-derivative solves almost all the insufficiencies that are indicated to the classical fractional derivatives. In particular we have the following result.

Theorem 1 (See [12]) Let f and g be N-differentiable at a point $t > 0$ and $\alpha \in (0,1]$. Then, we have the following relations:

a) $N_1^{\alpha}(af + bg)(t) = aN_1^{\alpha}(f)(t) + bN_1^{\alpha}(g)(t)$.

b) $N_1^\alpha(t^p) = e^{t^{-\alpha}} p t^{p-1},\ p \in \mathbb{R}$.
c) $N_1^\alpha(\lambda) = 0,\ \lambda \in \mathbb{R}$.
d) $N_1^\alpha(fg)(t) = f N_1^\alpha(g)(t) + g N_1^\alpha(f)(t)$.
e) $N_1^\alpha(\frac{f}{g})(t) = \frac{gN_1^\alpha(f)(t) - fN_1^\alpha(g)(t)}{g^2(t)}$.
f) If, in addition, f is differentiable, then $N_1^\alpha(f) = e^{t^{-\alpha}} f'(t)$.
g) If f is differentiable and $\alpha = n$ integer, then we have $N_1^n(f)(t) = e^{t^{-n}} f'(t)$.

Remark 2 The relations a), c), d) and (e) are similar to the classical results mathematical analysis, these relationships are not established (or do not occur) for fractional derivatives of global character (see [18] and [26] and bibliography there). The relation c) is maintained for the fractional derivative of Caputo. Cases c), f) and g) are typical of this non conformable local fractional derivative.

Now we will present the equivalent result, for N_1^α, of the well-known chain rule of classic calculus and that is basic in the second method of Lyapunov to study of stability of perturbed motion.

Theorem 2 (See [12]) Let $\alpha \in (0, 1]$. If g is N-differentiable at $t > 0$ and f is differentiable at $g(t)$, then $N_1^\alpha(f \circ g)(t) = f'(g(t)) N_1^\alpha g(t)$.

Definition 2 The non conformable fractional integral of order α is defined by the expression ${}_N J_{t_0}^\alpha f(t) ={}_{N_1} J_{t_0}^\alpha f(t) = \int_{t_0}^t \frac{f(s)}{e^{s^{-\alpha}}} ds$.

The following statement is analogous to the one known from the ordinary calculus.

Theorem 3 Let f be N-differentiable function in (t_0, ∞) with $\alpha \in (0, 1]$. Then for all $t > t_0$ we have

a) If f is differentiable ${}_N J_{t_0}^\alpha \left(N_1^\alpha f(t)\right) = f(t) - f(t_0)$.
b) $N_1^\alpha \left({}_N J_{t_0}^\alpha f(t)\right) = f(t)$.

Proof

a) From the given definition, we have

$$ {}_N J_{t_0}^\alpha \left(N_1^\alpha f(t)\right) = \int_{t_0}^t \frac{N_1^\alpha f(s)}{e^{s^{-\alpha}}} ds = \int_{t_0}^t \frac{f'(s) e^{s^{-\alpha}}}{e^{s^{-\alpha}}} ds = f(t) - f(t_0). $$

b) Analogously, we have

$$ N_1^\alpha \left({}_N J_{t_0}^\alpha f(t)\right) = e^{t^{-\alpha}} \frac{d}{dt} \left[\int_{t_0}^t \frac{f(s)}{e^{s^{-\alpha}}} ds \right] = f(t). $$

2. Preliminary results.

The equation (2) is equivalent to the system:

$$ \left. \begin{aligned} N_1^\alpha x &= y - F(x), \\ N_1^\alpha y &= -a(t) g(x), \end{aligned} \right\} \tag{3} $$

with $F(x) ={}_{N_1} J_{t_0}^\alpha f(t) = \int_{t_0}^t \frac{f(s)}{e^{s^{-\alpha}}} ds$. The regularity of functions involved in this system ensures existence and uniqueness of solutions of (3). The condition a) shows that $(0, 0)$ is the only point of equilibrium of this system and the condition b) ensures that results obtained are in global sense. From [21], obtain that condition c) is consistent with common sense.

