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NOTE ON THE CERTAIN SPECIAL FUNCTIONS REPRESENTABLE AS Φ

RECEP ŞAHİN, OĞUZ YAĞCI

Abstract. Here,we will give relationship between Sumudu and Laplace transform for a great number of the special functions which can be stated in the sense of the Φ (or ${}_1F_1$). Also, we present fractional derivative operators for the special functions expressible as Φ.

1. Introduction

In 1836, E. E. Kummer investigated many detailed solutions for Confluent (or Kummer) hypergeometric function (CHF) which can be defined as the following equation [4]:

$$ {}_1F_1(a;b;z) = \Phi(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \tag{1.1}$$

$$(\Re(b) > \Re(a) > 0;\ |z| < \infty)\,.$$

Also, you can see the properties of the (CHF) for furher reading see [2, 6].

Definition 1. *The certain special functions which can be expressed as in terms of the (CHF) defined as*[6]*;*

$$J_\xi(z) = \frac{\left(\frac{z}{2}\right)^\xi}{\Gamma(\xi+1)} \exp(\pm iz)\, \Phi\left(\xi + \frac{1}{2}; 2\xi+1; \mp 2iz\right), \tag{1.2}$$

$$M_{\xi,\zeta}(z) = z^{\zeta+\frac{1}{2}} \exp(-\frac{1}{2}z)\, \Phi\left(\zeta - \xi + \frac{1}{2}; 2\zeta+1; z\right), \tag{1.3}$$

$$M_{\xi,\zeta}(z) = z^{\zeta+\frac{1}{2}} \exp(\frac{1}{2}z)\, \Phi\left(\zeta + \xi + \frac{1}{2}; 2\zeta+1; -z\right), \tag{1.4}$$

$$\gamma\,(a,z) = a^{-1} z^a\, \Phi\,(a; a+1; -z)\,, \tag{1.5}$$

$$Erf(z) = z\ \Phi\left(\frac{1}{2}; \frac{3}{2}; -z^2\right), \tag{1.6}$$

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$$L_{\xi}^{(a)}(z)=\frac{\Gamma\left(\xi+a+1\right)}{\Gamma\left(\xi+1\right)\Gamma\left(a+1\right)}\,\Phi\left(-\xi;a+1;z\right), \tag{1.7}$$

and

$$c_{\xi}\left(\zeta,z\right)=(-z)^{-\xi}(\zeta-\xi+1)_{\xi}\,\Phi\left(-\xi;\zeta-\xi+1;z\right). \tag{1.8}$$

In this paper, we will show the relationship betwenn the Sumudu and Laplace transform for the some special functions which can be expressed as with regard to the (CHF) (1.1). Besides, we will give the fractional calculus operator for these speacial functions.

2. Main Results

In this section, we will investigate the relation between Sumudu and Laplace transform for the some special function such as $J_{\xi}(z)$, $M_{\xi,\zeta}(z)$, $\gamma\left(a,z\right)$, $Erf(z)$,$L_{\xi}^{(a)}(z)$, and $c_{\xi}\left(\zeta,z\right)$ which can be stated as (CHF) (1.1). For detailed reading see [1, 5, 8].

Theorem 1. *Assume that* $\lambda,\xi,\zeta\in\mathbb{C}$ *and* $|w|<1$*. The Sumudu transform of the certain special functions* (1.2)-(1.8) *are obtained:*

$$\begin{aligned}\mathfrak{S}\left\{z^{\lambda-1}\,J_{\xi}(wz)\right\}=&\;\frac{\left(\frac{w}{2}\right)^{\xi}\,\Gamma\left(\lambda+\xi\right)}{\Gamma\left(\lambda+1\right)\,\left(1\pm iw\right)^{\lambda+\xi}}\\&\times\,{}_2F_1\left[\xi+\frac{1}{2},\lambda+\xi;2\xi+1;\left(\frac{\mp2iw}{1\pm iw}\right)\right]\end{aligned} \tag{2.1}$$

$$\begin{aligned}\mathfrak{S}\left\{z^{\lambda-1}\,M_{\xi,\zeta}(wz))\right\}=&\;\frac{(w)^{\zeta+\frac{1}{2}}\,\Gamma\left(\lambda+\zeta+\frac{1}{2}\right)}{\left(1+\frac{w}{2}\right)^{\lambda+\zeta+\frac{1}{2}}}\\&\times\,{}_2F_1\left[\zeta-\xi+\frac{1}{2},\lambda+\zeta+\frac{1}{2};2\zeta+1;\left(\frac{2w}{2+w}\right)\right],\end{aligned} \tag{2.2}$$

$$\begin{aligned}\mathfrak{S}\left\{z^{\lambda-1}\,M_{\xi,\zeta}(wz))\right\}=&\;\frac{(w)^{\zeta+\frac{1}{2}}\,\Gamma\left(\lambda+\zeta+\frac{1}{2}\right)}{\left(1-\frac{w}{2}\right)^{\lambda+\zeta+\frac{1}{2}}}\\&\times\,{}_2F_1\left[\zeta+\xi+\frac{1}{2},\lambda+\zeta+\frac{1}{2};2\zeta+1;\left(-\frac{2w}{2-w}\right)\right],\end{aligned} \tag{2.3}$$

