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NEW SOLUTIONS FOR CONFORMABLE FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS USING FIRST INTEGRAL METHOD

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Abstract. The main purpose of this paper is to obtain the exact solutions of conformable time fractional Generalized Zakharov-Kuznetsov-Benjamin-Bona-Mahony Equation and Sine-Gordon Equation. The first integral method is used as a tool to get the exact solutions of considered fractional partial differential equations. Authors give the solutions for the case $r = 1$. But one can easily find the solutions for different values of r. Conformable fractional derivative is an applicable and well behaved definition that satisfies the basic properties of Newtonian concept derivative. Also conformable fractional derivative has some advantages over the other fractional derivative definitions.

1. Introduction

Integer order derivatives had been researched till L'Hospital's letter to Leibnitz. In this letter L'Hospital asked the question how the rational order derivative can be evaluated. At the beginning Leibnitz answered that it will be a great paradox. But it is understood that this question was a great beginning of an important research area which has many applications in different branches of science such as physics, chemistry engineering and social sciences. After this beginning many types of fractional derivative introduced by scientists to model the nonlinear real world problems. Riemann-Liouville, Caputo, Grunwald-Letnikov, Erdelyi-Cober, Reisz definitions are most popular ones [1,2,3]. But the models involving these fractional derivatives cannot be solved analytically. Furthermore these derivatives do not satisfy the basic properties which are used commonly in classical calculus. For instance both derivatives do not satisfy derivative of the quotient of two functions, derivative of product of two functions, chain rule and etc. But recently a new definition of fractional derivative and integral which satisfies some basic properties are expressed by Khalil et al.[4] called "conformable fractional derivative and integral".

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Definition 1.1. *Let $f : [0, \infty) \to \mathbb{R}$ be a function, $t > 0$ and $\alpha \in (0, 1)$. Then the fractional derivative of f of order α, T_α or f^α is defined as*

$$f^\alpha(t) = T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - (f)(t)}{\varepsilon}.$$

If f is α-differentiable in some $(0, a)$, $a > 0$ and $\lim_{t \to 0^+} f^\alpha(t)$ exists, then $f^\alpha(0)$ is defined by $f^\alpha(0) = \lim_{t \to 0^+} f^\alpha(t)$. The fractional integral of f starting form $\alpha \geq 0$ is defined as:

$$I^a_\alpha(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$$

where the integral is the usual Riemann improper integral and $\alpha \in (0, 1)$. Some useful properties of conformable fractional derivative is expressed in the following theorem [4,5].

Theorem 1.2. *Let $\alpha \in (0, 1]$ and suppose f, g are α-differantiable at $t > 0$. Then*

1. $T_\alpha(cf + dg) = cT_\alpha(f) + dT_\alpha(g)$ *for all* $a, b \in \mathbb{R}$,
2. $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$,
3. $T_\alpha(\lambda) = 0$ *for all constant functions* $f(t) = \lambda$,
4. $T_\alpha(f.g) = fT_\alpha(g) + gT_\alpha(f)$,
5. $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$,
6. *If f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha} \frac{df(t)}{dt}$.

2. Basic of First Integral Method

Let us present first integral method [6] step by step.

Step 1. Consider a general nonlinear conformable time fractional partial differential equation of the form,

$$F\left(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, ...\right) = 0 \tag{1}$$

where $u(x, y, t)$ is the solution of nonlinear time fractional partial differential equation and $\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}$ indicates two times sequential conformable fractional derivative of the function $u(x, y, t)$.

Step 2. By using the wave transformation

$$u(x, y, t) = U(\xi), \xi = x + y - m\frac{t^\alpha}{\alpha}$$

where m is an arbitrary constant, the nonlinear partial differential Eq.(1) turns into integer order nonlinear ordinary differential equation

$$G(U, U_\xi, U_{\xi\xi}, U_{\xi\xi\xi}, ...) = 0 \tag{2}$$

where the derivatives are with respect to independent variable ξ.

Step 3. Introducing independent variables

$$X(\xi) = U(\xi), Y(\xi) = U_\xi(\xi), Y_\xi(\xi) = U_{\xi\xi}(\xi) \tag{3}$$

led to a system of nonlinear ordinary differential equation

$$\frac{\partial X(\xi)}{\partial \xi} = Y(\xi), \frac{\partial Y(\xi)}{\partial \xi} = S(X(\xi), Y(\xi)). \tag{4}$$

By the qualitative theory of ordinary differential equations, if the integrals to Eq.(4) can be found under the same conditions, then general solution to Eq.(4) can be solved directly. However, in general, it is difficult to realize this even for one first integral, because for a given plane autonomous system. There is no systematic theory which can tell us how to find its first integral.

We apply the Division Theorem to obtain one first integral to Eq.(4), which reduces Eq.(2) to a first-order integrable ordinary differential equation. An exact solution to Eq.(1) is then obtained by solving this equation.

