CONTROLLABILITY AND OBSERVABILITY OF LINEAR IMPULSIVE DESCRIPTOR SYSTEMS: A DRAZIN INVERSE APPROACH

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ABSTRACT. For continuous-time linear descriptor system with impulse, a controllability and observability problem is solved. A solution of linear impulsive descriptor systems (LIDS) with the regular pencil is derived by using the Drazin inverse approach. Necessary and sufficient conditions for time-invariant LIDS are obtained in term of Gramian matrices and rank conditions. Few numerical examples are also given which illustrate the effectiveness of the new results.

1. INTRODUCTION

Linear differential-algebraic systems (LDAS) are considered as a most reliable source for appropriate mathematical model and their adequate solutions in various engineering systems, including aircraft stabilization, chemical engineering systems, loss less transition lines, etc. (see e.g.[3, 12, 13, 20] and the references therein). Depending on the area of application, these models are called singular or implicit or descriptor systems. The popularity of LDAS is continuously increasing as these are general enough to provide a solid understanding of inner dynamics for underlying physical problems [4, 5, 6, 8, 15] with different solution techniques [7, 18, 27, 29]. In LDAS, the problem of controllability and observability began to attract the attention of mathematicians and engineers as it began to play a significant role in control theory and engineering problems. Many contributions on controllability problem have been made in recent years, see for example [1, 19]. Also, many practical problems need to be described by differential equations with impulsive conditions. An application-based work in the field of Biology is done by Miron [23] in which he has discussed three types of biological applications along with the HIV problem which shows effectively the use of impulsive differential equations in real-world problems. Moving ahead in this field, in the last few years, controllability and observability of linear impulsive differential systems are immensely motivated problems for advancement. For time-invariant system, the results on controllability and observability are found by [10, 19, 20, 21, 22, 26]. Recently, some results on

singular systems with impulsive control have been reported, see [9, 11, 24, 28, 30] and references therein. Few of above-mentioned manuscripts, researchers analyze the existence and the stability of solutions of descriptor systems with impulse. To the best of our knowledge, no controllable and observable criteria have been derived for LIDS. An important point for finding the solution of descriptor system can be effectively understood by the following example:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \dot{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} x + f;$$

which implies that $x_2 = -f_2$, $x_1 = \dot{x}_2 - f_1 = -\dot{f}_2 - f_1$ and $f_3 = 0$. In particular, not for all initial values or all inhomogeneities there exist a solution. Furthermore, x_3 is not restricted at all, hence the uniqueness of the solution is not there. Finally, x_1 contains the derivative of the inhomogeneity so that the solution is less smooth than the inhomogeneity which could lead to non-existence of solutions if the inhomogeneity is not sufficiently smooth.

Motivated by these considerations, in this paper, we have adopted the Drazin inverse approach [14, 15] to find the solution of LIDS rather than other defined techniques. Our objective is to obtain the necessary and sufficient conditions for the controllability and observability of LIDS in terms of Gramian matrices and rank conditions.

Such type of systems can be seen in the problems of pendulum motion and especially in the field of electrical engineering, particularly in electrical circuits. As both the steady-state and transient state stability of a system is dependent on the impulsive stability of under consideration system. These impulsive conditions may occur on abrupt time, repetitive or during the sparkling behavior in any system.

The rest of this paper is organized as follows. In section 2, we recall some basic notations and definitions related to the Drazin inverse method. In section 3 and 4, we obtain necessary and sufficient conditions for controllability and observability of LIDS respectively. Also, some numerical examples are included to illustrate the effectiveness of these results.

2. Preliminaries

Consider the following linear descriptor impulsive system

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), \ t \neq t_i \\ x(t_i^+) = (1 + c_i)x(t_i), t = t_i, \ i = 1, 2, \cdots, \\ y(t) = Cx(t), \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where $t \in J := [t_0, \infty)$, $x(\cdot) \in \mathbb{R}^n$ is the system state vector, $u(\cdot) \in \mathbb{R}^m$ is the control input vector, $y(\cdot) \in \mathbb{R}^p$ is the output vector and $c_i \in \mathbb{R}$ are constants. It is worth noting that $c_i's$ are non zero scalers because the second equation of system (1) becomes trivial for any point of the sequence $\{t_k\}$. The singular matrix E and the matrices A, B, and C are constant matrices of appropriate dimensions. Throughout in this paper, to study system (1), we suppose that following assumptions hold:

(H$_1$): $0 \leq t_0 < t_1 < t_2 \cdots t_k < \cdots$ such that $\lim_{k\to\infty} t_k = \infty$.

(H$_2$): The state variable is left continuous at each impulsive time t_k namely

$$x(t_k) = x(t_k^-) = \lim_{h\to 0^+} x(t_k - h) \text{ and } x(t_k^+) = \lim_{h\to 0^+} x(t_k + h).$$

(H₃): The pencil of matrices (E, A) of equation (1) is regular, that is there exist $c \in \mathbb{C}$, $\det(Ec - A) \neq 0$.

Definition 2.1 [5] If A is an $n \times n$ matrix of complex numbers, then index of a matrix A, denoted by $Ind\,(A)$, is the smallest non-negative integer q such that

$$rankA^q = rankA^{q+1}. \tag{2}$$

Definition 2.2 [16] A matrix $E^D \in \mathbb{R}^{n \times n}$ is called the Drazin inverse of a matrix $E \in \mathbb{R}^{n \times n}$, if it satisfy the following conditions

$$EE^D = E^D E, \ E^D E E^D = E^D \text{ and } E^D E^{q+1} = E^q, \tag{3}$$

where q is the index of a matrix. The Drazin inverse of a square matrix always exists and is unique. The Drazin inverse was introduced in [7].

Algorithm 2.3 [16] To compute $E^D \in \mathbb{R}^{n \times n}$ of a matrix $E \in \mathbb{R}^{n \times n}$, following steps are required:

(1) Find the pair of matrices $V \in \mathbb{R}^{n \times r}$, $W \in \mathbb{R}^{r \times n}$, such that $rankV = rankW = rankE = r$ and

$$E = VW. \tag{4}$$

(2) Compute the nonsingular matrix

$$WEV \in \mathbb{R}^{r \times r}. \tag{5}$$

(3) The desired Drazin inverse matrix is given by

$$E^D = V\,(WEV)^{-1}\,W. \tag{6}$$

Example 2.4 Consider a matrix

$$E = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}. \tag{7}$$

Clearly $\det E = 0$. So, for the Drazin inverse E^D of matrix E as per Algorithm 2, let us choose V and W such that

$$E = VW = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{8}$$

and $rank(E) = rank(V) = rank(W) = 1$. Moreover

$$E^2 = \begin{bmatrix} 9 & 0 \\ 0 & 0 \end{bmatrix}. \tag{9}$$

So, $rank(E^2) = rank(E) = 1 = q(\text{index of } E)$. From Algorithm 2, it follows that

$$E^D = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

If the $Ind\,(E) = 1$, the Drazin inverse E^D is called the group inverse and it is denoted by $E^\sharp$ (see, e.g., [2, p. 118]). In general, the Drazin inverse can be expressed explicitly in terms of the Jordan canonical form of E

$$E = S \begin{pmatrix} J & 0 \\ 0 & N \end{pmatrix} S^{-1}, \ E^D = S \begin{pmatrix} J^{-1} & 0 \\ 0 & 0 \end{pmatrix} S^{-1}, \tag{11}$$

where J contains the Jordan blocks corresponding to nonzero eigenvalues and N is nilpotent with $N^k = 0$ and $N^{k-1} \neq 0$. With this representation of E^D we can immediately see that [25]

$$\mathcal{R}(E^D) = \mathcal{R}(E^q), \ \mathcal{N}(E^D) = \mathcal{N}(E^q) \text{ and } \mathbb{R}^n = \mathcal{R}(E^D) \oplus \mathcal{N}(E^D). \tag{12}$$