$$\mathfrak{S}\left\{z^{\lambda-1}\,\gamma\left(a,wz)\right)\right\}=\;a^{-1}\,w^{a}\Gamma\left(\lambda+a\right)\,{}_2F_1\left[a,\lambda+a;a+1;-w\right], \tag{2.4}$$

$$\mathfrak{S}\left\{z^{\lambda-1}\;Erf(wz)\right\}=\;w\Gamma\left(\lambda+1\right)\,{}_3F_1\left[\frac{1}{2},\frac{\lambda+1}{2},\frac{\lambda+2}{2};\frac{3}{2};-4w^{2}\right], \tag{2.5}$$

$$\mathfrak{S}\left\{z^{\lambda-1}\,L_{\xi}^{(a)}(wz)\right\}=\;\frac{\Gamma\left(\lambda\right)}{\left(\xi+a+1\right)B\left(\xi+1,a+1\right)}\,{}_2F_1\left[-\xi,\lambda;a+1;w\right], \tag{2.6}$$

and

$$\begin{aligned}\mathfrak{S}\left\{z^{\lambda-1}\;c_{\xi}\left(\zeta,wz\right)\right\}=&\;(\zeta-\xi+1)_{\xi}\,(-w)^{-\xi}\Gamma\left(\lambda-\zeta\right)\\&\times\,{}_2F_1\left[-\zeta,\lambda-\zeta;\zeta-\xi+1;w\right]\end{aligned} \tag{2.7}$$

, respectively.

Proof. Using the Sumudu transfrom [8] to defined in (1.2),

$$\mathfrak{S}\left\{z^{\lambda-1}\, J_{\xi}(wz)\right\} = \int_0^{\infty} z^{\lambda-1}\, \exp(-z)\, J_{\xi}(wz)dz, \tag{2.8}$$

and interchanging order of integration and summation (2.8). Taking advantage of the well-known integration formula for the Gamma function [2, 6]:

$$\Gamma\left(s\right) = \int_0^{\infty} t^{s-1}\, \exp(-t)\, dt,$$

we can be easily yield the required result (2.1). In the same way, we can get the equation (2.2)-(2.7). □

Theorem 2. *Assume that* $\lambda, \xi, \zeta, \delta \in \mathbb{C}$ *and* $|w| < 1$*. The Laplace transform of the certain special functions* (1.2)-(1.8) *are obtained:*

$$\begin{aligned}\mathfrak{L}\left\{z^{\lambda-1}\; J_{\xi}(wz);\delta\right\} = &\frac{\left(\frac{w}{2}\right)^{\xi}\Gamma\left(\lambda+\xi\right)}{\Gamma\left(\lambda+1\right)\left(\delta \pm iw\right)^{\lambda+\xi}}\\ &\times\; {}_2F_1\left[\xi+\frac{1}{2},\lambda+\xi;2\xi+1;\left(\frac{\mp 2iw}{\delta \pm iw}\right)\right],\end{aligned} \tag{2.9}$$

$$\begin{aligned}\mathfrak{L}\left\{z^{\lambda-1}\; M_{\xi,\zeta}(wz);\delta\right\} = &\frac{(w)^{\zeta+\frac{1}{2}}\,\Gamma\left(\lambda+\zeta+\frac{1}{2}\right)}{\left(\delta+\frac{w}{2}\right)^{\lambda+\zeta+\frac{1}{2}}}\\ &\times\; {}_2F_1\left[\zeta-\xi+\frac{1}{2},\lambda+\zeta+\frac{1}{2};2\zeta+1;\left(\frac{2w}{2\delta+w}\right)\right],\end{aligned} \tag{2.10}$$

$$\begin{aligned}\mathfrak{L}\left\{z^{\lambda-1}\; M_{\xi,\zeta}(wz);\delta\right\} = &\frac{(w)^{\zeta+\frac{1}{2}}\,\Gamma\left(\lambda+\zeta+\frac{1}{2}\right)}{\left(\delta-\frac{w}{2}\right)^{\lambda+\zeta+\frac{1}{2}}}\\ &\times{}_2\,F_1\left[\zeta+\xi+\frac{1}{2},\lambda+\zeta+\frac{1}{2};2\zeta+1;\left(-\frac{2w}{2\delta-w}\right)\right],\end{aligned} \tag{2.11}$$

$$\mathfrak{L}\left\{z^{\lambda-1}\; \gamma\left(a,wz\right);\delta\right\} = a^{-1}w^{a}\Gamma\left(\lambda+a\right)\; {}_2F_1\left[a,\lambda+a;a+1;-\frac{w}{\delta}\right], \tag{2.12}$$

$$\mathfrak{L}\left\{z^{\lambda-1}\, Erf(wz);\delta\right\} = w\,\Gamma\left(\lambda+1\right)\; {}_3F_1\left[\frac{1}{2},\frac{\lambda+1}{2},\frac{\lambda+2}{2};\frac{3}{2};-\frac{4w^2}{\delta^2}\right], \tag{2.13}$$

$$\mathfrak{L}\left\{z^{\lambda-1}\, L_{\xi}^{(a)}(wz);\delta\right\} = \frac{\Gamma\left(\lambda\right)}{\left(\xi+a+1\right)B\left(\xi+1,a+1\right)}\, {}_2F_1\left[-\xi,\lambda;a+1;\frac{w}{\delta}\right], \tag{2.14}$$

and

$$\begin{aligned}\mathfrak{L}\left\{z^{\lambda-1}\; c_{\xi}\left(\zeta,wz\right);\delta\right\} = &\left(\zeta-\xi+1\right)_{\xi}\left(-w\right)^{-\xi}\Gamma\left(\lambda-\zeta\right)\\ &\times\; {}_2F_1\left[-\zeta,\lambda-\zeta;\zeta-\xi+1;\frac{w}{\delta}\right]\end{aligned} \tag{2.15}$$

,respectively.

