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INTEGRAL REPRESENTATIONS AND OPERATIONAL RELATIONS INVOLVING SOME QUADRUPLE HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. In the present work, we establish a new integral representations of Euler type for the quadruple hypergeometric functions $X_i^{(4)}(.)$, $i = 38, 40, 45, 48, 50$. We obtain some operational relations include these quadruple functions.

1. Introduction

The multiple hypergeometric functions arise in a variety appliecations in various fields. For example, many auxiliary algebraic and integral transformations in various physical models give rise to these functions. The studies in mathematical physics, atomic and molecular physics, quantum mechanics, quantum chemistry, dynamics, networks, etc. have led to increasing interest in the study of hypergeometric functions of several variables (see e.g., [1], [11], [14], [15], [16]). Many applications and properties of these functions are found in the work of various researchers (see e.g., [2], [3], [4], [5], [6], [7], [9], [10], [12], [13], [17], [20]). More recently, Younis and Bin-Saad [25] discovered the existence of twenty additional complete quadruple hypergeometric functions $X_{31}^{(4)}, X_{32}^{(4)}, ..., X_{50}^{(4)}$. For the purpose of the present work, we choose the functions $X_{38}^{(4)}, X_{40}^{(4)}, X_{45}^{(4)}, X_{48}^{(4)}, X_{50}^{(4)}$ defined, respectively, by the following:

$$X_{38}^{(4)}(a_1, a_1, a_2, a_2, a_1, a_3, a_2, a_3; c, c, c, c; x, y, z, u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{2p+q}(a_3)_{n+q}}{(c)_{m+n+p+q}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{1}$$

$$X_{40}^{(4)}(a_1, a_1, a_1, a_2, a_1, a_1, a_2, a_3; c_1, c_2, c_1, c_3; x, y, z, u) = \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{2m+2n+p}(a_2)_{p+q}(a_3)_q}{(c_1)_{m+p}(c_2)_n(c_3)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \tag{2}$$

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$$X_{45}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_3,c_3;x,y,z,u\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+2n+p}(a_2)_p(a_3)_q(a_4)_q}{(c_1)_m(c_2)_n(c_3)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{3}$$

$$X_{48}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_2,a_2;c_1,c_2,c_3,c_4;x,y,z,u\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n+p+q}(a_2)_{n+p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}, \tag{4}$$

$$X_{50}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_1,a_1,a_2;c_1,c_2,c_3,c_1;x,y,z,u\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+2n+2p+q}(a_2)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}. \tag{5}$$

Here, $(a)_m$ is the Pochhammer symbol defined by

$$(a)_m=\frac{\Gamma(a+m)}{\Gamma(a)}=a(a+1)...(a+m-1)$$

for $m\geq 1$, $(a)_0=1$, where the notation Γ is used for the gamma function.

The Gauss hypergeometric function is defined by (see [23])

$${}_2F_1\left(a,b;c;x\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!},\,(|x|<1). \tag{6}$$

Appell's functions F_1,F_2,F_3,F_4 and Horn's functions H_3,H_4 are respectively defined by [23]

$$F_1\left(a,b,c;d;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{7}$$

$$F_2\left(a,b,c;d,e;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{8}$$

$$F_3\left(a,b,c,d;e;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n(c)_m(d)_n}{(e)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{9}$$

$$F_4\left(a,b;c,d;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(d)_n}\frac{x^m}{m!}\frac{y^n}{n!}, \tag{10}$$

$$H_3(a,b;c;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{11}$$

and

$$H_4(a,b;c,d;x,y)=\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(b)_n}{(c)_m(d)_n}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{12}$$

Exton [10] introduced twenty distinct triple hypergeometric functions, which he denoted by $X_1, X_2, ..., X_{20}$. We present below the definitions of some of these functions

$$X_1\left(a_1, a_2; c_1, c_2; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+2n+p}(a_2)_p}{(c_1)_m(c_2)_{n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{13}$$

$$X_2\left(a_1, a_2; c_1, c_2, c_3; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+2n+p}(a_2)_p}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{14}$$

$$X_4\left(a_1, a_2; c_1, c_2, c_3; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n+p}(a_2)_{n+p}}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{15}$$

$$X_{13}\left(a_1, a_2, a_3; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p}{(c)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{16}$$

$$X_{17}\left(a_1, a_2, a_3; c_1, c_2, c_3; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}, \tag{17}$$

$$X_{18}\left(a_1, a_2, a_3, a_4; c; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{2m+n}(a_2)_n(a_3)_p(a_4)_p}{(c)_{m+n+p}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!}. \tag{18}$$

In [13], Lauricella generalized the hypergeometric functions in two variables to n-variable hypergeometric functions. In the case of $n = 3$, he defined and studied three variable Lauricella functions $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$, $F_D^{(3)}$. From them, the triple hypergeometric function $F_C^{(3)}$ is defined by

$$F_C^{(3)}\left(a, b; c_1, c_2, c_3; x, y, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a)_{m+n+p}(b)_{m+n+p}}{(c_1)_m(c_2)_n(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p}. \tag{19}$$

Lauricella hypergeometric function of four variables $F_C^{(4)}$ is as below ([13], [23])

$$F_C^{(4)}\left(a, b; c_1, c_2, c_3, c_4; x, y, z, u\right)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b)_{m+n+p+q}}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p} \frac{u^q}{q!}. \tag{20}$$

Sharma and Parihar [19] defined eighty three complete hypergeometric functions of four variables denoted these by $F_1^{(4)}, F_2^{(4)}, ..., F_{83}^{(4)}$. One of them is presented as follows:

$$F_{14}^{(4)}\left(a_1, a_1, a_1, a_2, b_1, b_1, b_1, b_2; c_1, c_2, c_3, c_1; x, y, z, u\right)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n+p}(b_2)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}. \tag{21}$$

Generalized Horn's function ${}^{(p)}H_4^{(n)}$ [23] defined by

$${}^{(p)}H_4^{(n)}\left(a, b_{p+1}, ..., b_n; c_1, ..., c_n; x_1, ..., x_n\right)$$
$$= \sum_{m_1,...,m_n=0}^{\infty} \frac{(a)_{2(m_1+...+m_p)+m_{p+1}+...+m_n}(b_{p+1})_{m_{p+1}}...(b_n)_{m_n}}{(c_1)_{m_1}...(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}...\frac{x_n^{m_n}}{m_n!}. \tag{22}$$

Very recently, Bin-Saad et al. ([5], [6]) investigated that there exist ten additional complete quadruple hypergeometric functions, which are $X_1^{(4)}, X_2^{(4)}, ..., X_{10}^{(4)}$ and they had not been included in Exton's [9] and Sharma's and Parihar's [19] set. Here, we give the definition of one of these functions

$$X_1^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_1,a_2,a_2;c_2,c_1,c_1,c_3;x,y,z,u\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+2n+p+q}(a_2)_{p+q}}{(c_1)_{n+p}(c_2)_m(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}. \tag{23}$$