Premultiplying first equation of (1) by $(Ec - A)^{-1}$, we have

$$
\begin{cases}
\bar{E}\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t), \ t \neq t_i \\
x(t_i^+) = (1 + c_i)x(t_i), \ t = t_i, \ i = 1, 2, \cdots, \\
y(t) = Cx(t) + Du(t), \\
x(t_0) = x_0,
\end{cases}
\tag{13}
$$

where

$$
\bar{E} = (Ec - A)^{-1}E, \bar{A} = (Ec - A)^{-1}A \text{ and } \bar{B} = (Ec - A)^{-1}B.
\tag{14}
$$

Remark 2.5 The matrices $\bar{E}$ and $\bar{A}$ defined in equation (14) satisfy the following properties:

(1) $\bar{E}\bar{A} = \bar{A}\bar{E}, \ \bar{A}^D\bar{E} = (\bar{E}\bar{A})^D, \ \bar{E}^D\bar{A} = (\bar{A}\bar{E})^D, \bar{E}^D\bar{A}^D = \bar{A}^D\bar{E}^D$ and $\overline{AE} = \overline{EA}$.
(2) $\bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D$.
(3) If $\det E \neq 0$, then $E^D = E^{-1}$.

Lemma 2.6 The matrices $\bar{E}$ and $\bar{A}$ defined in equation (14) satisfy the following equalities:

(1) $\overline{AE} = \overline{EA}, \ \bar{A}^D\bar{E} = (\overline{EA})^D, \ \bar{E}^D\bar{A} = (\overline{AE})^D$ and $\bar{A}^D\bar{E}^D = \bar{E}^D\bar{A}^D$;
(2) $\mathcal{N}(\bar{A}) \cap \mathcal{N}(\bar{E}) = \{0\}$;
(3) $\bar{E} = T \begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} T^{-1}, \bar{E}^D = T \begin{pmatrix} J^{-1} & 0 \\ 0 & 0 \end{pmatrix} T^{-1}, \det T \neq 0, J \in \mathbb{R}^{n_1 \times m_1}$, is nonsingular, $N \in \mathbb{R}^{n_2 \times m_2}$ is nilpotent, $n_1 + n_2 = n$;
(4) $(I - \bar{E}\bar{E}^D)\bar{A}\bar{A}^D = \mathbb{I} - \bar{E}\bar{E}^D$, $\overline{E}^D(I - \bar{E}\bar{E}^D) = 0$ and $(\mathbb{I} - \bar{E}\bar{E}^D)(\bar{E}\bar{A}^D)^q = 0$, where $q = Ind(E)$.

Remark 2.7 From equation (3) and Lemma 2.6, it follows that

$$
\mathcal{N}(\bar{A}^q) \cap \mathcal{N}(\bar{E}^q) = \{0\} \text{ and } \mathcal{N}(\bar{A}^D) \cap \mathcal{N}(\bar{E}^D) = \{0\}.
\tag{15}
$$

Consider a linear non-autonomous system without impulse

$$
\begin{cases}
\bar{E}\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t) \\
x(0) = x_0.
\end{cases}
\tag{16}
$$

By using the Drazin inverse approach for the system (16), Campbell et. al., [4, Theorem 7], obtain the following solution of initial value problem (16):

Lemma 2.8 Let $q = Ind(E)$. Then the system (16) has a unique solution if and only if x_0 is of the following form, that is

$$
x_0 = \bar{E}\bar{E}^D v + (\bar{E}\bar{E}^D - \mathbb{I}_n) \sum_{r=0}^{q-1} [\bar{E}\bar{A}^D]^r \bar{A}^D\bar{B}u^{(r)}(0),
\tag{17}
$$

for some vector v. A particular solution of $\bar{E}\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t)$ is

$$
\begin{aligned}
x(t) &= e^{\bar{E}^D\bar{A}t} \int_a^t e^{-\bar{E}^D\bar{A}s} \bar{E}^D Bu(s)ds \\
&+ \sum_{r=0}^{q-1} (\bar{E}\bar{E}^D - \mathbb{I}_n)(\bar{E}\bar{A}^D)^r \bar{A}^D\bar{B}u^{(r)}(t),
\end{aligned}
\tag{18}
$$

where a is arbitrary. The general solution of $\bar{E}\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t)$ is

$$
\begin{aligned}
x\,(t) &= e^{\bar{E}^D\bar{A}t}\bar{E}\bar{E}^D v + e^{\bar{E}^D\bar{A}t}\int_a^t e^{-\bar{E}^D\bar{A}s}\bar{E}^D Bu(s)ds \\
&\quad + \sum_{r=0}^{q-1}(\bar{E}\bar{E}^D - \mathbb{I}_n)(\bar{E}\bar{A}^D)^r\bar{A}^D\bar{B}u^{(r)}(t).
\end{aligned}
\tag{19}
$$

The solution satisfying $x\,(0) = x_0$ is found by setting $v = x_0$ and $a = 0$. By using Lemma 2.8, one can easily obtain the following solution of LIDS (13).

Theorem 2.9 The solution of system (1)(or (13)) is given by

$$
x(t) = \begin{cases}
e^{\bar{E}^D\bar{A}(t-t_0)}\bar{E}^D\bar{E}x_0 + \displaystyle\int_{t_0}^t e^{\bar{E}^D\bar{A}(t-\tau)}\bar{E}^D\bar{B}u(\tau)d\tau \\
\quad + \displaystyle\sum_{r=0}^{q-1}(\bar{E}\bar{E}^D - \mathbb{I}_n)(\bar{E}\bar{A}^D)^r\bar{A}^D\bar{B}u^{(r)}(t), \text{ for } t \in [t_0, t_1]; \\
\displaystyle\prod_{i=1}^k (1+c_i)e^{\bar{E}^D\bar{A}(t-t_0)}\bar{E}^D\bar{E}x_0 \\
\quad + \displaystyle\sum_{j=1}^k\prod_{i=j}^k(1+c_i)\int_{t_{j-1}}^{t_j} e^{\bar{E}^D\bar{A}(t-\tau)}\bar{E}^D\bar{B}u(\tau)d\tau \\
\quad + \displaystyle\int_{t_k}^t e^{\bar{E}^D\bar{A}(t-\tau)}\bar{E}^D\bar{B}u(\tau)d\tau + \sum_{r=0}^{q-1}(\bar{E}\bar{E}^D - \mathbb{I}_n) \\
\quad \times(\bar{E}\bar{A}^D)^r\bar{A}^D\bar{B}u^{(r)}(t), \text{ for } t \in (t_k, t_{k+1}], \ k = 1, 2, 3, \cdots.
\end{cases}
\tag{20}
$$

3. Controllability

Definition 3.1 The system (1)(or (13)) is called controllable on $[t_0, t_f]$ with $t_f > 0$, if given any initial state x_0, there exist an input signal $u(\cdot)$ such that the corresponding solution of the system (1)(or (13)) satisfies $x(t_f) = 0$.