Proof. Applying the Laplace transfrom [1, 5] to defined in (1.2),

$$\mathfrak{L}\left\{z^{\lambda-1}\, J_{\xi}(wz);\delta\right\} = \int_0^\infty z^{\lambda-1}\exp(-\delta z)\, J_{\xi}(wz)\, dz \tag{2.16}$$

and shifting order of integration and summation (2.16). Taking advantage of the well-known Laplace transfrom equation for power function [1, 5]:

$$\int_0^\infty t^{\lambda+n-1}\, \exp\left(-\delta t\right) dt = \frac{\Gamma\left(\lambda+n\right)}{\delta^{\lambda+n}},$$

we can be easily yield the desired result (2.9). In the same way, we can get the equation (2.10)-(2.15). □

As you see that we can be easily seen the relation between Laplace transform and Sumudu transform for the certain special functions which are expressible as (CHF), if we take $\delta = 1$ in the $Theorem$ 2.

3. Fractional Calculus Operators

In this section, we derive certain fractional derivative operators for the some special functions such as $\gamma\left(a,z\right)$, $Erf(z)$, $L_{\xi}^{(a)}(z)$, and $c_{\xi}\left(\zeta,z\right)$ which can be stated as (CHF) (1.1). In accordance with this purpose, we evoke the derivative operators $\left\{D_{0+}^{\mu,\delta,\kappa}f\right\}(y)$ and $\left\{D_{\infty-}^{\mu,\delta,\kappa}f\right\}(y)$, which are given as integral operators $\left\{I_{0+}^{\mu,\delta,\kappa}f\right\}(y)$ and $\left\{I_{\infty-}^{\mu,-\delta,\kappa}f\right\}(y)$. In addition to these fractional derivative operators, we give following definition for the relationship between Riemann-Liouville, Erdėlyi-Kober and Weyl fractional operartors [3, 7].

Definition 2. *For $y > 0, \Re(\mu) > 0$ and $\mu,\delta,\kappa \in \mathbb{C}$, the left sided fractional integral and derivative operators $\left\{D_{0+}^{\mu,\delta,\kappa}f\right\}(y)$ and $\left\{I_{0+}^{\mu,\delta,\kappa}f\right\}(y)$, and also the right sided fractional integral and derivative operators $\left\{D_{\infty-}^{\mu,\delta,\kappa}f\right\}(y)$ and $\left\{I_{\infty-}^{\mu,-\delta,\kappa}f\right\}(y)$ are defined by* [3, 7]*;*

$$\left\{I_{0+}^{\mu,\delta,\kappa}f\right\}(y) = \frac{y^{-\mu-\delta}}{\Gamma\left(\mu\right)}\int_0^y {}_2F_1(\mu+\delta,-\kappa;\mu;1-\frac{t}{y})\, f(t)\, dt, \tag{3.1}$$

$$\left\{D_{0+}^{\mu,\delta,\kappa}f\right\}(y) = \left\{I_{0+}^{-\mu,-\delta,\mu+\kappa}f\right\}(y) = \left(\frac{d}{dy}\right)^m \left\{I_{0+}^{-\mu+\kappa,-\delta-\kappa,\mu+\kappa-m}f\right\}(y), \tag{3.2}$$

$$\left\{I_{\infty-}^{\mu,\delta,\kappa}f\right\}(y) = \frac{1}{\Gamma\left(\mu\right)}\int_0^y \left(t-y\right)^{-\mu} t^{-\mu-\delta}\ {}_2F_1(\mu+\delta,-\kappa;\mu;1-\frac{y}{t})\, f(t)\, dt, \tag{3.3}$$

and

$$\left\{D_{\infty-}^{\mu,\delta,\kappa}f\right\}(y) = \left\{I_{\infty-}^{-\mu,-\delta,\mu+\kappa}f\right\}(y) = \left(-\frac{d}{dy}\right)^m \left\{I_{0+}^{-\mu+\kappa,-\delta-\kappa,\mu+\kappa-m}f\right\}(y), \tag{3.4}$$

where $m = [\Re(\mu)] + 1$, respectively. Together with above fractional derivative operators, we have the following equalities:

$$RL_{0+}^{\mu} = D_{0+}^{\mu,-\mu,\kappa}\ \ and\ \ EK_{0+}^{\mu,\rho} = D_{0+}^{\mu,0,\kappa} \tag{3.5}$$

and

$$W_{\infty-}^{\mu} = D_{\infty-}^{\mu,-\mu,\kappa}\ \ and\ \ EK_{\infty-}^{\mu,\kappa} = D_{\infty-}^{\mu,0,\kappa} \tag{3.6}$$

where RL, EK, W is the Riemann-Liouville, Erdėlyi-Kober and Weyl fractional calculus operators, respectively [3, 7].