The quadruple hypergeometric function $X_{39}^{(4)}$ is given by [25]

$$X_{39}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_3,c_4;x,y,z,u\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+2n+p}(a_2)_{p+q}(a_3)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!}. \tag{24}$$

This paper is divided into three sections. Section 2 deals with the integrals of Euler-type involving the Gaussian hypergeometric function ${}_2F_1$, Appell's functions F_3 and F_4, Horn's functions H_3 and H_4, Exton's triple functions X_1, X_2 and X_4, Lauricell's triple function $F_C^{(3)}$ and the hypergeometric functions of four variables ${}^{(3)}H_4^{(4)}, F_C^{(4)}, F_{14}^{(4)}, X_1^{(4)}$ and $X_{39}^{(4)}$. In Section 3, some interesting and useful operational relations between the hypergometric functions of four variables $X_{38}^{(4)}, X_{40}^{(4)}, X_{45}^{(4)}, X_{48}^{(4)}, X_{50}^{(4)}$ and the hypergeometric functions of one variable ${}_2F_1$, the hypergeometric functions of two variables F_1, F_2, F_3, F_4, H_4 and the hypergeometric functions of three variables $X_1, X_2, X_{13}, X_{17}, X_{18}, F_C^{(3)}$ have been studied. In Section 4, we give a number of special cases of the results of Section 3.

2. Integral representations of Euler-Type

In this section we establish three new integral representations of Euler-type for each functions $X_i^{(4)}(i=38,40,45,48,50)$ as follows:

$$X_{38}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c,c,c,c;x,y,z,u\right)=\frac{\Gamma(c)}{2^{c-1}\Gamma(a_3)\Gamma(c-a_3)}$$
$$\times\int_{-1}^{1}(1+\alpha)^{a_3-1}(1-\alpha)^{c-a_3-1}\left[1-\frac{1}{2}(1+\alpha)y\right]^{-a_1}\left[1-\frac{1}{2}(1+\alpha)u\right]^{-a_2}$$
$$\times F_3\left(\frac{a_1}{2},\frac{a_2}{2},\frac{a_1+1}{2},\frac{a_2+1}{2};c-a_3;\frac{2(1-\alpha)x}{\left[1-\frac{1}{2}(1+\alpha)y\right]^2},\frac{2(1-\alpha)z}{\left[1-\frac{1}{2}(1+\alpha)u\right]^2}\right)d\alpha$$
$$\left(\Re(a_3)>0,\Re(c-a_3)>0\right), \tag{25}$$

$$X_{38}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c,c,c,c;x,y,z,u\right)=\frac{2M^{a_1}\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}$$
$$\times\int_0^{\infty}cosh\alpha\left(sinh^2\alpha\right)^{a_1-\frac{1}{2}}\left(1+Msinh^2\alpha\right)^{-(a_1+a_2)}H_3\left(a_1+a_2,a_3;c;\right.$$
$$\left.\frac{M^2x\ sinh^4\alpha}{\left(1+Msinh^2\alpha\right)^2}+\frac{z}{\left(1+Msinh^2\alpha\right)^2},\frac{My\ sinh^2\alpha}{(1+Msinh^2\alpha)}+\frac{u}{(1+Msinh^2\alpha)}\right)d\alpha$$
$$\left(\Re(a_1)>0,\Re(a_2)>0,M>0\right), \tag{26}$$

$$X_{38}^{(4)}\left(a_1,a_1,a_2,a_2,a_1,a_3,a_2,a_3;c,c,c,c;x,y,z,u\right)=\frac{\Gamma(a_1+a_2+a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$
$$\times\int_0^1\int_0^1\alpha^{a_1-1}\left(1-\alpha\right)^{a_2-1}\beta^{a_1+a_2-1}\left(1-\beta\right)^{a_3-1}{}_2F_1\left(\frac{a_1+a_2+a_3}{2},\right.$$
$$\frac{a_1+a_2+a_3+1}{2};c;4\alpha^2\beta^2x+4\alpha\beta(1-\beta)y+4(1-\alpha)^2\beta^2z+$$
$$\left.4(1-\alpha)\beta(1-\beta)u\right)d\alpha d\beta$$
$$\left(\Re(a_i)>0,(i=1,2,3)\right), \tag{27}$$

$$X_{40}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_1,c_3;x,y,z,u\right)=\frac{\Gamma(a_1+a_3)}{\Gamma(a_1)\Gamma(a_3)}\int_{-\frac{1}{2}}^{\frac{1}{2}}$$
$$\times\left(\frac{1}{2}+\alpha\right)^{a_1-1}\left(\frac{1}{2}-\alpha\right)^{a_3-1}X_1^{(4)}\left(a_1+a_3,a_1+a_3,a_1+a_3,a_1+a_3,a_1+a_3,\right.$$
$$\left.a_1+a_3,a_2,a_2;c_2,c_1,c_1,c_3;\left(\frac{1}{2}+\alpha\right)^2y,\left(\frac{1}{2}+\alpha\right)^2x,\left(\frac{1}{2}+\alpha\right)z,\left(\frac{1}{2}-\alpha\right)u\right)d\alpha$$
$$\left(\Re(a_1)>0,\Re(a_3)>0\right), \tag{28}$$

$$X_{40}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_1,c_3;x,y,z,u\right)=\frac{2\Gamma(c_1)}{\Gamma(a)\Gamma(c_1-a)}\int_0^{\frac{\pi}{2}}$$
$$\times\left(sin^2\alpha\right)^{a-\frac{1}{2}}\left(cos^2\alpha\right)^{c_1-a-\frac{1}{2}}X_{39}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;a,c_2,c_1-a,c_3;\right.$$
$$\left.x\ sin^2\alpha,y,z\ sin^2\alpha,u\right)d\alpha$$
$$\left(\Re(a)>0,\Re(c_1-a)>0\right), \tag{29}$$

$$X_{40}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;c_1,c_2,c_1,c_3;x,y,z,u\right)=\frac{\Gamma(c_1)(S-R)^{2+a_1+a_2-2c_1}}{\Gamma(a_1)\Gamma(c_1-a_1)}$$
$$\times\int_R^S(\alpha-R)^{a_1-1}\left[(S-R)(S-\alpha)+(\alpha-R)^2x)\right]^{c_1-a_1-1}\left[(S-R)-(\alpha-R)z)\right]^{-a_2}$$
$$\times{}_2F_1\left(\frac{1+a_1-c_1}{2},\frac{a_1-c_1}{2}+1;c_2;\frac{4(S-R)^2(\alpha-R)^2y}{\left[(S-R)(S-\alpha)+(\alpha-R)^2x)\right]^2}\right)$$
$$\times{}_2F_1\left(a_2,a_3;c_3;\frac{(S-R)u}{\left[(S-R)-(\alpha-R)z)\right]}\right)d\alpha$$
$$\left(\Re(a_1)>0,\Re(c_1-a_1)>0,R<S\right), \tag{30}$$