For the necessary and sufficient condition of controllability of the system (1)(or (13)) we define the following $(k+1)\,n \times n$ controllabilty Gramian matrices as:
For $j = 1, 2, \cdots, k$

$$
\begin{cases}
G_{j-1}(t_0, t_{j-1}, t_j) := \displaystyle\int_{t_{j-1}}^{t_j} (e^{\bar{E}^D\bar{A}(t_0-\tau)}\bar{E}^D\bar{B})(e^{\bar{E}^D\bar{A}(t_0-\tau)}\bar{E}^D\bar{B})^* d\tau, \\
\text{and} \\
G_k(t_0, t_k, t_f) := \displaystyle\int_{t_k}^{t_f} (e^{\bar{E}^D\bar{A}(t_0-\tau)}\bar{E}^D\bar{B})(e^{\bar{E}^D\bar{A}(t_0-\tau)}\bar{E}^D\bar{B})^* d\tau,
\end{cases}
\tag{21}
$$

for $j = 1, 2, \cdots, k$ and here "*" denotes the conjugate transpose.

Remark 3.2 The matrices $G_{j-1}(t_0, t_{j-1}, t_j)$ and $G_k(t_0, t_k, t_f)$ defined in (21) are symmetric and positive semi-definite.

Theorem 3.3 *(i)* If there exist at least $l \in \{1, \cdots, k+1\}$ such that $G_{l-1}(t_0, t_{l_{-1}}, t_l)$ is invertible then the system (1)(or (13)) is controllable on $[t_0, t_f](t_f \in (t_k, t_{k+1}])$.

(ii) Assume that $c_i \neq -1 \, (i = 1, 2, \cdots, k)$, if the system (1)(or (13)) is controllable on $[t_0, t_f](t_f \in (t_k, t_{k+1}])$, then

$$rank\,(G_0 \; G_1 \cdots G_k) = n. \tag{22}$$

Proof *(i)* Consider that there exist at least one $l \in \{1, \cdots, k+1\}$ such that the controllability Gramian $G_{l-1}(t_0, t_{l_{-1}}, t_l)$ is invertible.

For a given $x_0 \in \mathbb{R}^n$, we choose the input $u(t)$ of the following form

$$u(t) = \begin{cases} \left(e^{\bar{E}^D \bar{A}(t_0 - t)} \bar{E}^D \bar{B} \right)^* G_0^{-1}(t_0, t_0, t_1) \left(-e^{\bar{E}^D \bar{A}(t_f - t_0)} \bar{E}^D \bar{E} x_0 \right) \\ \times a_l \left(e^{\bar{E}^D \bar{A}(t_0 - t)} \bar{E}^D \bar{B} \right)^* G_{l-1}^{-1}(t_0, t_{l-1}, t_l) \\ \times \bar{E}^D \bar{E} x_0 \qquad \text{if } t \in (t_{l-1}, t_l], \ l \in \{2, \cdots, k+1\}; \\ 0 \qquad \qquad \text{if } t \in [t_0, t_f] \backslash (t_{l-1}, t_l], \end{cases} \tag{23}$$

where a_l's are constants such that $\prod\limits_{j=1}^{k}(1 + c_j) + a_l \prod\limits_{j=l}^{k}(1 + c_j) = 0$. At $t = t_f$, we can write equation (20) as follows

$$x(t_f) = \begin{cases} e^{\bar{E}^D \bar{A}(t_f - t_0)} \bar{E}^D \bar{E} x_0 + \int\limits_{t_0}^{t_f} e^{\bar{E}^D \bar{A}(t_f - \tau)} \bar{E}^D \bar{B} u(\tau) d\tau \\ + \sum\limits_{r=0}^{q-1} (\bar{E}^D \bar{E} - \mathbb{I}_n)(\bar{E}\bar{A}^D)^r \bar{A}^D \bar{B} u^{(r)}(t_f), \ t_f \in [t_0, t_1]; \\ \prod\limits_{i=1}^{k}(1 + c_i) e^{\bar{E}^D \bar{A}(t_f - t_0)} \bar{E}^D \bar{E} x_0 \\ + \sum\limits_{j=1}^{k} \prod\limits_{i=j}^{k}(1 + c_i) \int\limits_{t_{j-1}}^{t_j} e^{\bar{E}^D \bar{A}(t_f - \tau)} \bar{E}^D \bar{B} u(\tau) d\tau \\ + \int\limits_{t_k}^{t_f} e^{\bar{E}^D \bar{A}(t_f - \tau)} \bar{E}^D \bar{B} u(\tau) d\tau + \sum\limits_{r=0}^{q-1}(\bar{E}^D \bar{E} - \mathbb{I}_n) \\ \times (\bar{E}\bar{A}^D)^r \bar{A}^D \bar{B} u^{(r)}(t_f), \ t_f \in (t_k, t_{k+1}], \ k = 1, 2, 3, \cdots. \end{cases} \tag{24}$$

Firstly, premultiplying equation (24) by $\bar{E}^D$ and using equation (23) again in (24), this yields into the following equation

$$\bar{E}^D x(t_f) = 0. \tag{25}$$

Premultiplying equation (25) by $\bar{A}^D$

$$\bar{A}^D \bar{E}^D x(t_f) = 0. \tag{26}$$

Since $\bar{A}^D \bar{E}^D = \bar{E}^D \bar{A}^D$, this implies that $\bar{E}^D x(t_f) \in \ker(\bar{A}^D)$ and $\bar{A}^D x(t_f) \in \ker(\bar{E}^D)$. Finally, we obtain the following equation

$$x(t_f) = 0 \tag{27}$$

and we conclude that the system (1)(or (13)) is controllable.