Theorem 3. *The left-sided fractional derivative of the certain special functions* (1.5)-(1.8) *are obtained:*

$$\begin{aligned}\left\{D_{0+}^{\mu,\delta,\kappa}w^{\lambda-1}\ \gamma\left(a,wz\right)\right\}(\rho) &= a^{-1}z^{a}\rho^{\lambda+a+\delta-1}\\ &\times\ \frac{\Gamma\left(\lambda+a\right)\Gamma\left(\lambda+a+\mu+\delta+\kappa\right)}{\Gamma\left(\lambda+a+\delta\right)\Gamma\left(\lambda+a+\kappa\right)}\\ &\times\ {}_{3}F_{3}\left[\begin{array}{ll}\lambda+a,\lambda+a+\mu+\delta+\kappa,a; & \\ & \\ \lambda+a+\delta,\lambda+a+\kappa,a+1; & -z\rho\end{array}\right]\end{aligned} \tag{3.7}$$

$$\left(\rho>0;\Re(\mu)\geq\ 0;\ \Re(\lambda+a)>-\min\left\{0,\ \Re(\mu+\delta+\kappa)\right\}\right),$$

$$\begin{aligned}\left\{D_{0+}^{\mu,\delta,\kappa}w^{\lambda-1}\ Erf(wz)\right\}(\rho) &=\ z\rho^{\lambda+\delta}\\ &\times\ \frac{\Gamma\left(\lambda+1\right)\Gamma\left(\lambda+1+\mu+\delta+\kappa\right)}{\Gamma\left(\lambda+1+\delta\right)\Gamma\left(\lambda+1+\kappa\right)}\\ &\times\ \times\ {}_{5}F_{5}\left[\begin{array}{ll}\frac{\lambda+1}{2},\frac{\lambda+2}{2},\frac{\lambda+1+\mu+\delta+\kappa}{2},\frac{\lambda+2+\mu+\delta+\kappa}{2},\frac{3}{2}; & \\ & \\ \frac{\lambda+1+\delta}{2},\frac{\lambda+2+\delta}{2},\frac{\lambda+1+\kappa}{2},\frac{\lambda+2+\kappa}{2},\frac{1}{2}; & -\left(z\rho\right)^{2}\end{array}\right]\end{aligned} \tag{3.8}$$

$$\left(\rho>0;\ \Re(\mu)\geq 0;\ \Re(\lambda+1)>-\min\left\{0,\ \Re(\mu+\delta+\kappa)\right\}\right),$$

$$\begin{aligned}\left\{D_{0+}^{\mu,\delta,\kappa}w^{\lambda-1}\ L_{\xi}^{(a)}(wz)\right\}(\rho) &= \frac{\rho^{\lambda+\delta-1}}{\left(\xi+a+1\right)B\left(\xi+1,a+1\right)}\\ &\times\ \frac{\Gamma\left(\lambda\right)\Gamma\left(\lambda+\mu+\delta+\kappa\right)}{\Gamma\left(\lambda+\delta\right)\Gamma\left(\lambda+\kappa\right)}\\ &\times\ {}_{3}F_{3}\left[\begin{array}{ll}\lambda,\lambda+\mu+\delta+\kappa,\xi; & \\ & \\ \lambda+\delta,\lambda+\kappa,a+1; & z\rho\end{array}\right]\end{aligned} \tag{3.9}$$

$$\left(\rho>0;\ \Re(\mu)\geq 0;\ \Re(\lambda)>-\min\left\{0,\ \Re(\mu+\delta+\kappa)\right\}\right),$$

and

$$\begin{aligned}\left\{D_{0+}^{\mu,\delta,\kappa}w^{\lambda-1}\ c_{\xi}\left(\zeta,wz\right)\right\}(\rho) &= (-z)^{-\xi}\left(\zeta-\xi+1\right)_{\xi}\rho^{\lambda+\xi+\delta-1}\\ &\times\ \frac{\Gamma\left(\lambda+\xi\right)\Gamma\left(\lambda+\xi+\mu+\delta+\kappa\right)}{\Gamma\left(\lambda+\xi+\delta\right)\Gamma\left(\lambda+\xi+\kappa\right)}\\ &\times\ {}_{3}F_{3}\left[\begin{array}{ll}\lambda+\xi,\lambda+\xi+\mu+\delta+\kappa,-\xi; & \\ & \\ \lambda+\xi+\delta,\lambda+\xi+\kappa,\zeta-\xi+1; & z\rho\end{array}\right]\end{aligned} \tag{3.10}$$

$$\left(\rho>0;\ \Re(\mu)\geq 0;\Re(\lambda-\xi)>-\min\left\{0,\ \Re(\mu+\delta+\kappa)\right\}\right),$$

Proof. Using the left sided hypergeometric fractional transfrom (3.2) to defined in (1.5), regulating order of integration and summation, we get,

$$\begin{aligned}&\left\{D_{0+}^{\mu,\delta,\kappa} w^{\lambda-1}\ \gamma\left(a, wz\right)\right\}(\rho)\\&\quad= a^{-1}\, z^{a}\, \Phi\left(a; a+1; -z\right)\, \left\{D_{0+}^{\mu,\delta,\kappa} w^{\lambda+a+n-1}\right\}(\rho)\end{aligned} \tag{3.11}$$

and taking advantage of the well-known fractional derivative operator equality for the power function [7]:

$$\left\{D_{0+}^{\mu,\delta,\kappa} w^{\lambda+n-1}\right\}(\rho) = |\, \rho^{\lambda+n+\delta-1}\, \frac{\Gamma\left(\lambda+n\right)\Gamma\left(\lambda+n+\mu+\delta+\kappa\right)}{\Gamma\left(\lambda+n+\delta\right)\Gamma\left(\lambda+n+\kappa\right)}. \tag{3.12}$$