$$X_{45}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_3,c_3;x,y,z,u\right)=\frac{\Gamma(a_1+a_2)}{2^{a_1+a_2-1}\Gamma(a_1)\Gamma(a_2)}$$
$$\times\int_{-1}^{1}\left(1+\alpha\right)^{a_1-1}\left(1-\alpha\right)^{a_2-1}F_{14}^{(4)}\left(a_1+a_2,a_1+a_2,a_1+a_2,a_3,a_1+a_2,a_1+a_2,\right.$$
$$\left.a_1+a_2,a_4;c_3,c_1,c_2,c_3;\frac{1}{4}\left(1+\alpha\right)\left(1-\alpha\right)z,\frac{1}{4}\left(1+\alpha\right)^2x,\frac{1}{4}\left(1+\alpha\right)^2y,u\right)d\alpha$$

$$(\Re(a_1)>0,\Re(a_2)>0)\,, \tag{31}$$

$$X_{45}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_3,c_3;x,y,z,u\right)=\frac{\Gamma(c_3)}{\Gamma(a_2)\Gamma(a_3)\Gamma(c_3-a_2-a_3)}$$
$$\times\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\frac{1}{2}+\alpha\right)^{a_2-1}\left(\frac{1}{2}-\alpha\right)^{c_3-a_2-1}\left(\frac{1}{2}+\beta\right)^{a_3-1}\left(\frac{1}{2}-\beta\right)^{c_3-a_2-a_3-1}$$
$$\times\left[1-\left(\frac{1}{2}+\alpha\right)z\right]^{-a_1}\left[1-\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}+\beta\right)u\right]^{-a_4}$$
$$\times F_4\left(\frac{a_1}{2},\frac{a_1+1}{2};c_1,c_2;\frac{4x}{\left[1-\left(\frac{1}{2}+\alpha\right)z\right]^2},\frac{4y}{\left[1-\left(\frac{1}{2}+\alpha\right)z\right]^2}\right)d\alpha d\beta$$
$$(\Re(a_1)>0,\Re(a_3)>0,\Re(c_3-a_3)>0,\Re(c_3-a_2-a_3)>0)\,, \tag{32}$$

$$X_{45}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,a_4;c_1,c_2,c_3,c_3;x,y,z,u\right)=\frac{\Gamma(c_3)}{2^{c_3-2}\Gamma(a_4)\Gamma(c_3-a_4)}$$
$$\times\int_{-1}^{1}\left[(1+\alpha)^2\right]^{a_4-\frac{1}{2}}\left[(1-\alpha)^2\right]^{c_3-a_4-\frac{1}{2}}\left(1+\alpha^2\right)^{c_3-a_3}\left[(1+\alpha^2)-\frac{1}{2}(1+\alpha)^2u)\right]^{-a_3}$$
$$\times X_2\left(a_1,a_2;c_1,c_2,c_3-a_4;x,y,\frac{(1-\alpha)^2z}{2(1+\alpha^2)}\right)d\alpha$$
$$(\Re(a_4)>0,\Re(c_3-a_4)>0)\,, \tag{33}$$

$$X_{48}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_2,a_2;c_1,c_2,c_3,c_4;x,y,z,u\right)=\frac{2\Gamma(a_1+a_2)}{\Gamma(a_1)\Gamma(a_2)}\int_0^{\frac{\pi}{2}}$$
$$\times\left(sin^2\alpha\right)^{a_1-\frac{1}{2}}\left(cos^2\alpha\right)^{a_2-\frac{1}{2}}F_C^{(4)}\left(\frac{a_1+a_2}{2},\frac{a_1+a_2+1}{2};c_1,c_2,c_3,c_4;4x\ sin^4\alpha,\right.$$
$$\left.y\ sin^2 2\alpha,z\ sin^2 2\alpha,u\ sin^2 2\alpha\right)d\alpha$$
$$(\Re(a_1)>0,\Re(a_2)>0)\,, \tag{34}$$

$$X_{48}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_2,a_2;c_1,c_2,c_3,c_4;x,y,z,u\right)=\frac{2M^{a_2}\Gamma(c_2)}{\Gamma(a_2)\Gamma(c_2-a_2)}\int_0^{\infty}$$
$$\times cosh\alpha\left(sinh^2\alpha\right)^{a_2-\frac{1}{2}}\left(1+Msinh^2\alpha\right)^{a_1-c_2}\left[(1+Msinh^2\alpha)-My\ sinh^2\alpha\right]^{-a_1}$$
$$\times X_4\left(a_1,1+a_2-c_2;c_1,c_3,c_4;\frac{\left(1+Msinh^2\alpha\right)^2x}{\left[(1+Msinh^2\alpha)-My\ sinh^2\alpha\right]^2},\right.$$
$$\left.\frac{-Mz\left(1+Msinh^2\alpha\right)sinh^2\alpha}{\left[(1+Msinh^2\alpha)-My\ sinh^2\alpha\right]},\frac{-Mu\left(1+Msinh^2\alpha\right)sinh^2\alpha}{\left[(1+Msinh^2\alpha)-My\ sinh^2\alpha\right]}\right)d\alpha$$
$$(\Re(a_1)>0,\Re(c_2-a_2)>0,M>0)\,, \tag{35}$$

$$X_{48}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_2,a_2;c_1,c_2,c_3,c_4;x,y,z,u\right)=\frac{\Gamma(c_3)}{\Gamma(a_1)\Gamma(a_2)}$$
$$\times\frac{\Gamma(c_4)}{\Gamma(c_3-a_1)\Gamma(c_4-a_2)}\int_0^{\infty}\int_0^{\infty}\alpha^{a_1-1}(1+\alpha)^{1+a_2-c_3-c_4}\beta^{a_2-1}(1+\beta)^{1+a_1-c_3-c_4}$$

$$\times\left[(1+\alpha)+\alpha\beta z\right]^{c_4-a_2-1}\left[(1+\beta)+\alpha\beta u\right]^{c_3-a_1-1} H_4\left(1+a_1-c_3, 1+a_2-c_4;\right.$$

$$\left. c_1, c_2; \frac{(1+\beta)^2 x}{\left[(1+\beta)+\alpha\beta u\right]^2}, \frac{\alpha\beta(1+\alpha)(1+\beta)y}{\left[(1+\alpha)+\alpha\beta z\right]\left[(1+\beta)+\alpha\beta u\right]}\right) d\alpha d\beta$$

$$\left(\Re(a_1)>0, \Re(a_2)>0, \Re(c_3-a_1)>0, \Re(c_4-a_2)>0\right), \tag{36}$$

$$X_{50}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_1,a_1,a_2;c_1,c_2,c_3,c_1;x,y,z,u\right)=\frac{\Gamma(c_1)}{\Gamma(a)\Gamma(c_1-a)(S-R)^{c_1-1}}$$

$$\times\int_R^S (\alpha-R)^{a-1}(S-\alpha)^{c_1-a-1}\,{}^{(3)}H_4^{(4)}\left(a_1,a_2;a,c_2,c_3,c_1-a;\frac{(\alpha-R)x}{(S-R)},y,z,\right.$$