(ii) Suppose that, if $rank\,(G_0\ G_1\cdots\ G_k)\neq n$ than there exist a nonzero vector $z\in\mathbb{R}^n$ such that

$$z^*G_{j-1}(t_0,t_{j-1},t_j)z=0 \tag{28}$$

and

$$z^*G_k(t_0,t_k,t_f)z=0, \tag{29}$$

for $t\in(t_j,t_{j+1}]$, $j=1,2,\cdots,k$. From both equations (28) and (29), we observe that

$$\int_{t_{j-1}}^{t_j}z^*(e^{\bar{E}^D\bar{A}(t_0\ -\tau)}\bar{E}^D\bar{B})(e^{\bar{E}^D\bar{A}(t_0\ -\tau)}\bar{E}^D\bar{B})^*zd\tau=0 \tag{30}$$

and

$$\int_{t_k}^{t_f}z^*\left(e^{\bar{E}^D\bar{A}(t_0-\tau)}\bar{E}^D\bar{B}\right)\left(e^{\bar{E}^D\bar{A}(t_0-\tau)}\bar{E}^D\bar{B}\right)^*zd\tau=0. \tag{31}$$

But the integrand in both equations (30) and (31) are non-negative and one can get easily

$$\|\,z^*e^{\bar{E}^D\bar{A}(t_0-t)}\bar{E}^D\bar{B}\,\|^2=0. \tag{32}$$

Thus we have

$$z^*\left(e^{\bar{E}^D\bar{A}(t_0-t)}\bar{E}^D\bar{B}\right)=0. \tag{33}$$

Since the system (1)(or (13)) is controllable, then it follows from equation (24) that is

$$0=x(t_f)=\begin{cases}e^{\bar{E}^D\bar{A}(t_f-t_0)}\bar{E}^D\bar{E}x_0+\displaystyle\int_{t_0}^{t_f}e^{\bar{E}^D\bar{A}(t_f-\tau)}\bar{E}^D\bar{B}u(\tau)d\tau\\[2mm]
\quad+\displaystyle\sum_{r=0}^{q-1}(\bar{E}^D\bar{E}-\mathbb{I}_n)(\bar{E}\bar{A}^D)^r\bar{A}^D\bar{B}u^{(r)}(t_f),\ t_f\in[t_0,t_1]\,;\\[3mm]
\displaystyle\prod_{i=1}^k(1+c_i)e^{\bar{E}^D\bar{A}(t_f-t_0)}\bar{E}^D\bar{E}x_0\\[3mm]
\quad+\displaystyle\sum_{j=1}^k\prod_{i=j}^k(1+c_i)\int_{t_{j-1}}^{t_j}e^{\bar{E}^D\bar{A}(t_f-\tau)}\bar{E}^D\bar{B}u(\tau)d\tau\\[3mm]
\quad+\displaystyle\int_{t_k}^{t_f}e^{\bar{E}^D\bar{A}(t_f-\tau)}\bar{E}^D\bar{B}u(\tau)d\tau+\sum_{r=0}^{q-1}(\bar{E}^D\bar{E}-\mathbb{I}_n)\\[3mm]
\quad(\bar{E}\bar{A}^D)^r\bar{A}^D\bar{B}u^{(r)}(t_f),t_f\in(t_k,t_{k+1}],k=1,2,3,\cdots.\end{cases} \tag{34}$$

Substitute $x_0 = z$ in equation (34) and from the semigroup property of exponential matrix, we have

$$
0 = \begin{cases}
\bar{E}^D \bar{E} z + \int\limits_{t_0}^{t_f} e^{\bar{E}^D \bar{A}(t_0 - \tau)} \bar{E}^D \bar{B} u(\tau) d\tau + \sum\limits_{r=0}^{q-1} (\bar{E}^D \bar{E} - \mathbb{I}_n) \\
\times e^{\bar{E}^D \bar{A}(t_0 - t_f)} (\bar{E} \bar{A}^D)^k \bar{A}^D \bar{B} u^{(k)}(t_f), \ t_f \in [t_0, t_1] \\
\prod\limits_{i=1}^{k} (1 + c_i) \bar{E}^D \bar{E} z \\
+ \sum\limits_{j=1}^{k} \prod\limits_{i=j}^{k} (1 + c_i) \int\limits_{t_{j-1}}^{t_j} e^{\bar{E}^D \bar{A}(t_0 - \tau)} \bar{E}^D \bar{B} u(\tau) d\tau \\
+ \int\limits_{t_k}^{t_f} e^{\bar{E}^D \bar{A}(t_0 - \tau)} \bar{E}^D \bar{B} u(\tau) d\tau + \sum\limits_{r=0}^{q-1} (\bar{E}^D \bar{E} - \mathbb{I}_n) e^{\bar{E}^D \bar{A}(t_0 - t_f)} \\
\times (\bar{E} \bar{A}^D)^k \bar{A}^D \bar{B} u^{(k)}(t_f), \ t_f \in (t_k, t_{k+1}], \ k = 1, 2, 3, \cdots.
\end{cases} \tag{35}
$$

Premultiplying equation (35) by $\bar{E}^D$ we can write it as

$$
0 = \begin{cases}
\bar{E}^D \bar{E}^D \bar{E} z + \bar{E}^D \int\limits_{t_0}^{t_f} e^{\bar{E}^D \bar{A}(t_0 - \tau)} \bar{E}^D \bar{B} u(\tau) d\tau, \ t_f \in [t_0, t_1] ; \\
\prod\limits_{i=1}^{k} (1 + c_i) \bar{E}^D \bar{E}^D \bar{E} z \\
+ \sum\limits_{j=1}^{k} \prod\limits_{i=j}^{k} (1 + c_i) \bar{E}^D \int\limits_{t_{j-1}}^{t_j} e^{\bar{E}^D \bar{A}(t_0 - \tau)} \bar{E}^D \bar{B} u(\tau) d\tau \\
+ \bar{E}^D \int\limits_{t_k}^{t_f} e^{\bar{E}^D \bar{A}(t_0 - \tau)} \bar{E}^D \bar{B} u(\tau) d\tau, \ t_f \in (t_k, t_{k+1}], \ k = 1, 2, 3, \cdots
\end{cases} \tag{36}
$$

and it follows that

$$
0 = \begin{cases}
\bar{E}^D \left(z + \int\limits_{t_0}^{t_f} e^{\bar{E}^D \bar{A}(t_0 - \tau)} \bar{E}^D \bar{B} u(\tau) d\tau \right), \ t_f \in [t_0, t_1] ; \\
\bar{E}^D \left\{ \prod\limits_{i=1}^{k} (1 + c_i) z \right. \\
+ \sum\limits_{j=1}^{k} \prod\limits_{i=j}^{k} (1 + c_i) \int\limits_{t_{j-1}}^{t_j} e^{\bar{E}^D \bar{A}(t_0 - \tau)} \bar{E}^D \bar{B} u(\tau) d\tau \\
+ \left. \int\limits_{t_k}^{t_f} e^{\bar{E}^D \bar{A}(t_0 - \tau)} \bar{E}^D \bar{B} u(\tau) d\tau \right\}, \ t_f \in (t_k, t_{k+1}], \ k = 1, 2, 3, \cdots.
\end{cases} \tag{37}
$$