Let β be a given real. We indicate by Ω_β the following open set:
$\Omega_\beta \equiv \mathbb{R}^2$ if $\beta \equiv 0$;
$\Omega_\beta = \left\{(x,y) : y > F(x) - \beta^{-1}\right\}$ if $\beta>0$;
$\Omega_\beta = \left\{(x,y) : y < F(x) - \beta^{-1}\right\}$ if $\beta<0$. Let $F_g(\mathbb{R}) = \{f \in C(\mathbb{R}) : for\ x \geq 0, f(x) - \beta A g(x) > 0$ and for $x \leq 0,\ f(x) - \beta A g(x) < 0\}$.
Consider the positive definite function V_β given by

$$V_\beta(t,x,y) = \frac{1}{a(t)} W_\beta(x,y) + G(x), \tag{4}$$

with $G(x) = {}_{N_1} J^{\alpha}_{t_0} g(x)$ and $W_\beta(x,y) = \int_0^{y-F(x)} \frac{s ds}{\beta s+1}$. It can be immediately verified that the derivative of V related to the system (3) is

$$\begin{aligned} N_1^\alpha V_\beta(t,x,y) \quad \leq \quad & -\frac{N_1^\beta a(t)}{a^2(t)} W_\beta(x,y) - \frac{(y-F(x))^2\,(f(x) - \beta a(t) g(x))}{a(t)\,[\beta(y-F(x)) + 1]} - \\ & -\frac{N_1^\alpha a(t)}{a^2(t)} \frac{(y-F(x))^2}{2} - \frac{(y-F(x))^2}{a(t)} f(x) \end{aligned} \tag{5}$$

Because $\beta(y - F(x)) + 1 > 0$ for all $(x,y) \in \Omega_\beta$, it follow that the sign of $N_1^\alpha V_\alpha(t,x,y)$ is the same of $f(x) - \beta a(t) g(x)$. We observe that if $f(x) \in F_g(\mathbb{R})$, then $f(x) - \beta a(t) g(x) > 0$. In view of this fact we have the following result.

Lemma 1 Under conditions a)-c) if $f(x) \in F_g(\mathbb{R})$ and $N_1^\alpha a(t) > 0$, then all solutions of system (3) are continuable to the future, i.e., for all $t \geq t_0 \geq 0$.

Proof It is known (see [17]) that a solution $(x(t), y(t))$ of system (3) is not continuable to $+\infty$ if there is a certain $T>t_0$ such that

$$\lim_{t \to T^-} \left(x^2(t) + y^2(t)\right) = +\infty \tag{6}$$

Let $(x(t), y(t))$ be a solution of (3) satisfying (6). From (5) we obtain that V_β is a decreasing function along solutions of system (5). So, we have that

$$A^{-1} W_\beta(x(t), y(t)) \leq V_\beta(t, x(t), y(t)) \leq V_\beta(t_0, x_0, y_0),$$

where $(x_0, y_0) = (x(t_0), y(t_0))$. From here and condition b) we obtain that *x(t)* is equibounded, i.e., there is $M>0$ such that

$$|x(t)| < M \text{ for } t_0 \leq t \leq T. \tag{7}$$

This completes the proof.

3. Main Results.

Now we will establish various results on the oscillatory character of system (3). For this we will redefine the functions $b(t)$ and $c(t)$ used in the classic case. By this way, we have $b(t) = \exp\left(-_{N_1} J_0^\alpha \left(\frac{N_1^\alpha a(s)^-}{a(s)}\right)(t)\right)$, $c(t) = a(0) \exp\left(-_{N_1} J_0^\alpha \left(\frac{N_1^\alpha a(s)^+}{a(s)}\right)(t)\right)$,

$N_1^\alpha a(t)^- = \max(-N_1^\alpha a(t), 0)$, $N_1^\alpha a(t)^+ = \max(N_1^\alpha a(t), 0)$ and $N_1^\alpha a(t) = N_1^\alpha a(t)^+ - N_1^\alpha a(t)^-$.

So, we now sate the following theorem.

Theorem 4 Let the following conditions hold:

1) ${}_{N_1}J_0^\alpha \left(\frac{N_1^\alpha a(t)^-}{a(t)}\right)(+\infty) < \infty$.
2) $xg(x)>0, \quad x \neq 0$.
3) There is $N > 0$ such that $|F(x)| \leq N$ for $x \in \mathbb{R}$. Then all solutions of the system (3) are oscillatory if and only if

$$ {}_{N_1}J_{t_0}^\alpha a(t) g\left[\pm k(t-t_0)\right](+\infty) = \pm\infty, \tag{8} $$

for all $k>0$ and all $t_0 \geq 0$.