$$\left.\frac{(S-\alpha)u}{(S-R)}\right) d\alpha$$

$$\left(\Re(a)>0, \Re(c_1-a)>0, R<S\right), \tag{37}$$

$$X_{50}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_1,a_1,a_2;c_1,c_2,c_3,c_1;x,y,z,u\right)=\frac{\Gamma(a_1+a_2)}{2^{a_1+a_2-2}\Gamma(a_1)\Gamma(a_2)}$$

$$\times\int_{-1}^{1}\left[(1+\alpha)^2\right]^{a_1-\frac{1}{2}}\left[(1-\alpha)^2\right]^{a_2-\frac{1}{2}}\left(1+\alpha^2\right)^{-(a_1+a_2)} F_C^{(3)}\left(\frac{a_1+a_2}{2},\frac{a_1+a_2+1}{2};\right.$$

$$\left. c_1,c_2,c_3;\frac{(1+\alpha)^4 x}{(1+\alpha^2)^2}+\frac{(1+\alpha)^2(1-\alpha)^2 u}{(1+\alpha^2)^2},\frac{(1+\alpha)^4 y}{(1+\alpha^2)^2},\frac{(1+\alpha)^4 z}{(1+\alpha^2)^2}\right) d\alpha$$

$$\left(\Re(a_1)>0, \Re(a_2)>0\right), \tag{38}$$

$$X_{50}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_1,a_1,a_2;c_1,c_2,c_3,c_1;x,y,z,u\right)=\frac{\Gamma(c_3)}{\Gamma(a_1)\Gamma(c_3-a_1)}$$

$$\times\int_0^\infty \alpha^{a_1-1}(1+\alpha)^{2c_3-a_1-1}\left[(1+\alpha)+\alpha^2 z\right]^{c_3-a_1-1} X_1\left(1+a_1-c_3,a_2;c_2,c_1;\right.$$

$$\left.\frac{\alpha^2(1+\alpha)^2 y}{\left[(1+\alpha)+\alpha^2 z\right]^2},\frac{\alpha^2(1+\alpha)^2 x}{\left[(1+\alpha)+\alpha^2 z\right]^2},\frac{-\alpha(1+\alpha)u}{\left[(1+\alpha)+\alpha z\right]}\right) d\alpha$$

$$\left(\Re(a_1)>0, \Re(c_3-a_1)>0\right). \tag{39}$$

Proof. To prove these integral representations (25) to (39), it is enough to expand functions inside of integral to the series and then, changing the order of the integral and the summation, and finally to use the following integral representations of the Beta function and their various associated Eulerian integrals (see, for example, [8], [21], [22], [24]):

$$B(a,b)=\begin{cases}\displaystyle\int_0^1 \alpha^{a-1}(1-\alpha)^{b-1}\,dt & \left(\Re(a)>0,\Re(b)>0\right),\\[2mm] \displaystyle\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} & (a,b\in\mathbb{C}\setminus\mathbb{Z}_0^-),\end{cases} \tag{40}$$

$$B(a,b)=\int_0^1 \alpha^{a-1}(1-\alpha)^{b-1}\,d\alpha=(S-R)^{1-a-b}\int_R^S(\alpha-R)^{a-1}(S-\alpha)^{b-1}\,d\alpha$$

$$\left(\Re(a)>0,\Re(b)>0,R<S\right), \tag{41}$$

$$B(a,b) = 2\int_0^{\frac{\pi}{2}} (sin\alpha)^{2a-1}(cos\alpha)^{2b-1}\ d\alpha = \int_0^{\infty} \frac{\alpha^{a-1}}{(1+\alpha)^{a+b}}\ d\alpha$$
$$(\Re(a) > 0, \Re(b) > 0)\,, \tag{42}$$

$$B(a,b) = 2^{1-a-b}\int_{-1}^{1} (1+\alpha)^{a-1}(1-\alpha)^{b-1}\ d\alpha = 2M^a \int_0^{\infty} \frac{cosh\alpha\,(sinh\alpha)^{2a-1}}{(1+Msinh^2\alpha)^{a+b}}\ d\alpha$$
$$(\Re(a) > 0, \Re(b) > 0, M > 0)\,. \tag{43}$$

3. Operational Relations

For the purpose of the present Section, we recall the following operational relations (see [2], [18]):

$$D_\alpha^n \alpha^r = \frac{\Gamma(r+1)}{\Gamma(r-n+1)}\alpha^{r-n}, D_\alpha^{-n}\alpha^r = \frac{\Gamma(r+1)}{\Gamma(r+n+1)}\alpha^{r+n},$$
$$n \in \mathbb{N}\cup\{0\}, r \in \mathbb{C} - \{-1,-2,...\}, \tag{44}$$

where D_α, D_α^{-1} are the derivative and integral operator, respectively.

Now, we give numerous operational connections among the quadruple functions $X_i^{(4)}(.)(i = 38, 40, 45, 48, 50)$ by using above formulas

$$\left[1 - \left(D_\alpha^2 \beta^{-1} D_\beta^{-1} \gamma^{-1} D_\gamma^{-1} \alpha^2\right) u\right]^{-a} X_{13}\left(a_1, a_2, a_3; c; \beta x, \beta y, \alpha\beta z\right)\left(\alpha^{a_3-1}\beta^{c-1}\gamma^{a-1}\right)$$
$$= \alpha^{a_3-1}\beta^{c-1}\gamma^{a-1} X_{38}^{(4)}\left(a_1, a_1, a_3, a_3, a_1, a_2, a_3, a_2; c, c, c, c; \beta x, \beta y, u, \alpha\beta z\right), \tag{45}$$

$$\left[1 - \left(D_{\alpha_1} D_{\alpha_2} \beta^{-1} D_\beta^{-1} \gamma^{-1} D_\gamma^{-1} \alpha_1\alpha_2\right) u\right]^{-a} X_{18}\left(a_1, a_2, \frac{a_3}{2}, \frac{a_3+1}{2}; c; \beta x, \alpha_1\beta y,\right.$$
$$\left.4\alpha_2^2\beta z\right)\left(\alpha_1^{a_2-1}\alpha_2^{a_3-1}\beta^{c-1}\gamma^{a-1}\right) = \alpha_1^{a_2-1}\alpha_2^{a_3-1}\beta^{c-1}\gamma^{a-1} X_{38}^{(4)}\left(a_1, a_1, a_3, a_3, a_1, a_2,\right.$$
$$\left.a_3, a_2; c, c, c, c; \beta x, \alpha_1\beta y, \alpha_2^2\beta z, u\right), \tag{46}$$