Since $\bar{A}^D \bar{E}^D = \bar{E}^D \bar{A}^D$, then equation (37) implies that

$$
\begin{aligned}
&\bar{E}^D \left(z + \int_{t_0}^{t_f} e^{\bar{E}^D \bar{A}(t_0-\tau)} \bar{E}^D \bar{B} u(\tau) d\tau \right) \in \ker(\bar{A}^D), \\
&\bar{A}^D \left(z + \int_{t_0}^{t_f} e^{\bar{E}^D \bar{A}(t_0-\tau)} \bar{E}^D \bar{B} u(\tau) d\tau \right) \in \ker(\bar{E}^D),
\end{aligned}
\tag{38}
$$

for $t_f \in [t_0, t_1]$ and

$$
\begin{aligned}
&\bar{E}^D \left\{ \prod_{i=1}^{k}(1+c_i)z + \sum_{j=1}^{k}\prod_{i=j}^{k}(1+c_i) \int_{t_{j-1}}^{t_j} e^{\bar{E}^D \bar{A}(t_0-\tau)} \bar{E}^D \bar{B} u(\tau) d\tau \right. \\
&\left. \int_{t_k}^{t_f} e^{\bar{E}^D \bar{A}(t_0-\tau)} \bar{E}^D \bar{B} u(\tau) d\tau \right\} \in \ker(\bar{A}^D), \\
&\bar{E}^D \left\{ \prod_{i=1}^{k}(1+c_i)z + \sum_{j=1}^{k}\prod_{i=j}^{k}(1+c_i) \int_{t_{j-1}}^{t_j} e^{\bar{E}^D \bar{A}(t_0-\tau)} \bar{E}^D \bar{B} u(\tau) d\tau \right. \\
&\left. \int_{t_k}^{t_f} e^{\bar{E}^D \bar{A}(t_0-\tau)} \bar{E}^D \bar{B} u(\tau) d\tau \right\} \in \ker(\bar{E}^D),
\end{aligned}
\tag{39}
$$

for $t_f \in (t_k, t_{k+1}]$, $k = 1, 2, 3, \cdots$.

By using Remark 2.7, we can write both the equations (38) and (39) as

$$
0 = \begin{cases}
\left(z + \int_{t_0}^{t_f} e^{\bar{E}^D \bar{A}(t_0-\tau)} \bar{E}^D \bar{B} u(\tau) d\tau \right), \text{ for } t_f \in [t_0, t_1]; \\
\left\{ \prod_{i=1}^{k}(1+c_i)z + \sum_{j=1}^{k}\prod_{i=j}^{k}(1+c_i) \int_{t_{j-1}}^{t_j} e^{\bar{E}^D \bar{A}(t_0-\tau)} \bar{E}^D \bar{B} u(\tau) d\tau \right. \\
\left. + \int_{t_k}^{t_f} e^{\bar{E}^D \bar{A}(t_0-\tau)} \bar{E}^D \bar{B} u(\tau) d\tau \right\}, \text{ for } t_f \in (t_k, t_{k+1}], \ k = 1, 2, 3, \cdots.
\end{cases}
\tag{40}
$$

Premultiplying (37) by z^* and using equation (33), we have

$$
\|z\|^2 = 0;
\tag{41}
$$

which is a contradiction to the fact that $z \neq 0$. Hence proof completes.

For our next result of controllability, let us define the following matrix

$$
Q_c := \left[\bar{E}^D \bar{B} | (\bar{E}^D \bar{A}) \bar{E}^D \bar{B} | \cdots | (\bar{E}^D \bar{A})^{n-1} \bar{E}^D \bar{B} \right].
\tag{42}
$$

Theorem 3.4 Assume that $c_i \neq -1$ $(i = 1, 2, \cdots, k)$ then the system (1)(or (13)) is controllable on $[t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$ if and only if $rank\ Q_c = n$.

Proof From Cayley Hamilton's theorem, we can write

$$e^{\bar{E}^D \bar{A}(t-s)} = \sum_{i=0}^{n-1} c_i(t-s)(\bar{E}^D \bar{A})^i. \tag{43}$$

Now, suppose that the system (1)(or (13)) is controllable on $[t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$.

If the rank condition does not holds, then there exist $z \in \mathbb{R}^n$ with $z \neq 0$ such that

$$z^*(\bar{E}^D \bar{A})^m \bar{E}^D \bar{B} = 0, \ m = 0, 1, \cdots, n-1; \tag{44}$$

which gives

$$z^* G_{j-1}(t_0, t_{j-1}, t_j) = \int_{t_{j-1}}^{t_j} z^*(e^{\bar{E}^D \bar{A}(t_0 - \tau)} \bar{E}^D \bar{B})(e^{\bar{E}^D \bar{A}(t_0 - \tau)} \bar{E}^D \bar{B})^* d\tau$$

$$= \int_{t_{j-1}}^{t_j} z^* \sum_{i=0}^{n-1} c_i(t_0 - \tau)(\bar{E}^D \bar{A})^i (e^{\bar{E}^D \bar{A}(t_0 - \tau)} \bar{E}^D \bar{B})^* d\tau = 0, \tag{45}$$

for $j = 1, 2, \cdots, k$ and

$$z^* G_k(t_0, t_k, t_f) = \int_{t_k}^{t_f} z^* \sum_{i=0}^{n-1} c_i(t_0 - \tau)(\bar{E}^D \bar{A})^i (e^{\bar{E}^D \bar{A}(t_0 - \tau)} \bar{E}^D \bar{B})^* d\tau = 0. \tag{46}$$

It follows that $rank\, (G_0 \ G_1 \cdots G_k) < n$; which is a contradiction to our supposition, therefore $rank Q_c = n$.

Conversely, suppose that $rank Q_c = n$ but the system (1)(or (13)) is not controllable on $[t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$, then from Theorem 3.3 there exist $z \in \mathbb{R}^n$ with $z \neq 0$ such that

$$z^*(e^{\bar{E}^D \bar{A}(t_0 - t)} \bar{E}^D \bar{B}) = 0. \tag{47}$$

In particular, at $t = t_0$ equation (47) follows that $z^* \bar{E}^D \bar{B} = 0$. Taking Caputo fractional derivative for equation (47), we have

$$z^* \bar{E}^D \bar{A}(e^{\bar{E}^D \bar{A}(t_0 - t)} \bar{E}^D \bar{B}) = 0. \tag{48}$$

For $t = t_0$, we have $z^* \bar{E}^D \bar{A}(\bar{E}^D \bar{B}) = 0$.

Repeating this argument $(n-1)$ times, we have

$$z^* (\bar{E}^D \bar{A})^m (\bar{E}^D \bar{B}) = 0 \text{ for } m = 0, 1, \cdots, n-1. \tag{49}$$

Therefore, it follows that

$$z^* \left[(\bar{E}^D \bar{B}) | (\bar{E}^D \bar{A}) (\bar{E}^D \bar{B}) | \cdots | (\bar{E}^D \bar{A})^{n-1} (\bar{E}^D \bar{B}) \right] = 0; \tag{50}$$

which implies that the rank conditions fails. Hence the system (1)(or (13)) is controllable on $[t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$.