Applying the equation (3.12) in the equation (3.11), and making some arragenments. We can be easily yield the required result (3.7). In the same way, we can get the equation (3.8)-(3.10). □

If we put $\delta = -\mu$ and $\delta = 0$ and using the equalities (3.5) in $Theorem\, 3$, respectively, we can obtain the $Corollary\, 1$ and $Corollary\, 2$.

Corollary 1. *The left sided Riemann-Lioville fractional derivative operators of the certain special functions* (1.5)-(1.8) *are obtained:*

$$\begin{aligned}\left\{RL_{0+}^{\mu}\ w^{\lambda-1}\ \gamma\left(a, wz\right)\right\}(\rho) &= a^{-1} z^{a} \rho^{\lambda+a-\mu-1}\\&\times \frac{\Gamma\left(\lambda+a\right)}{\Gamma\left(\lambda+a-\mu\right)}\\&\times\ {}_{2}F_{2}\left[\begin{array}{cc}\lambda+a, a; & \\ & \\ \lambda+a-\mu, a+1; & -z\rho\end{array}\right],\end{aligned} \tag{3.13}$$

$$\begin{aligned}\left\{RL_{0+}^{\mu} w^{\lambda-1}\ Erf(wz)\right\}(\rho) &= z\, \rho^{\lambda-\mu}\, \frac{\Gamma\left(\lambda+1\right)}{\Gamma\left(\lambda+1-\mu\right)}\\&\times\ {}_{3}F_{3}\left[\begin{array}{cc}\frac{\lambda+1}{2}, \frac{\lambda+2}{2}, \frac{3}{2}; & \\ & \\ \frac{\lambda+1-\mu}{2}, \frac{\lambda+2-\mu}{2}, \frac{1}{2}; & -\left(z\rho\right)^{2}\end{array}\right]\end{aligned} \tag{3.14}$$

$$\begin{aligned}\left\{D_{0+}^{\mu,\delta,\kappa} w^{\lambda-1}\, L_{\xi}^{(a)}(wz)\right\}(\rho) &= \frac{\rho^{\lambda-\mu-1}}{\left(\xi+a+1\right) B\left(\xi+1, a+1\right)}\\&\times\ \frac{\Gamma\left(\lambda\right)}{\Gamma\left(\lambda-\mu\right)}\\&\times\ {}_{2}F_{2}\left[\begin{array}{cc}\lambda, \xi; & \\ & \\ \lambda-\mu, a+1; & z\rho\end{array}\right]\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\left\{D_{0+}^{\mu,\delta,\kappa} w^{\lambda-1}\, c_{\xi}\left(\zeta, wz\right)\right\}(\rho) &= \left(-z\right)^{-\xi}\left(\zeta-\xi+1\right)_{\xi}\rho^{\lambda+\xi-\mu-1}\\&\times\ \frac{\Gamma\left(\lambda+\xi\right)}{\Gamma\left(\lambda+\xi-\mu\right)}\\&\times\ {}_{2}F_{2}\left[\begin{array}{cc}\lambda+\xi, -\xi; & \\ & \\ \lambda+\xi-\mu, \zeta-\xi+1; & z\rho\end{array}\right]\end{aligned} \tag{3.16}$$

, respectively.

Corollary 2. *The left-sided Erdėlyi-Kober fractional derivative operators of the certain special functions* (1.5)-(1.8) *are obtained:*

$$\begin{aligned}\left\{EK_{0+}^{\mu,\kappa}w^{\lambda-1}\ \gamma\left(a,wz\right)\right\}(\rho) &= a^{-1}z^{a}\rho^{\lambda+a-1}\\ &\times\ \frac{\Gamma\left(\lambda+a+\mu+\kappa\right)}{\Gamma\left(\lambda+a+\kappa\right)}\\ &\times\ {}_2F_2\left[\begin{matrix}\lambda+a+\mu+\kappa, a; & \\ & \\ \lambda+a+\kappa, a+1; & -z\rho\end{matrix}\right]\end{aligned}\tag{3.17}$$

$$\begin{aligned}\left\{EK_{0+}^{\mu,\kappa}w^{\lambda-1}\ Erf(wz)\right\}(\rho) &=\ z\,\rho^{\lambda}\frac{\Gamma\left(\lambda+1+\mu+\kappa\right)}{\Gamma\left(\lambda+1+\kappa\right)}\\ &\times\ {}_3F_3\left[\begin{matrix}\frac{\lambda+1+\mu+\kappa}{2}, \frac{\lambda+2+\mu+\kappa}{2}, \frac{3}{2}; & \\ & \\ \frac{\lambda+1+\kappa}{2}, \frac{\lambda+2+\kappa}{2}, \frac{1}{2}; & -\left(z\rho\right)^{2}\end{matrix}\right]\end{aligned}\tag{3.18}$$