Proof Necessity. We suppose that all solution of (3) are oscillatory, but condition (8) is not satisfy for some $k > 0$. We shall construct a non-oscillatory solution of system (3). Making in (8) $s = \pm k(t-t_0)$, then we have

$$ \pm k_{N_1} J_{t_0}^\alpha a(t) g[\pm k(t-t_0)](+\infty) =_{N_1} J_{t_0}^\alpha a(\pm\frac{s}{k} + t_0) g(s)(\pm\infty). $$

Thus, it follows that

$$ {}_{N_1}J_{t_0}^\alpha a(\pm\frac{s}{k} + t_0) g(s)(\pm\infty) = M < +\infty, $$

for some $k > 0$ and some $t_0 \geq 0$. We consider a solution of system (3), $(x(t), y(t))$ such that $x(t_0) = 0$, $y(t_0) = A$ with $A>k+N$. While that $y(t) > k+N$ we have $N_1^\alpha x(t) \geq k>0$. From this inequality, after integration between t_0 and t, we obtain $x(t) \geq k(t-t_0)$. Then, there is $x^{-1}(s)$ such that $x^{-1}(s) \leq \frac{s}{k} + t_0$. We also have that $0<b_1 \leq b(t) \leq 1$ for $0 \leq t< +\infty$, for some b_1.

Since $a(t) = b(t)c(t)$, then we obtain

$$ \begin{aligned} M &= {}_{N_1}J_{t_0}^\alpha a(t) g[k(t-t_0)](+\infty) =_{N_1} J_{t_0}^\alpha b(t)c(t) g[k(t-t_0)](+\infty) \geq \\ &\geq b_{1\,N_1} J_{t_0}^\alpha c(t) g[k(t-t_0)](+\infty) \end{aligned} $$

and hence it follows that

$$ N_1 J_{t_0}^\alpha c(t) g[k(t-t_0)](+\infty) \leq \frac{M}{b_1} \equiv M_1. $$

From the second equation of system (3) we deduce that

$$ \frac{N_1^\alpha y(t)}{b(t)} = c(t) g(x(t)). \tag{9} $$

Thus, it is clear that $N_1^\alpha y(t) \geq \frac{N_1^\alpha y(t)}{b(t)} = c(t)g(x(t))$. Integrating (9) between t_0 and t and taking into account the above inequality, we have

$$ \begin{aligned} y(t) &\geq y(t_0) - N_1 J_{t_0}^\alpha c(s) g(x(s))(t) \geq A - \frac{1}{k} N_1 J_{t_0}^\alpha c(s) g(x(s)) N_1^\alpha x(t)(t) = \\ &= A - \frac{1}{k} N_1 J_{t_0}^\alpha c(x^{-1}(s)) g(s)(x(t)). \end{aligned} $$

Since $x^{-1}(s) \leq \frac{s}{k} + t$, then we have $c(x^{-1}(s)) \leq c(\frac{s}{k} + t_0)$. Hence, we obtain

$$y(t) \geq A - \frac{1}{k}\,{}_{N_1}J_0^{\alpha} c\left(\frac{s}{k} + t_0\right) g(s)(x(t)) \geq A - \frac{M_1}{k}.$$

Taking A such that $A - \frac{M_1}{k} \geq k + N$ for $t \geq t_0$, then we have that $x(t) \geq k(t - t_0) \to +\infty$ as $t \to +\infty$. This is a contradictory with the initial supposition. Thus, we have the necessity of condition (8). Next, the case $x \leq 0$ can be proved in a similar way.

Sufficiency. Let $(x(t), y(t))$ be the solution of system (3) leaving a point $B(x_0, F(x_0))$, at $t = 0$. Suppose that $(x(t), y(t))$ does not traverse the $y - axis$. Then $(x(t), y(t))$ stays in the region $R_2 = \{(x, y) : x \geq 0, y < F(x)\}$ as long as the solution is defined for $t \geq 0$. Hence, $N_1^{\alpha} x(t) < 0$ and therefore $x(t) \leq x(t_0)$. Let $F_1 = \max_{0 \leq x \leq x_0} |F(x)|$, then the solution $(x(t), y(t))$ does not traverse the curve