Example 3.5 Consider the following impulsive system

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t), t \neq t_i, \\ x(t_0) = 1, \\ y(t) = Cx(t) \ , t \in t \in [0, 8.5] \\ x(t_i^+) = (\frac{1}{2})x(t_i), \ t = t_i, \ t_i = \frac{(4i-3)}{2}, i = 1, 2, \cdots. \end{cases} \tag{51}$$

where the matrices E, A and B are defined as

$$E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \ A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}, \ B = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \text{ and}$$
$$C = \begin{pmatrix} 1 & 3 \end{pmatrix}. \tag{52}$$

Clearly, E is not an invertible matrix and $rank(E) = 1$. Also, the pencil of matrices (E, A) is regular for $c = 1$, that is

$$(Ec - A)^{-1} = (E - A)^{-1} = \frac{1}{4} \begin{pmatrix} 4 & -3 \\ -4 & 2 \end{pmatrix}. \tag{53}$$

From equation (53) system (51) can be written as $\bar{E}\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t)$, where

$$\bar{E} = \begin{pmatrix} 0 & -\frac{3}{4} \\ 0 & \frac{1}{2} \end{pmatrix}, \ \bar{A} = \frac{1}{4} \begin{pmatrix} -4 & -3 \\ 0 & 2 \end{pmatrix} \text{ and } \bar{B} = \begin{pmatrix} -\frac{7}{4} \\ \frac{2}{4} \end{pmatrix}. \tag{54}$$

Using the Algorithm of Drazin inverse we have obtained the following matrix

$$\bar{E}^D = \frac{1}{4} \begin{pmatrix} 0 & -3 \\ 0 & 2 \end{pmatrix}. \tag{55}$$

For $n = 2$, the rank condition of Theorem 3.4 becomes

$$Q_c = \begin{pmatrix} \bar{E}^D \bar{B} | \ \bar{E}^D \bar{A} \ \bar{E}^D \bar{B} \end{pmatrix} = \begin{pmatrix} -1.75 & 0.005 \\ 0.5 & -0.0013 \end{pmatrix}; \tag{56}$$

which implies that $rankQ_c = 2$. So, the system (51) is controllable on $[t_0, t_f] = [0, 8.5]$.

4. Observability

Definition 4.1 System (1)(or (13)) is called state observable on $[t_0, t_f]$ if any initial state x_0 can be uniquely determined by the corresponding input $u(t)$ and system output $y(t)$, for $t \in [t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$.

Let us define the following observability Gramian matrix for the necessary and sufficient conditions of observability for the system (1)(or (13)):

$$M(t_0, t_f) := M(t_0, t_0, t_1) + \sum_{j=1}^{k-1} \prod_{j=1}^{i} (1 + c_j) M(t_0, t_i, t_{i+1})$$
$$+ \prod_{j=1}^{k} (1 + c_j) M(t_0, t_k, t_f), \tag{57}$$

where

$$M(t_0, t_i, t_{i+1}) := \int_{t_i}^{t_{i+1}} \left(e^{\bar{E}^D \bar{A}(t-t_0)} \bar{E}^D \bar{E} \right)^* C^* C \left(e^{\bar{E}^D \bar{A}(t-t_0)} \bar{E}^D \bar{E} \right) d\tau, \tag{58}$$

for $i = 1, 2, \cdots, k$ and

$$M(t_0, t_k, t_f) := \int_{t_k}^{t_f} \left(e^{\bar{E}^D \bar{A}(t-t_0)} \bar{E}^D \bar{E} \right)^* C^* C \left(e^{\bar{E}^D \bar{A}(t-t_0)} \bar{E}^D \bar{E} \right) d\tau. \tag{59}$$

Theorem 4.2 Assume that $1 + c_i \geq 0$, $i = 1, 2, \cdots, k$. Then, the impulsive system (1)(or (13)) is observable on $[t_0, t_f]$, $t_f \in (t_k, t_{k+1}]$ if and only if the observability Gramian matrix $M(t_0, t_f)$ is invertible.

Proof Consider that the observability Gramian matrix $M(t_0, t_f)$ is invertible. From equation (13) and (20), we obtain the corresponding output of the system (1)(or (13)) as follows

$$
y(t) = \begin{cases}
Ce^{\bar{E}^D \bar{A}(t-t_0)}\bar{E}^D \bar{E}x_0 + \int\limits_{t_0}^{t} Ce^{\bar{E}^D \bar{A}(t-\tau)}\bar{E}^D \bar{B}u(\tau)d\tau \\[3mm]
+C\sum\limits_{r=0}^{q-1}(\bar{E}^D \bar{E} - \mathbb{I}_n)(\bar{E}\bar{A}^D)^r \bar{A}^D \bar{B}u^{(r)}(t), \ t \in [t_0, t_1];
\end{cases}
\tag{60}
$$

and

$$
y(t) = \begin{cases}
\prod\limits_{i=1}^{k}(1 + c_i)Ce^{\bar{E}^D \bar{A}(t-t_0)}\bar{E}^D \bar{E}x_0 \\[3mm]
+\sum\limits_{j=1}^{k}\prod\limits_{i=j}^{k}(1 + c_i)\int\limits_{t_{j-1}}^{t_j} Ce^{\bar{E}^D \bar{A}(t-\tau)}\bar{E}^D \bar{B}u(\tau)d\tau \\[3mm]
+\int\limits_{t_k}^{t} Ce^{\bar{E}^D \bar{A}(t-\tau)}\bar{E}^D \bar{B}u(\tau)d\tau + \sum\limits_{r=0}^{q-1}(\bar{E}^D \bar{E} - \mathbb{I}_n) \\[3mm]
\times C(\bar{E}\bar{A}^D)^r \bar{A}^D \bar{B}u^{(r)}(t), \ t \in (t_k, t_{k+1}], k = 1, 2, 3, \cdots .
\end{cases}
\tag{61}
$$

From Definition 4.1, observability of (13) is equivalent to the observability of $y(t)$ given by

$$
y(t) = \begin{cases}
Ce^{\bar{E}^D \bar{A}(t-t_0)}\bar{E}^D \bar{E}x_0, & \text{for } t \in [t_0, t_1]; \\[3mm]
\prod\limits_{i=1}^{k}(1 + c_i)Ce^{\bar{E}^D \bar{A}(t-t_0)} \\[2mm]
\times \bar{E}^D \bar{E}x_0, \ t \in (t_k, t_{k+1}], \ k = 1, 2, 3, \cdots ,
\end{cases}
\tag{62}
$$

as $u(t) = 0$.