$$\left[1 - \left(D_{\alpha_1}^2 \beta^{-1} D_\beta^{-1} \gamma_1^{-1} D_{\gamma_1}^{-1} \alpha_1^2\right) z\right]^{-a}\left[1 - \left(D_{\alpha_2}^2 \beta^{-1} D_\beta^{-1} \gamma_2^{-1} D_{\gamma_2}^{-1} \alpha_2^2\right) u\right]^{-b}$$
$$\times F_1\left(a_1, a_2, a_3; c; \alpha_1\beta x, \alpha_2\beta y\right)\left(\alpha_1^{a_2-1}\alpha_2^{a_3-1}\beta^{c-1}\gamma_1^{a-1}\gamma_2^{b-1}\right)$$
$$= \alpha_1^{a_2-1}\alpha_2^{a_3-1}\beta^{c-1}\gamma_1^{a-1}\gamma_2^{b-1} X_{38}^{(4)}\left(a_2, a_2, a_3, a_3, a_2, a_1, a_3, a_1;\right.$$
$$\left.c, c, c, c; z, \alpha_1\beta x, u, \alpha_2\beta y\right), \tag{47}$$

$$\left[1 - \left(D_{\alpha_1} D_{\alpha_2} \beta^{-1} D_\beta^{-1} \gamma^{-1} D_\gamma^{-1} \alpha_1\alpha_2\right) u\right]^{-a} X_1\left(a_1, a_2, ; c_2, c_1; x, y, \alpha_1 z\right)$$
$$\times\left(\alpha_1^{a_2-1}\alpha_2^{a_3-1}\beta^{c_3-1}\gamma^{a-1}\right) = \alpha_1^{a_2-1}\alpha_2^{a_3-1}\beta^{c_3-1}\gamma^{a-1} X_{40}^{(4)}\left(a_1, a_1, a_1, a_2,\right.$$
$$\left.a_1, a_1, a_2, a_3; c_1, c_2, c_1, c_3; y, x, \alpha_1 z, u\right), \tag{48}$$

$$\left[1-\left(D_{\alpha}^{2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^{2}\right)u\right]^{-a}X_{17}\left(a_1,a_2,a_3;c_1,c_2,c_3;\alpha^2x,\alpha\beta y,z\right)$$
$$\times\left(\alpha^{a_1-1}\beta^{c_2-1}\gamma^{a-1}\right)=\alpha^{a_1-1}\beta^{c_2-1}\gamma^{a-1}X_{40}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_2,a_3;\right.$$
$$\left.c_2,c_1,c_2,c_3;u,\alpha^2x,\alpha\beta y,z\right), \tag{49}$$

$$\left[1-\left(D_{\alpha}^{2}\beta_1^{-1}D_{\beta_1}^{-1}\gamma_1^{-1}D_{\gamma_1}^{-1}\alpha^{2}\right)z\right]^{-a}\left[1-\left(D_{\alpha}^{2}\beta_2^{-1}D_{\beta_2}^{-1}\gamma_2^{-1}D_{\gamma_2}^{-1}\alpha^{2}\right)u\right]^{-b}$$
$$\times F_2\left(a_1,a_2,a_3;c_1,c_2;\alpha\beta_1x,y\right)\left(\alpha^{a_2-1}\beta_1^{c_1-1}\beta_2^{c_3-1}\gamma_1^{a-1}\gamma_2^{b-1}\right)$$
$$=\alpha^{a_2-1}\beta_1^{c_1-1}\beta_2^{c_3-1}\gamma_1^{a-1}\gamma_2^{b-1}X_{40}^{(4)}\left(a_2,a_2,a_2,a_1,a_2,a_2,a_1,a_3;\right.$$
$$\left.c_1,c_3,c_1,c_2;z,u,\alpha\beta_1x,y\right), \tag{50}$$

$$\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha_1\alpha_2\right)u\right]^{-a}X_{2}\left(a_1,a_2,;c_1,c_2,c_3;x,y,\beta z\right)$$
$$\times\left(\alpha_1^{a_3-1}\alpha_2^{a_4-1}\beta^{c_3-1}\gamma^{a-1}\right)=\alpha_1^{a_3-1}\alpha_2^{a_4-1}\beta^{c_3-1}\gamma^{a-1}X_{45}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,\right.$$
$$\left.a_1,a_2,a_4;c_1,c_2,c_3,c_3;x,y,\beta z,u\right), \tag{51}$$

$$\left[1-\left(D_{\alpha}^{2}\beta_1^{-1}D_{\beta_1}^{-1}\gamma_1^{-1}D_{\gamma_1}^{-1}\alpha^{2}\right)z\right]^{-a}\left[1-\left(D_{\alpha}^{2}\beta_2^{-1}D_{\beta_2}^{-1}\gamma_2^{-1}D_{\gamma_2}^{-1}\alpha^{2}\right)u\right]^{-b}$$
$$\times F_3\left(a_1,a_2,a_3,a_4;c_1;\alpha x,y\right)\left(\alpha^{a_1-1}\beta_1^{c_2-1}\beta_2^{c_3-1}\gamma_1^{a-1}\gamma_2^{b-1}\right)$$
$$=\alpha^{a_1-1}\beta_1^{c_2-1}\beta_2^{c_3-1}\gamma_1^{a-1}\gamma_2^{b-1}X_{45}^{(4)}\left(a_1,a_1,a_1,a_2,a_1,a_1,a_3,a_4;\right.$$
$$\left.c_2,c_3,c_1,c_1;z,u,\alpha x,y\right), \tag{52}$$

$$\left[1-\left(D_{\alpha_1}^{2}\beta_1^{-1}D_{\beta_1}^{-1}\gamma_1^{-1}D_{\gamma_1}^{-1}\alpha_1^{2}\right)z\right]^{-a}\left[1-\left(D_{\alpha_2}D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\gamma_2^{-1}D_{\gamma_2}^{-1}\alpha_2\alpha_3\right)u\right]^{-b}$$
$$\times H_4\left(a_1,a_2;c_1,c_2;\alpha_1^2x,\alpha_1\beta_2y\right)\left(\alpha_1^{a_1-1}\alpha_2^{a_3-1}\alpha_3^{a_4-1}\beta_1^{c_3-1}\beta_2^{c_2-1}\gamma_1^{a-1}\gamma_2^{b-1}\right)$$
$$=\alpha_1^{a_1-1}\alpha_2^{a_3-1}\alpha_3^{a_4-1}\beta_1^{c_3-1}\beta_2^{c_2-1}\gamma_1^{a-1}\gamma_2^{b-1}X_{45}^{(4)}\left(a_1,a_1,a_1,a_3,a_1,a_1,a_2,\right.$$
$$\left.a_4;c_1,c_3,c_2,c_2;\alpha_1^2x,z,\alpha_1\beta_2y,u\right), \tag{53}$$

$$\left[1-\left(D_{\alpha}^{2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^{2}\right)u\right]^{-a}F_C^{(3)}\left(a_1,a_2;c_1,c_2,c_3;\alpha x,\alpha y,\alpha z\right)$$
$$\times\left(\alpha^{a_1-1}\beta^{c_4-1}\gamma^{a-1}\right)=\alpha^{a_1-1}\beta^{c_4-1}\gamma^{a-1}X_{48}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_2,\right.$$
$$\left.a_2;c_4,c_1,c_2,c_3;u,\alpha x,\alpha y,\alpha z\right), \tag{54}$$