$$V_{\beta}(t, x(t), y(t)) = \overline{V_{\beta}}(x_0, F(x_0)) = \frac{1}{A} \quad {}_{N_1}J_0^{\alpha}\left(\frac{s}{\alpha s + 1}\right)(F(x_0) + N_1) + G(x_0)$$

as t increases. Therefore, the orbit $(x(t), y(t))$ traverses the $y - axis$ at $C(0, y_C)$. Since $N_1^{\alpha} x = 0$ and $N_1^{\alpha} y < 0$ on the curve $y = F(x)$ in the region $x > 0, F(0) = 0$ implies that $y_C \leq 0$. Thus the orbit traverses the negative $y - axis$ at some finite time t_1. We choose $x(t_1) = 0$, $y(t_1) = y_C$. In the region $R_3 = \{(x, y) : x \leq 0, y < F(x)\}$, $N_1^{\alpha} x(t) \leq y_C$, so we have $x(t) \leq y_C(t - t_0)$ from here $x^{-1}(s) \geq \frac{s}{y_C} + t_0$ and $\frac{N_1^{\alpha} y}{C_1} \geq -d(t) g(x(t))$. It follows then, for all $t > t_1$, that

$$y(t) \geq y_C - \frac{b_1}{y_C}\,{}_{N_1}J_{t_0}^{\alpha} c(s) g(x(s))(N_1^{\alpha} x(s))(t).$$

Hence, we have

$$y(t) \geq y_C - \frac{b_1}{y_C}\,{}_{N_1}J_{t_0}^{\alpha} c\left(\frac{r}{y_0} + t_0\right) g(r)(x(t)). \tag{10}$$

Since $y(t) < F(x(t))$ if $x(t) \to +\infty$, then from above inequality we have that $y(t) \to +\infty$, and the orbit $(x(t), y(t))$ traverses the curve $y = F(\mathtt{x})$. Now consider the region $R_3 = \{(x, y) : x < 0, y > F(x)\}$, here $x\prime(t) > 0, y\prime(t) > 0$, the analysis of phases velocities show the existence of a point $D(0, y_D)$ on the $y - axis$ positive. If $x(t)$ is bounded, i.e., $x(t_1) \geq x(t) \geq M$ we have that $x(t) \to M^-$ while that $y(t)$ is increasing. Again an analysis of phases velocities show that there is a finite time $t\prime$ such that $y(t\prime) = F(x(t\prime))$. This completes the proof of the theorem.

Remark 3 The simple case $N_1^{\alpha}(N_1^{\alpha} x) - N_1^{\alpha} x + \alpha t^{-(\alpha+1)} x = 0$, with non-oscillatory solution $x(t) = e^t$, shows that positivity of f is probably necessary in some sense. This is an open problem.

Theorem 5 Under assumptions of Lemma 2 the following conditions:

1) $N_1^{\alpha} a(t) > 0$ for $t \geq 0$,
2) $|F(x)| \leq N$ for some $N > 0$ and $x \in \mathbb{R}$,
3) $G(\infty) = \infty$,

hold. Then the solutions of the equation (2) are bounded if and only if the condition (8) is fulfilled.

Proof We suppose that condition (8) is fulfilled. Then all solutions of are oscillatory. In this case $c(t) = a(t)$ for all $t \geq t_0 \geq 0$. We taking in account the function V_{β} defined in (4) and his total derivative (5) we have that:

$$V_\beta(t, x(t), y(t)) \leq V_\beta(t_0, x(t_0), y(t_0)).$$

From Theorem 3 there are $t_2 \geq t_1 \geq t_0$ such that $x(t_1) > 0$, $x(t_2) < 0$, and $y(t_1) = F(x(t_1))$, $y(t_2) = F(x(t_2))$. Also we obtain, from decreasing of functions V_β, that:

$$V_\beta(t, x(t), y(t)) \leq V_\beta(t_1, x(t_1), y(t_1)) = G(x(t_1))$$

and consequently:

$$G(x(t)) \leq G(x(t_1)).$$

From this we obtain that $x(t) \leq x(t_1)$. Similarly, we can obtain that $x(t_2) \leq x(t)$. So, putting $M = \max(-x(t_2), x(t_1))$ we have $|x(t)| \leq M$ for $t \geq \max\{t_2, t_1\}$. This prove the sufficiency. In Theorem 3 we proved that if the condition is not true, there are unbounded solutions of equation (2). Thus the proof of theorem is finished.