Now, multiplying $\left(e^{\bar{E}^D \bar{A}(t-t_0)}\bar{E}^D \bar{E}\right)^* C^*$ to the both sides of equation (62) and integrating it with respect to t from t_0 to t_f, we can write

$$
\begin{aligned}
\int\limits_{t_0}^{t_f} \left(e^{\bar{E}^D \bar{A}(\tau-t_0)}\bar{E}^D \bar{E}\right)^* C^* y(\tau)d\tau = &\left[\int\limits_{t_0}^{t_1} \left(e^{\bar{E}^D \bar{A}(\tau-t_0)}\bar{E}^D \bar{E}\right)^* C^* \right.\\
\times Ce^{\bar{E}^D \bar{A}(\tau-t_0)}\bar{E}^D \bar{E}d\tau &+ \sum\limits_{i=1}^{k-1}\prod\limits_{j=1}^{i}(1 + c_j)\int\limits_{t_i}^{t_{i+1}} \left(e^{\bar{E}^D \bar{A}(\tau-t_0)}\bar{E}^D \bar{E}\right)^* C^* \\
\times Ce^{\bar{E}^D \bar{A}(\tau-t_0)}\bar{E}^D \bar{E}d\tau &+ \prod\limits_{j=1}^{k}(1 + c_j)\int\limits_{t_k}^{t_f} \left(e^{\bar{E}^D \bar{A}(\tau-t_0)}\bar{E}^D \bar{E}\right)^* C^* \\
\times Ce^{\bar{E}^D \bar{A}(\tau-t_0)}\bar{E}^D \bar{E}d\tau &\Big] x_0
\end{aligned}
\tag{63}
$$

and it yields that

$$
\int_{t_0}^{t_f} \left(e^{\bar{E}^D \bar{A}(\tau - t_0)} \bar{E}^D \bar{E} \right)^* C^* y(\tau) d\tau = \Big[M(t_0, t_0, t_1)
$$
$$
+ \sum_{j=1}^{k-1} \prod_{i=j}^{k} (1 + c_i) M(t_0, t_j, t_{j+1}) + \prod_{i=j}^{k} (1 + c_j) M(t_0, t_k, t_f) \Big] x_0. \tag{64}
$$

Obviously left hand side of equation (64) depends on $y(t)$ and is a linear algebraic equation in x_0, $t \in [t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$. Since the matrix $M(t_0, t_f)$ is invertible then the initial state x_0 can be uniquely determined by the corresponding system output $y(t)$ for $t \in [t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$.

Conversely, we consider that if the observability Gramian $M(t_0, t_f)$ is not invertible, then there exist a non zero vector $z \in \mathbb{R}^n$ such that

$$
z^* M(t_0, t_f) z = 0. \tag{65}
$$

Since $(1 + c_i) \geq 0$ and $M(t_0, t_i, t_{i+1})$ and $M(t_0, t_k, t_f)$ are positive semidefinite matrices, therefore, for $i = 0, 1, \cdots, k - 1$

$$
\begin{aligned}
z^* M(t_0, t_i, t_{i+1}) z &= 0 \text{ and} \\
z^* M(t_0, t_k, t_f) z &= 0.
\end{aligned} \tag{66}
$$

Consider $z = x_0$. Then from equation (62) and equation (66), we can write

$$
\int_{t_0}^{t_f} y^*(\tau) y(\tau) d\tau = \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} y^*(\tau) y(\tau) d\tau + \int_{t_k}^{t_f} y^*(\tau) y(\tau) d\tau
$$
$$
= \int_{t_0}^{t_1} x_0^* \left(e^{\bar{E}^D \bar{A}(\tau - t_0)} \bar{E}^D \bar{E} \right)^* C^* C e^{\bar{E}^D \bar{A}(\tau - t_0)} \bar{E}^D \bar{E} x_0 d\tau
$$
$$
+ \sum_{i=1}^{k-1} \left[\prod_{j=1}^{k} (1 + c_j) \right]^2 \left[\int_{t_i}^{t_{i+1}} x_0^* \left(e^{\bar{E}^D \bar{A}(\tau - t_0)} \bar{E}^D \bar{E} \right)^* C^* \right. \tag{67}
$$
$$
\left. \times C e^{\bar{E}^D \bar{A}(\tau - t_0)} \bar{E}^D \bar{E} x_0 d\tau \right] + \left[\prod_{j=1}^{k} (1 + c_j) \right]^2
$$
$$
\times \left[\int_{t_k}^{t_f} x_0^* \left(e^{\bar{E}^D \bar{A}(\tau - t_0)} \bar{E}^D \bar{E} \right)^* C^* C e^{\bar{E}^D \bar{A}(\tau - t_0)} \bar{E}^D \bar{E} x_0 d\tau \right];
$$

which returns into the following equation

$$
\int_{t_0}^{t_f} y^*(\tau) y(\tau) d\tau = x_0^* M(t_0, t_0, t_1) x_0 + \sum_{i=1}^{k-1} \left[\prod_{j=1}^{k} (1 + c_j) \right]^2
$$
$$
\times x_0^* M(t_0, t_i, t_{i+1}) x_0 + \left[\prod_{j=1}^{k} (1 + c_j) \right]^2 x_0^* M(t_0, t_k, t_f) x_0 = 0. \tag{68}
$$

Therefore, we have

$$\int\limits_{t_0}^{t_f} \|y(\tau)\|^2 d\tau = 0 \tag{69}$$

and it follows that

$$0 = y(t) = \begin{cases} C\left(e^{\bar{E}^D \bar{A}(t-t_0)} \bar{E}^D \bar{E}\right) x_0, & \text{for } t \in [t_0, t_1], \\ \prod\limits_{j=1}^{l}(1+c_j)C\left(e^{\bar{E}^D \bar{A}(t-t_0)} \bar{E}^D \bar{E}\right) x_0, & \text{for } t \in (t_l, t_{l+1}], \\ l = 1, 2, \cdots, k-1. \\ \prod\limits_{j=1}^{k}(1+c_j)C\left(e^{\bar{E}^D \bar{A}(t-t_0)} \bar{E}^D \bar{E}\right) x_0, & \text{for } t \in (t_k, t_f]. \end{cases} \tag{70}$$

From equation (70), it implies that system is unobservable on $[t_0, t_f]\,(t_f \in (t_k, t_{k+1}])$; which is a contradiction.