$$\begin{aligned}\left\{EK_{0+}^{\mu,\kappa}w^{\lambda-1}\ L_{\xi}^{(a)}(wz)\right\}(\rho) &= \frac{\rho^{\lambda-1}}{\left(\xi+a+1\right)B\left(\xi+1,a+1\right)}\\ &\times\ \frac{\Gamma\left(\lambda+\mu+\kappa\right)}{\Gamma\left(\lambda+\kappa\right)}\\ &\times\ {}_2F_2\left[\begin{matrix}\lambda+\mu++\kappa, \xi; & \\ & \\ \lambda+\kappa, a+1; & z\rho\end{matrix}\right],\end{aligned}\tag{3.19}$$

and

$$\begin{aligned}\left\{EK_{0+}^{\mu,\kappa}w^{\lambda-1}\ c_{\xi}\left(\zeta,wz\right)\right\}(\rho) &= (-z)^{-\xi}\left(\zeta-\xi+1\right)_{\xi}\rho^{\lambda+\xi-1}\\ &\times\ \frac{\Gamma\left(\lambda+\xi+\mu+\kappa\right)}{\Gamma\left(\lambda+\xi+\kappa\right)}\\ &\times\ {}_2F_2\left[\begin{matrix}\lambda+\xi+\mu+\kappa, -\xi; & \\ & \\ \lambda+\xi+\kappa, \zeta-\xi+1; & z\rho\end{matrix}\right]\end{aligned}\tag{3.20}$$

, respectively.

Theorem 4. *The right-sided fractional derivative of the certain special functions* (1.5)-(1.8) *are obtained:*

$$\begin{aligned}\left\{D_{\infty-}^{\mu,\delta,\kappa}w^{\lambda-1}\ \gamma\left(a,\frac{z}{w}\right)\right\}(\rho) &= a^{-1}z^{a}\rho^{\lambda+a+\delta-1}\\ &\times\ \frac{\Gamma\left(1-\lambda-a-\delta\right)\Gamma\left(1-\lambda+\mu+\kappa\right)}{\Gamma\left(1-\lambda-a\right)\Gamma\left(1-\lambda-a+\kappa-\delta\right)}\\ &\times\ {}_3F_3\left[\begin{matrix}1-\lambda-a-\delta, 1-\lambda-a+\mu+\kappa, a; & \\ & \\ 1-\lambda-a, 1-\lambda-a+\kappa-\delta, a+1; & \frac{z}{\rho}\end{matrix}\right]\end{aligned}\tag{3.21}$$

$$\left(\rho>0; \Re(\mu)\geq 0;\ \Re(\lambda+a)>-\min\left\{0,\Re(\mu+\delta+\kappa)\right\}\right),$$

$$\left\{D_{\infty-}^{\mu,\delta,\kappa}w^{\lambda-1}\,Erf(\frac{z}{w})\right\}(\rho)=\ z\,\rho^{\lambda+\delta}\frac{\Gamma\left(-\lambda-\delta\right)\Gamma\left(-\lambda+\mu+\kappa\right)}{\Gamma\left(-\lambda\right)\Gamma\left(-\lambda+\kappa-\delta\right)}$$
$$\times\ {}_5F_5\left[\begin{array}{ll}\frac{-\lambda-\delta}{2},\frac{1-\lambda-\delta}{2},\frac{-\lambda+\mu+\kappa}{2},\frac{1-\lambda+\mu+\kappa}{2},\frac{1}{2}; & \\ \frac{-\lambda}{2},\frac{1-\lambda}{2},\frac{-\lambda+\kappa-\delta}{2},\frac{1-\lambda+\kappa-\delta}{2},\frac{3}{2}; & -\left(\frac{z}{\rho}\right)^2\end{array}\right] \tag{3.22}$$

$$\left(\rho>0;\,\Re(\mu)\geq 0;\,\Re(\lambda+1)>-\min\left\{0,\Re(\mu+\delta+\kappa)\right\}\right),$$

$$\left\{D_{\infty-}^{\mu,\delta,\kappa}w^{\lambda-1}\ L_{\xi}^{(a)}(\frac{z}{w})\right\}(\rho)=\frac{\rho^{\lambda+\delta-1}}{\left(\xi+a+1\right)B\left(\xi+1,a+1\right)}$$
$$\times\ \frac{\Gamma\left(1-\lambda+\delta\right)\Gamma\left(1-\lambda+\mu+\kappa\right)}{\Gamma\left(1-\lambda\right)\Gamma\left(1-\lambda+\kappa-\delta\right)}$$
$$\times\ {}_3F_3\left[\begin{array}{ll}1-\lambda+\delta,,1-\lambda+\mu+\kappa,-\xi; & \\ 1-\lambda,1-\lambda+\kappa-\delta,a+1; & \frac{z}{\rho}\end{array}\right] \tag{3.23}$$

$$\left(\rho>0;\,\Re(\mu)\geq 0;\,\Re(\lambda)>-\min\left\{0,\Re(\mu+\delta+\kappa)\right\}\right),$$