$$\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha_1\alpha_2\right)u\right]^{-a}X_{4}\left(a_1,a_2,;c_1,c_2,c_3;\alpha_1^2x,\alpha_1\alpha_2y,\right.$$
$$\left.\alpha_1\alpha_2z\right)\left(\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{c_4-1}\gamma^{a-1}\right)=\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{c_4-1}\gamma^{a-1}X_{48}^{(4)}\left(a_1,a_1,a_1,a_1,\right.$$
$$\left.a_1,a_2,a_2,a_2;c_1,c_2,c_3,c_4;\alpha_1^2x,\alpha_1\alpha_2y,\alpha_1\alpha_2z,u\right), \tag{55}$$

$$\left[1-\left(D_{\alpha_1}^{2}\beta_1^{-1}D_{\beta_1}^{-1}\gamma_1^{-1}D_{\gamma_1}^{-1}\alpha_1^{2}\right)z\right]^{-a}\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\gamma_2^{-1}D_{\gamma_2}^{-1}\alpha_1\alpha_2\right)u\right]^{-b}$$
$$\times F_4\left(a_1,a_2;c_1,c_2;\alpha_1\alpha_2x,\alpha_1\alpha_2y\right)\left(\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta_1^{c_3-1}\beta_2^{c_4-1}\gamma_1^{a-1}\gamma_2^{b-1}\right)$$

$$= \alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta_1^{c_3-1}\beta_2^{c_4-1}\gamma_1^{a-1}\gamma_2^{b-1}X_{48}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_2,a_2,a_2;\right.$$
$$\left. c_3,c_1,c_2,c_4;z,\alpha_1\alpha_2 x,\alpha_1\alpha_2 y,u\right), \quad (56)$$

$$\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_\beta^{-1}\gamma^{-1}D_\gamma^{-1}\alpha_1\alpha_2\right)u\right]^{-a}F_C^{(3)}\left(\frac{a_1}{2},\frac{a_1+1}{2};c_1,c_2,c_3;4\alpha_1^2\beta x,\right.$$
$$\left.4\alpha_1^2 y,4\alpha_1^2 z\right)\left(\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{c_1-1}\gamma^{a-1}\right)=\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta^{c_1-1}\gamma^{a-1}X_{50}^{(4)}\left(a_1,a_1,a_1,\right.$$
$$\left.a_1,a_1,a_1,a_1,a_2;c_1,c_2,c_3,c_1;\alpha_1^2\beta x,\alpha_1^2 y,\alpha_1^2 z,u\right), \quad (57)$$

$$\left[1-\left(D_\alpha^2\beta^{-1}D_\beta^{-1}\gamma^{-1}D_\gamma^{-1}\alpha^2\right)u\right]^{-a}X_1\left(a_1,a_2;c_2,c_1;\alpha^2 x,\alpha^2 y,\alpha z\right)$$
$$\times\left(\alpha^{a_1-1}\beta^{c_3-1}\gamma^{a-1}\right)=\alpha^{a_1-1}\beta^{c_3-1}\gamma^{a-1}X_{50}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_1,a_1,\right.$$
$$\left.a_2;c_1,c_2,c_3,c_1;\alpha^2 y,\alpha^2 x,u,\alpha z\right), \quad (58)$$

$$\left[1-\left(D_\alpha^2\beta_1^{-1}D_{\beta_1}^{-1}\gamma_1^{-1}D_{\gamma_1}^{-1}\alpha^2\right)y\right]^{-a}\left[1-\left(D_\alpha^2\beta_2^{-1}D_{\beta_2}^{-1}\gamma_2^{-1}D_{\gamma_2}^{-1}\alpha^2\right)z\right]^{-b_1}$$
$$\times\left[1-\left(D_\alpha^2\beta_3^{-1}D_{\beta_3}^{-1}\gamma_3^{-1}D_{\gamma_3}^{-1}\alpha^2\right)u\right]^{-b_2}{}_2F_1\left(a_1,a_2;c_1;\alpha\beta_3 x,\right)\left(\alpha^{a_1-1}\beta_1^{c_2-1}\right.$$
$$\left.\beta_2^{c_3-1}\beta_3^{c_1-1}\gamma_1^{a-1}\gamma_2^{b_1-1}\gamma_3^{b_2-1}\right)=\alpha^{a_1-1}\beta_1^{c_2-1}\beta_2^{c_3-1}\beta_3^{c_1-1}\gamma_1^{a-1}\gamma_2^{b_1-1}\gamma_3^{b_2-1}$$
$$\times X_{50}^{(4)}\left(a_1,a_1,a_1,a_1,a_1,a_1,a_1,a_2;c_1,c_2,c_3,c_1;u,y,z,\alpha\beta_3 x\right). \quad (59)$$

Proof. To prove (45), let us denote, for simplicity, the left-hand side of relation (45) by $\bigvee$. Then, by considering the definition of Exton's triple function X_{13} and using the formulas (44), one gets:

$$\bigvee=\left[1-\left(D_\alpha^2\beta^{-1}D_\beta^{-1}\gamma^{-1}D_\gamma^{-1}\alpha^2\right)u\right]^{-a}X_{13}\left(a_1,a_2,a_3;c;\beta x,\beta y,\alpha\beta z\right)$$
$$\times\left(\alpha^{a_3-1}\beta^{c-1}\gamma^{a-1}\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_2)_{n+p}(a_3)_p(a)_q\ \beta^{-q}\gamma^{-q}}{(c)_{m+n+p}m!n!p!q!}x^m y^n z^p u^q$$
$$\times D_\alpha^{2q}D_\beta^{-q}D_\gamma^{-q}\left(\alpha^{a_3+p+2q-1}\beta^{c+m+n+p-1}\gamma^{a-1}\right)$$
$$=\alpha^{a_3-1}\beta^{c-1}\gamma^{a-1}\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{2m+n}(a_3)_{2q+p}(a_2)_{n+p}}{(c)_{m+n+p+q}}\frac{(\beta x)^m}{m!}\frac{(\beta y)^n}{n!}\frac{u^q}{q!}\frac{(\alpha\beta z)^p}{p!}$$
$$=\alpha^{a_3-1}\beta^{c-1}\gamma^{a-1}X_{38}^{(4)}\left(a_1,a_1,a_3,a_3,a_1,a_2,a_3,a_2;c,c,c,c;\beta x,\beta y,u,\alpha\beta z\right),$$

which proves the relation (45). Similarly, by applying the same techniques, we can easily obtain the other operational connections.

4. Special Cases

Among a large number of possible special cases of the main results (45) to (59), in this section, we consider some interesting relations.