For rank condition of observability, let us define

$$Q_o := \begin{bmatrix} C \\ C\left(\bar{E}\bar{A}^D\right) \\ C\left(\bar{E}\bar{A}^D\right)^2 \\ \vdots \\ C\left(\bar{E}\bar{A}^D\right)^{n-1} \end{bmatrix}. \tag{71}$$

Theorem 4.3 Let $(1+c_i) \geq 0, i = 1, 2, \cdots, k$. Then the LIDS (1)(or (13)) is observable on $[t_0, t_f]\,(t_f \in (t_k, t_{k+1}])$ if and only if

$$rank(Q_o) = n. \tag{72}$$

Proof Let $rank(Q_o) = n$ and we will proof that the system (1)(or (13)) is observable. Here, we assume contrarily that the system (1)(or (13)) is unobservable and $M(t_0, t_f)$ is not invertible, then there exist a nonzero vector z that is $z^*M(t_0, t_f)z = 0$. By using Theorem 4, we can write

$$z^*M(t_0, t_i, t_{i+1})z = \int\limits_{t_i}^{t_{i+1}} \left(Ce^{\bar{E}^D \bar{A}(\tau-t_0)} \bar{E}^D \bar{E}z\right)^* \left(Ce^{\bar{E}^D \bar{A}(\tau-t_0)} \bar{E}^D \bar{E}z\right) d\tau = 0 \tag{73}$$

for $i = 0, 1, \cdots, k-1$ and

$$z^*M(t_0, t_k, t_f)z = \int\limits_{t_k}^{t_f} \left(Ce^{\bar{E}^D \bar{A}(\tau-t_0)} \bar{E}^D \bar{E}z\right)^* \left(Ce^{\bar{E}^D \bar{A}(\tau-t_0)} \bar{E}^D \bar{E}z\right) d\tau = 0; \tag{74}$$

which implies that

$$Ce^{\bar{E}^D \bar{A}(t-t_0)} \bar{E}^D \bar{E}z = 0, \ t \in [t_0, t_f]\,(t_f \in (t_k, t_{k+1}]). \tag{75}$$

Obviously, for $t = t_0$ in equation (75) we have $C\bar{E}^D \bar{E}z = 0$.

Differentiating equation (75) $(n-1)$ times at $t = t_0$, it implies that

$$C\left(\bar{E}^D \bar{A}\right)^j \bar{E}^D \bar{E}z = 0 \text{ for } j = 1, 2, \cdots, n-1. \tag{76}$$

Since $\bar{E}^D \bar{A} = \bar{A}^D \bar{E} = \bar{E} \bar{A}^D$, it follows

$$C \left(\bar{E}^D \bar{A} \right)^j \bar{E}^D \bar{E} z = C \left(\bar{E}^D \bar{A} \right)^j z, \tag{77}$$

that is

$$\begin{aligned}
C \left(\bar{E}^D \bar{A} \right)^{j-1} \left(\bar{E}^D \bar{A} \right) \bar{E} \ \bar{E}^D z &= C \left(\bar{E}^D \bar{A} \right)^{j-1} \bar{E}^D \bar{E} \ \bar{A} \bar{E}^D z \\
&= C \left(\bar{E}^D \bar{A} \right)^{j-1} \bar{E}^D \bar{E} \bar{E}^D \bar{A} z = C \left(\bar{E}^D \bar{A} \right)^{j-1} \bar{E}^D \bar{A} z \\
&= C \left(\bar{E}^D \bar{A} \right)^j z = 0, \text{ for } j = 0, 1, 2, \cdots, n-1.
\end{aligned} \tag{78}$$

Therefore we can write

$$C \left(\bar{E} \bar{A}^D \right)^j z = 0 \text{ for } j = 0, 1, 2, \cdots, n-1. \tag{79}$$

Since $z \neq 0$, this implies that $rank(Q_o) < n$ and leads to a contradiction for our assumption that $rank(Q_o) = n$.

For the converse part, we assume that $rank(Q_o) < n$. Then there exist $z \neq 0$ such that $(Q_o)z = 0$, that is

$$C \left(\bar{E} \bar{A}^D \right)^j z = 0 \text{ for } j = 0, 1, 2, \cdots, n-1. \tag{80}$$

Equation (77) implies that

$$C(\bar{E}^D \bar{A})^i \bar{E}^D \bar{E} z = 0. \tag{81}$$

From both equations (43) and (81), it follows that

$$\begin{aligned}
M(t_0, t_i, t_{i+1})z &= \int_{t_i}^{t_{i+1}} \sum_{j=0}^{n-1} \gamma_j(\tau - t_0) \left(C e^{\bar{E}^D \bar{A}(\tau - t_0)} \bar{E}^D \bar{E} \right)^* \\
&\times \left(C \left(\bar{E} \bar{A}^D \right)^j \bar{E}^D \bar{E} \right) z d\tau = 0, \ i = 0, 1, 2, \cdots, k-1 \\
&\text{and}
\end{aligned} \tag{82}$$

$$\begin{aligned}
M(t_0, t_k, t_f)z &= \int_{t_k}^{t_f} \sum_{j=0}^{n-1} \gamma_j(\tau - t_0) \left(C e^{\bar{E}^D \bar{A}(\tau - t_0)} \bar{E}^D \bar{E} \right)^* \\
&\times \left(C \left(\bar{E} \bar{A}^D \right)^j \bar{E}^D \bar{E} \right) z d\tau = 0.
\end{aligned}$$

From equation (79) we note that $M(t_0, t_f)z = 0$, but we have assumed that $z \neq 0$. Thus $M(t_0, t_f)$ becomes a singular matrix which is a contradiction and completes our proof.

Example 4.4 Consider the following impulsive system

$$\begin{cases}
E \ \dot{x}(t) = Ax(t) + Bu(t), \\
x(t_0) = 1, \\
x(t_i^+) = (\frac{1}{2})x(t_i), \ t_i = \frac{(4i-3)}{2}, i = 1, 2, \cdots, \\
y(t) = Cx(t), t \in [0, 8.5],
\end{cases} \tag{83}$$

where

$$\begin{aligned}
E &= \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}, \ A = \begin{pmatrix} -2 & 3 \\ 4 & 7 \end{pmatrix}, \ B = \begin{pmatrix} 7 & 2 \end{pmatrix}^T, \\
C &= \begin{pmatrix} 7 & 2 \end{pmatrix}.
\end{aligned} \tag{84}$$

Clearly E is not invertible and $rank(E) = 1$. Also, the pencil of matrices (E, A) is regular for $c = 1$, that is

$$(E - A)^{-1} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -1 \end{pmatrix} \tag{85}$$

From equation (85), we can write system (83) as $\bar{E}\dot{x}(t) = \bar{A}x(t) + \bar{B}u(t)$, where

$$\bar{E} = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ -4 & -6 \end{pmatrix}, \ \bar{A} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{4} \\ -4 & -7 \end{pmatrix} \text{ and } \bar{B} = \begin{pmatrix} \frac{7}{4} \\ -2 \end{pmatrix}. \tag{86}$$

Algorithm 23, implies that

$$\bar{E}^D = \begin{pmatrix} \frac{2}{121} & \frac{3}{121} \\ -\frac{16}{121} & -\frac{24}{121} \end{pmatrix}. \tag{87}$$

and it follows from Theorem 4.3, that is

$$Q_o = \begin{pmatrix} C \\ C\bar{E}^D\bar{A} \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ \frac{117}{121} & \frac{351}{242} \end{pmatrix}. \tag{88}$$

It is easy to see that $rank(Q_o) = 2$. So, the system (83) is observable on $[t_0, t_f] = [0, 8.5]$.

Conclusion In this article, we have considered a linear impulsive differential-algebraic system. We have obtained its solution as well as its necessary and sufficient conditions for the controllability and observability. Adopting a new way for these results our results are more precise than the previously adopted techniques and will be very fruitful for the researchers.

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