and,

$$\left\{D_{\infty-}^{\mu,\delta,\kappa}w^{\lambda-1}\ c_{\xi}\left(\zeta,\frac{z}{w}\right)\right\}(\rho)=(-z)^{-\xi}\left(\zeta-\xi+1\right)_{\xi}\rho^{\lambda+\xi+\delta-1}$$
$$\times\ \frac{\Gamma\left(1-\lambda-\xi-\delta\right)\Gamma\left(1-\lambda-\xi+\mu+\kappa\right)}{\Gamma\left(1-\lambda-\xi\right)\Gamma\left(1-\lambda-\xi+\kappa-\delta\right)}$$
$$\times\ {}_3F_3\left[\begin{array}{ll}1-\lambda-\xi+\delta,,1-\lambda-\xi+\mu+\kappa,-\xi; & \\ 1-\lambda-\xi,1-\lambda-\xi+\kappa-\delta,\zeta-\xi+1; & \frac{z}{\rho}\end{array}\right] \tag{3.24}$$

$$\left(\rho>0;\,\Re(\mu)\geq 0;\,\Re(\lambda+\xi)>-\min\left\{0,\Re(\mu+\delta+\kappa)\right\}\right),$$

Proof. Using the right sided hypergeometric fractional transfrom (3.4) to defined in (1.5), interchanging order of integration and summation, we get,

$$\left\{D_{\infty-}^{\mu,\delta,\kappa}\,w^{\lambda-1}\,\gamma\left(a,wz\right)\right\}(\rho)$$
$$=\ a^{-1}\,z^{a}\,\Phi\left(a;a+1;-z\right)\left\{D_{\infty-}^{\mu,\delta,\kappa}\,w^{\lambda+a-n-1}\right\}(\rho) \tag{3.25}$$

and taking advantage of the well-known fractional derivative operator equality for the power function [7]:

$$\left\{D_{\infty-}^{\mu,\delta,\kappa}w^{\lambda-n-1}\right\}(\rho)=\rho^{\lambda-n+\delta-1}\frac{\Gamma\left(1-\lambda+n-\delta\right)\Gamma\left(1-\lambda+n+\mu+\kappa\right)}{\Gamma\left(1-\lambda+n\right)\Gamma\left(1-\lambda+n+\kappa-\delta\right)}. \tag{3.26}$$

Applying the equation (3.26) in the equation (3.25), and making some arragenments . We can be easily yield the required result (3.21). Similarly, we can get the equation (3.22)-(3.24). □

Applying into $\delta=-\mu$ and $\delta=0$ and taking advatage of the equalities (3.6) in *Theorem* 4, respectively. We can come to a conclusion *Corollary* 3 and *Corollary* 4.

Corollary 3. *The right sided Weyl fractional derivative of the certain special functions* (1.5)-(1.8) *are obtained:*

$$\begin{aligned}\left\{W^{\mu}_{\infty-}w^{\lambda-1}\ \gamma\left(a,\frac{z}{w}\right)\right\}(\rho) &= a^{-1}z^{a}\rho^{\lambda+a-\mu-1}\\ &\times\ \frac{\Gamma(1-\lambda-a+\mu)}{\Gamma(1-\lambda-a)}\\ &\times\ {}_2F_2\left[\begin{array}{ll}1-\lambda-a+\mu,a; & \\ & \\ 1-\lambda-a,a+1; & \frac{z}{\rho}\end{array}\right],\end{aligned} \tag{3.27}$$

$$\begin{aligned}\left\{W^{\mu}_{\infty-}w^{\lambda-1}\ Erf(\frac{z}{w})\right\}(\rho) &= z\rho^{\lambda-\mu}\frac{\Gamma(\lambda+\mu)}{\Gamma(-\lambda)}\\ &\times\ {}_3F_3\left[\begin{array}{ll}\frac{-\lambda+\mu}{2},\frac{1-\lambda+\mu}{2},\frac{1}{2}; & \\ & \\ \frac{-\lambda}{2},\frac{1-\lambda}{2},\frac{3}{2}; & -\left(\frac{z}{\rho}\right)^{2}\end{array}\right],\end{aligned} \tag{3.28}$$

$$\begin{aligned}\left\{W^{\mu}_{\infty-}w^{\lambda-1}\ H_{\zeta}(\frac{z}{w})\right\}(\rho) &= (2z)^{\zeta}\rho^{\lambda-\zeta-\mu-1}\frac{\Gamma(1-\lambda+\zeta+\mu)}{\Gamma(1-\lambda+\zeta)}\\ &\times\ {}_4F_2\left[\begin{array}{ll}\frac{1-\lambda+\zeta+\mu}{2},\frac{2-\lambda+\zeta+\mu}{2},\frac{1}{2}\zeta,\frac{1}{2}-\frac{1}{2}\zeta; & \\ & \\ \frac{1-\lambda+\zeta}{2},\frac{2-\lambda+\zeta}{2}; & -\left(\frac{z}{\rho}\right)^{-2}\end{array}\right],\end{aligned} \tag{3.29}$$