If in (45) and (54), we let $x = 0$, we shall obtain relationships between Exton's functions and Appelle's functions

$$\left[1-\left(D_{\alpha}^{2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^{2}\right)u\right]^{-a}F_{1}\left(a_{2},a_{1},a_{3};c;\beta y,\alpha\beta z\right) \times\left(\alpha^{a_{3}-1}\beta^{c-1}\gamma^{a-1}\right)=\alpha^{a_{3}-1}\beta^{c-1}\gamma^{a-1}X_{13}\left(a_{3},a_{2},a_{1};c;u,\alpha\beta z,\beta y\right) \tag{60}$$

and

$$\left[1-\left(D_{\alpha}^{2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^{2}\right)u\right]^{-a}F_{4}\left(a_{1},a_{2};c_{2},c_{3};\alpha y,\alpha z\right) \times\left(\alpha^{a_{1}-1}\beta^{c_{4}-1}\gamma^{a-1}\right)=\alpha^{a_{1}-1}\beta^{c_{4}-1}\gamma^{a-1}X_{4}\left(a_{1},a_{2};c_{4},c_{2},c_{3};u,\alpha y,\alpha z\right), \tag{61}$$

respectively. If we put $y = 0$ in (60) and (61), we get the following connections:

$$\left[1-\left(D_{\alpha}^{2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^{2}\right)u\right]^{-a}{}_{2}F_{1}\left(a_{2},a_{3};c;\alpha\beta z\right)\left(\alpha^{a_{3}-1}\beta^{c-1}\gamma^{a-1}\right) =\alpha^{a_{3}-1}\beta^{c-1}\gamma^{a-1}H_{3}\left(a_{3},a_{2};c;u,\alpha\beta z\right), \tag{62}$$

$$\left[1-\left(D_{\alpha}^{2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^{2}\right)u\right]^{-a}{}_{2}F_{1}\left(a_{1},a_{2};c_{3};\alpha z\right)\left(\alpha^{a_{1}-1}\beta^{c_{4}-1}\gamma^{a-1}\right) =\alpha^{a_{1}-1}\beta^{c_{4}-1}\gamma^{a-1}H_{4}\left(a_{1},a_{2};c_{4},c_{3};u,\alpha z\right). \tag{63}$$

A special case of (59) when $x = 0$ yields the symbolic representation for $F_C^{(3)}$

$$\left[1-\left(D_{\alpha}^{2}\beta_{1}^{-1}D_{\beta_{1}}^{-1}\gamma_{1}^{-1}D_{\gamma_{1}}^{-1}\alpha^{2}\right)y\right]^{-a}\left[1-\left(D_{\alpha}^{2}\beta_{2}^{-1}D_{\beta_{2}}^{-1}\gamma_{2}^{-1}D_{\gamma_{2}}^{-1}\alpha^{2}\right)z\right]^{-b_{1}} \times\left[1-\left(D_{\alpha}^{2}\beta_{3}^{-1}D_{\beta_{3}}^{-1}\gamma_{3}^{-1}D_{\gamma_{3}}^{-1}\alpha^{2}\right)u\right]^{-b_{2}}{}_{2}\left(\alpha^{a_{1}-1}\beta_{1}^{c_{2}-1}\beta_{2}^{c_{3}-1}\beta_{3}^{c_{1}-1}\gamma_{1}^{a-1}\gamma_{2}^{b_{1}-1}\gamma_{3}^{b_{2}-1}\right) =\alpha^{a_{1}-1}\beta_{1}^{c_{2}-1}\beta_{2}^{c_{3}-1}\beta_{3}^{c_{1}-1}\gamma_{1}^{a-1}\gamma_{2}^{b_{1}-1}\gamma_{3}^{b_{2}-1}\times F_{C}^{(3)}\left(\frac{a_{1}}{2},\frac{a_{1}+1}{2};c_{2},c_{3},c_{1};,4y,4z,4u\right). \tag{64}$$

On other hand, If in (58), we set $x = 0$ and in (49), we take $z = 0$, we find that

$$\left[1-\left(D_{\alpha}^{2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^{2}\right)u\right]^{-a}H_{3}\left(a_{1},a_{2};c_{1};\alpha^{2}y,\alpha z\right) \times\left(\alpha^{a_{1}-1}\beta^{c_{3}-1}\gamma^{a-1}\right)=\alpha^{a_{1}-1}\beta^{c_{3}-1}\gamma^{a-1}X_{1}\left(a_{1},a_{2};c_{3},c_{1};u,\alpha^{2}y,\alpha z\right) \tag{65}$$

and

$$\left[1-\left(D_{\alpha}^{2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha^{2}\right)u\right]^{-a}H_{4}\left(a_{1},a_{2};c_{1},c_{2};\alpha^{2}x,\alpha\beta y\right) \times\left(\alpha^{a_{1}-1}\beta^{c_{2}-1}\gamma^{a-1}\right)=\alpha^{a_{1}-1}\beta^{c_{2}-1}\gamma^{a-1}X_{1}\left(a_{1},a_{2};c_{1},c_{2};\alpha^{2}x,u,\alpha\beta y\right), \tag{66}$$

respectively.

Again, on choosing $z = 0$ in (51), we get the elegant relation

$$F_{4}\left(\frac{a_{1}}{2},\frac{a_{1}+1}{2};c_{1},c_{2};4x,4y,\right)\times\left[1-\left(D_{\alpha_{1}}D_{\alpha_{2}}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha_{1}\alpha_{2}\right)u\right]^{-a} \times\left(\alpha_{1}^{a_{3}-1}\alpha_{2}^{a_{4}-1}\beta^{c_{3}-1}\gamma^{a-1}\right)=\alpha_{1}^{a_{3}-1}\alpha_{2}^{a_{4}-1}\beta^{c_{3}-1}\gamma^{a-1}F_{4}\left(\frac{a_{1}}{2},\frac{a_{1}+1}{2};c_{1},c_{2};4x,4y,\right)$$

$$\times {}_2F_1\left(a_3,a_4;c_3;u\right), \tag{67}$$

which, $x=0$ and $y=0$ yields new operational representation for Gaussian function ${}_2F_1$

$$\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta^{-1}D_{\beta}^{-1}\gamma^{-1}D_{\gamma}^{-1}\alpha_1\alpha_2\right)u\right]^{-a}\left(\alpha_1^{a_3-1}\alpha_2^{a_4-1}\beta^{c_3-1}\gamma^{a-1}\right)$$
$$=\alpha_1^{a_3-1}\alpha_2^{a_4-1}\beta^{c_3-1}\gamma^{a-1}{}_2F_1\left(a_3,a_4;c_3;u\right). \tag{68}$$