Lemma 2 If in addition to conditions of previous the theorem we have that g(x) is not increasing function and $a(t) \to +\infty$ as $t \to +\infty$. Then condition (8) does not hold.

Proof If condition (8) is not valid, then there exits $k>0$ and $t_0 \geq 0$ such that

$${}_{N_1}J^{\alpha}_{t_0} a(t) g\left[k(t - t_0)\right](+\infty) = M^* < +\infty,$$

(the negative case is similar). From Lemma 2, the equation (2) has non-oscillatory solutions defined for $t \geq t_0 \geq 0$. We consider a solution $x = x(t)$ with this property, without loss of generality we can suppose that there exists $T_1 \geq t_0$ such that for some m, $a(t)>m$ if $t \geq T_1$ (the case $x(t)<-m<0$ is analogous). It is easy follow that for $m>0$ there exists $T_2 \geq t_0$ such that:

$$k(t - t_0)>m>0, t \geq T_2. \tag{11}$$

By using the above inequality and assumptions on g, we have

$$g[k(t - t_0)] \geq g(m)>0, t \geq T_2.$$

Therefore, we obtain that

$$a(t)g(m) \leq a(t)g[k(t - t_0)), t \geq T_2. \tag{12}$$

Let us consider $T = \max\{T_1, T_2\}$. After integration of (12) between T and $+\infty$, we obtain

$$g(m)_{N_1}J^{\alpha}_{T} a(t)(+\infty) \leq_{N_1} J^{\alpha}_{T} a(t)g[k(t - t_0)](+\infty) = M^* < +\infty,$$

hence

$${}_{N_1}J^{\alpha}_{T} a(t)(+\infty) \leq \frac{M^*}{g(m)} < +\infty. \tag{13}$$

Since $a(t) \to +\infty$ as $t \to +\infty$, then we have that

$${}_{N_1}J^{\alpha}_{T} a(t)(+\infty) = +\infty,$$

which is a contradiction to (13). Hence, the condition (8) holds. Thus, the proof is now complete.

Corollary 1 Under the conditions of Lemma 3 all solutions of equation (2) are oscillatory if the following conditions hold:

a) ${}_{N_1}J_0^\alpha\left(\frac{N_1^\alpha a(t)^-}{a(t)}\right)(+\infty)<\infty$.

b) There exists $N>0$ such that $F(x)\le N$ for $x\in\mathbb{R}$.

Proof Taking into account Lemma 3, Lemma 2 and Theorem 8, the proof can be easily completed We omit the details of the proof.

Corollary 2 Under condition of Lemma 3 all solutions of equation (2) are bounded if the following conditions hold:

a) $N_1^\alpha a(t)>0$ for all $t\ge 0$.

b) there exists $N>0$ such that $F(x)\le N$ for $x\in\mathbb{R}$.

Proof It is enough applying Lemma 3 and Theorem 3.

Theorem 6 Under condition Lemma 2 if the conditions

a) ${}_{N_1}J_0^\alpha\left(\frac{N_1^\alpha a(t)^-}{a(t)}\right)(+\infty)<\infty$,

b) $G(x)\to+\infty$ as $|x|\to+\infty$,

hold, then all solutions of equation (2) are bounded.

Proof By similar arguments to sufficiency of Theorem 3 we obtain that there exists $R>0$ such that $|x(t)|\le R$.

4. Conclusions

In this work, we study the oscillatory character of non-conformable equation (2) of order 2α by using the analysis of the phase plane. By this way, we extend known results for classic Liénard equation and fractional differential equations with the classical derivative of Caputo or Riemann-Liouville. In particular, in [13] we study the Equation of Liénard in the framework of the fractional derivative of Caputo, since it did not have a similar result to (5) the conclusions obtained were derived by methods less "usual"; in the case of [2] the study of an equation of order $1+\alpha$, was based on the properties of nonlocal fractional calculus generated by conformable derivatives. Hence, the results differ from ours; en [23] with the definition of conformable fractional derivative and using averaging functions obtain oscillation results for an equation of order α with $1<\alpha<2$; finally in [32] they also worked on a conformable fractional differential equation; our results are different from all of these, since the derivative used "does not return" the classical derivative when α tends to 1, so the equation studied cannot be reduced to the classical Liénard equation. The example presented in Remark 3 shows the consistency of our results.

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