$$\begin{aligned}\left\{W^{\mu}_{\infty-}w^{\lambda-1}\ L^{(a)}_{\xi}(\frac{z}{w})\right\}(\rho) &= \frac{\rho^{\lambda-\mu-1}}{(\xi+a+1)B(\xi+1,a+1)}\\ &\times\ \frac{\Gamma(1-\lambda-\mu)}{\Gamma(1-\lambda)}\\ &\times\ {}_2F_2\left[\begin{array}{ll}1-\lambda-\mu,-\xi; & \\ & \\ 1-\lambda; & \frac{z}{\rho}\end{array}\right],\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}\left\{W^{\mu}_{\infty-}w^{\lambda-1}\ c_{\xi}\left(\zeta,\frac{z}{w}\right)\right\}(\rho) &= (-z)^{-\xi}(\zeta-\xi+1)_{\xi}\rho^{\lambda+\xi-\mu-1}\\ &\times\ \frac{\Gamma(1-\lambda-\xi+\mu)}{\Gamma(1-\lambda-\xi)}\\ &\times\ {}_2F_2\left[\begin{array}{ll}1-\lambda-\xi+\mu,-\xi; & \\ & \\ 1-\lambda-\xi,\zeta-\xi+1; & \frac{z}{\rho}\end{array}\right]\end{aligned} \tag{3.31}$$

, respectively.

Corollary 4. *The right-sided Erdėlyi-Kober fractional derivative of the certain special functions*(1.5)-(1.8) *are obtained:*

$$\left\{EK_{\infty-}^{\mu,\kappa}w^{\lambda-1}\ \gamma\left(a,\frac{z}{w}\right)\right\}(\rho)=a^{-1}z^{a}\rho^{\lambda+a-1}\times\frac{\Gamma\left(1-\lambda+\mu+\kappa\right)}{\Gamma\left(1-\lambda-a+\kappa\right)}\times{}_{2}F_{2}\left[\begin{array}{cc}1-\lambda-a+\mu+\kappa,a; & \\ 1-\lambda-a+\kappa,a+1; & \frac{z}{\rho}\end{array}\right],\tag{3.32}$$

$$\left\{EK_{\infty-}^{\mu,\kappa}w^{\lambda-1}\ Erf(\frac{z}{w})\right\}(\rho)=z\rho^{\lambda}\frac{\Gamma\left(-\lambda+\mu+\kappa\right)}{\Gamma\left(-\lambda+\kappa-\delta\right)}\times{}_{3}F_{3}\left[\begin{array}{cc},\frac{-\lambda+\mu+\kappa}{2},\frac{1-\lambda+\mu+\kappa}{2},\frac{1}{2}; & \\ \frac{-\lambda+\kappa}{2},\frac{1-\lambda+\kappa}{2},\frac{3}{2}; & -\left(\frac{z}{\rho}\right)^{2}\end{array}\right],\tag{3.33}$$

$$\left\{EK_{\infty-}^{\mu,\kappa}w^{\lambda-1}\ L_{\xi}^{(a)}(\frac{z}{w})\right\}(\rho)=\frac{\rho^{\lambda-1}}{\left(\xi+a+1\right)B\left(\xi+1,a+1\right)}\times\frac{\Gamma\left(1-\lambda+\mu+\kappa\right)}{\Gamma\left(1-\lambda+\kappa\right)}\times{}_{2}F_{2}\left[\begin{array}{cc}1-\lambda+\mu+\kappa,-\xi; & \\ 1-\lambda+\kappa,a+1; & \frac{z}{\rho}\end{array}\right],\tag{3.34}$$

and

$$\left\{EK_{\infty-}^{\mu,\kappa}w^{\lambda-1}\ c_{\xi}\left(\zeta,\frac{z}{w}\right)\right\}(\rho)=(-z)^{-\xi}\left(\zeta-\xi+1\right)_{\xi}\rho^{\lambda+\xi-1}\times\frac{\Gamma\left(1-\lambda-\xi+\mu+\kappa\right)}{\Gamma\left(1-\lambda-\xi+\kappa\right)}\times{}_{2}F_{2}\left[\begin{array}{cc}1-\lambda-\xi+\mu+\kappa,-\xi; & \\ 1-\lambda-\xi+\kappa,\zeta-\xi+1; & \frac{z}{\rho}\end{array}\right]\tag{3.35}$$

, respectively.

References

[1] L. Debnath, D. Bhatta, Integral Transforms and Their Applications, Third edition, Chapman and Hall (CRC Press), Taylor and Francis Group, London and New York, 2014.

[2] A. Erdélyi,W. Magnus, F. Oberhettinger, F.G. Tricomi, Higher Transcendental Functions, Vol. I, McGraw-Hill Book Company, NewYork, Toronto and London, 1953.

[3] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies, Vol. 204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

[4] E. E. Kummer, Uber die hipergeometrische Riehe F(a;b;x), J.Reine Angew.Math. 15, 39-83, 1836.

[5] A. M. Mathai, R. K. Saxena, H. J. Haubold, The H-Function, Theory and Applications,Springer, Dordrecht, 2010.

[6] H. M. Srivastava, H. L. Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[7] H. M. Srivastava, R. K. Saxena, Operators of fractional integration and their applications, Appl. Math. Comput. 118, 1–52, 2001.

[8] G. K. Watugala, Sumudu transform: a new integral transform to solve differential equations and control engineering problems, International Journal of Mathematical Education in Science and Technology 24, no:1, 35-43, 1993.

Recep Şahin
Faculty of Science, Department of Mathematics, Kirikkale University, Kirikkale, Turkey
E-mail address: recepsahin@kku.edu.tr

O. Yağci
Kirikkale University, Kirikkale, Turkey
E-mail address: oguzyagci26@gmail.com