Further, formulas (47), (50), (53) and (56), with $x=y=0$, yield the following operational representations:

$$\left[1-\left(D_{\alpha_1}^2\beta^{-1}D_{\beta}^{-1}\gamma_1^{-1}D_{\gamma_1}^{-1}\alpha_1^2\right)z\right]^{-a}\left[1-\left(D_{\alpha_2}^2\beta^{-1}D_{\beta}^{-1}\gamma_2^{-1}D_{\gamma_2}^{-1}\alpha_2^2\right)u\right]^{-b}$$
$$\times\left(\alpha_1^{a_2-1}\alpha_2^{a_3-1}\beta^{c-1}\gamma_1^{a-1}\gamma_2^{b-1}\right)=\alpha_1^{a_2-1}\alpha_2^{a_3-1}\beta^{c-1}\gamma_1^{a-1}\gamma_2^{b-1}F_3\left(\frac{a_1}{2},\frac{a_2}{2},\right.$$
$$\left.\frac{a_1+1}{2},\frac{a_2+1}{2};c;4z,4u\right), \tag{69}$$

$$\left[1-\left(D_{\alpha}^2\beta^{-1}D_{\beta}^{-1}\gamma_1^{-1}D_{\gamma_1}^{-1}\alpha^2\right)z\right]^{-a}\left[1-\left(D_{\alpha}^2\beta^{-1}D_{\beta}^{-1}\gamma_2^{-1}D_{\gamma_2}^{-1}\alpha^2\right)u\right]^{-b}$$
$$\times\left(\alpha^{a_2-1}\beta^{c_3-1}\gamma_1^{a-1}\gamma_2^{b-1}\right)=\alpha^{a_2-1}\beta^{c_3-1}\gamma_1^{a-1}\gamma_2^{b-1}{}_2F_1\left(\frac{a_2}{2},\frac{a_2+1}{2};c_3;4z+4u\right), \tag{70}$$

$$\left[1-\left(D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\gamma_1^{-1}D_{\gamma_1}^{-1}\alpha_1^2\right)z\right]^{-a}\left[1-\left(D_{\alpha_2}D_{\alpha_3}\beta_2^{-1}D_{\beta_2}^{-1}\gamma_2^{-1}D_{\gamma_2}^{-1}\alpha_2\alpha_3\right)u\right]^{-b}$$
$$\times\left(\alpha_1^{a_1-1}\alpha_2^{a_3-1}\alpha_3^{a_4-1}\beta_1^{c_3-1}\beta_2^{c_2-1}\gamma_1^{a-1}\gamma_2^{b-1}\right)=\alpha_1^{a_1-1}\alpha_2^{a_3-1}\alpha_3^{a_4-1}\beta_1^{c_3-1}\beta_2^{c_2-1}\gamma_1^{a-1}$$
$$\times\gamma_2^{b-1}{}_2F_1\left(\frac{a_1}{2},\frac{a_1+1}{2};c_3;4z\right){}_2F_1\left(a_3,a_4;c_2;u\right), \tag{71}$$

$$\left[1-\left(D_{\alpha_1}^2\beta_1^{-1}D_{\beta_1}^{-1}\gamma_1^{-1}D_{\gamma_1}^{-1}\alpha_1^2\right)z\right]^{-a}\left[1-\left(D_{\alpha_1}D_{\alpha_2}\beta_2^{-1}D_{\beta_2}^{-1}\gamma_2^{-1}D_{\gamma_2}^{-1}\alpha_1\alpha_2\right)u\right]^{-b}$$
$$\times\left(\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta_1^{c_3-1}\beta_2^{c_4-1}\gamma_1^{a-1}\gamma_2^{b-1}\right)=\alpha_1^{a_1-1}\alpha_2^{a_2-1}\beta_1^{c_3-1}\beta_2^{c_4-1}\gamma_1^{a-1}\gamma_2^{b-1}$$
$$\times H_4\left(a_1,a_2;c_3,c_4;z,u\right). \tag{72}$$

References

[1] K.L. Bell and N. S. Scott, Coulomb functions (negative energies), Comput. Phys. Comm., Vol. 20, 447-58, 1980.

[2] M.G. Bin-Saad, Symbolic operational images and decomposition formulas of hypergeometric functions, J. Math. Anal. Appl., Vol. 376, 451-468, 2011.

[3] M.G. Bin-Saad and J.A. Younis, Some integral representations for certain quadruple hypergeometric functions, MATLAB J., Vol. 1, 61-68, 2018.

[4] M.G. Bin-Saad and J.A. Younis, On generating functions of quadruple hypergeometric function $X_7^{(4)}$, J. Sci. Arts., Vol. 44, 583-588, 2018.

[5] M.G. Bin-Saad, J.A. Younis and R. Aktas, Integral representations for certain quadruple hypergeometric series, Far East J. Math. Sci., Vol. 103, 21-44, 2018.

[6] M.G. Bin-Saad, J.A. Younis and R. Aktas, New quadruple hypergeometric series and their integral representations, Sarajevo Mathematical Journal, Vol. 14, 45-57, 2018.

[7] J. Choi, A. Hasanov and M. Turaev, Certain integral representations of Euler type for the Exton function X_2, J. Korean Soc. Math. Edu., Ser. B, Pure Appl. Math., Vol. 17, 347-354, 2010.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, Higher Transcendental Functions, Vol. I, McGraw-Hill Book Company, New York, Toronto and London, 1953.
[9] H. Exton, Multiple Hypergeometric Functions and Applications, Halsted Press, New York, London, Sydney and Toronto, 1976.
[10] H. Exton, Hypergeometric functions of three variables, J. Indian Acad. Math., Vol. 4, 113-119, 1982.
[11] F.I. Frankl, Selected Works in Gas Dynamics, Nauka, Moscow, 1973 (in Russian).
[12] A. Hasanov, H.M. Srivastava and M. Turaev, Decomposition formulas for some triple hypergeometric functions, J. Math. Anal. Appl., Vol. 324, 955-969, 2016.
[13] G. Lauricella, Sull funzioni ipergeometric a pi? variabili, Rend. Cric. Mat. Palermo, Vol. 7, 111-158, 1883.
[14] G. Lohöfer, Theory of an electromagnetically deviated metal sphere. I: Absorbed power, SIAM J. Appl. Math., Vol. 49, 567-581, 1989.
[15] A.W. Niukkanen, Generalised hypergeometric series ${}^{N}F(x_1, ..., x_N)$ arising in physical and quantum chemical applications, J. Phys. A: Math. Gen., Vol. 16, 1813-1825, 1983.
[16] R. Pastor-Satorras and A. Vespignani, Epidemic Dynamics and Endemic States in Complex Networks, Physical Review E, 63, DOI: 10.1103, 066117, 2001.
[17] M. A. Rakha and E. S. Application of basic hypergeometric series, Appl. Math. Comput., Vol. 148, 717-723, 2oo4.
[18] B. Ross, Fractional Calculus and its Applications, Lecture Notes in Mathematics. Vol. 457, Springer-Verlag, 1975.
[19] C. Sharma and C.L. Parihar, Hypergeometric functions of four variables (I), J. Indian Acad. Math., Vol. 11, 121-133, 1989.
[20] L.J. Slater, Generalized Hypergeometric Functions, Cambridge University Press, 1966.
[21] H.M. Srivastava and J. Choi, Series Associated with the Zeta and Related Functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.
[22] H.M. Srivastava and J. Choi, Zeta and q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London and New York, 2012.
[23] H.M. Srivastava and P.W. Karlsson, A Treatise on Generating Function, Ellis Horwood Lt1., Chichester, 1984.
[24] H.M. Srivastava and H.L. Manocha, Multiple Gaussian Hypergeometric Series, Halsted Press, Bristone, London, New York and Toronto, 1985.
[25] J.A. Younis and M.G. Bin-Saad, Integral representations involving new hypergeometric functions of four variables. J. Frac. Calc. Appl., Vol. 10, 77-91, 2019.

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