Theorem 2.1. ***[6] (Division Theorem)*** *Suppose that $P(w,z)$ and $Q(w,z)$ are polynomials in $\mathbb{C}(w,z)$ and $P(w,z)$ is irreducible in $\mathbb{C}(w,z)$. If $Q(w,z)$ vanishes at all zero points of $P(w,z)$, then there exists a polynomial $G(w,z)$ in $\mathbb{C}(w,z)$ such that $Q(w,z) = P(w,z)G(w,z)$.*

3. Application of Considered Method

3.1. **Analytical Solution of Fractional Generalized Zakharov-Kuznetsov-Benjamin-Bona-Mahony Equation.** Let us consider the generalized Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation

$$D_t^\alpha u + D_x u + aD_x u^3 + bD_x(D_x D_t^\alpha u + D_y^2 u) = 0. \tag{5}$$

By using the wave transformation $\xi = x + y - m\frac{t^\alpha}{\alpha}$ the function becomes to $u(x,y,t) = U(\xi)$. So Eq.(5) is converted into the ordinary differential equation

$$(1-m)U^{'} + 3aU^2U^{'} + b(1-m)U^{'''} = 0. \tag{6}$$

By integrating Eq.(5) once with respect to ξ, we acquire

$$(1-m)U + aU^3 + b(1-m)U^{''} = 0. \tag{7}$$

Using Eq.(3) and Eq.(4), we get

$$\frac{\partial X(\xi)}{\partial \xi} = Y(\xi), \frac{\partial Y(\xi)}{\partial \xi} = \frac{aX^3(\xi)}{b(m-1)} - \frac{X(\xi)}{b}. \tag{8}$$

According to the first integral method, we suppose that $X(\xi) = X$ and $Y(\xi) = Y$ are non-trivial solutions of Eq.(8) and

$$P(X,Y) = \sum_{i=0}^{r} a_i(X)Y^i$$

is an irreducible function in the complex domain $\mathbb{C}(X,Y)$, such that

$$P(X(\xi),Y(\xi)) = \sum_{i=0}^{r} a_i(X(\xi))Y(\xi)^i \tag{9}$$

where $a_i(X)(i = 0,1,2,?,r)$ are polynomials of X and $a_r(X) \neq 0$. Eq.(9) is the first integral for system (8); owing to the division theorem, there $g(X) + h(X)Y$ in $\mathbb{C}(X,Y)$ such that

$$\frac{\partial P}{\partial \xi} = \frac{\partial P}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial P}{\partial Y}\frac{\partial Y}{\partial \xi} = (g(X) + h(X)Y)\left(\sum_{i=0}^{r} a_i(X)Y^i\right) \tag{10}$$

Suppose that $r = 1$in Eq.(9). By equating the coefficients of $Y^i(i = 0,1,2)$ for both sides of Eq.(10), we have

$$a_1^{'}(X) = a_1(X)h(X), \tag{11}$$

$$a_0^{'}(X) = a_1(X)g(X) + a_0(X)h(X), \tag{12}$$

$$a_1(X)\frac{aX^3(\xi)}{b(m-1)} - a_1(X)\frac{X(\xi)}{b} = a_0(X)g(X). \tag{13}$$

From Eq.(11), if we choose $a_1(X)$ as a constant, then we find $h(X) = 0$. For convenience let $a_1(X) = 1$. From the Eq.(12) and Eq.(13) the degrees of the polynomials $g(X)$ and $a_0(X)$ can be found. So we have that $degg(g(X)) = 1$ and $degg(a_0(X)) = 2$. Supposing that $g(X) = AX + B$ led to

$$a_0(X) = \frac{A}{2}X^2 + BX + C$$

where C is arbitrary integration constant. Subrogating $a_0(X)$, $a_1(X)$ and $g(X)$ in Eq.(13) and equating the coefficients of $X^i (i = 0, 1, 2, 3)$ to zero, we get the following algebraic equation system

$$\frac{a}{b(m-1)} - \frac{A^2}{2} = 0,$$
$$\frac{3AB}{2} = 0,$$
$$B^2 + AC + \frac{1}{b} = 0,$$
$$BC = 0.$$

Solving the equation system yields to the solution

$$B = 0, b = -\frac{1}{AC}, a = \frac{A - Am}{2C}.$$

By using the solution procedure of the method the exact solution of Eq.(5) is obtained

$$u(x,y,t) = \frac{\sqrt{2C}\tanh\left(\frac{1}{2}\left(-\sqrt{2AC}\left(x + y - m\frac{t^\alpha}{\alpha}\right) + 2d\sqrt{2AC}\right)\right)}{\sqrt{A}}$$

where d is integration constant.

3.2. **Analytical Solution of Fractional Sine-Gordon Equation.** Regard the conformable time fractional the Sine-Gordon equation as

$$D_t^{2\alpha}u - D_x^2 u + u - \frac{1}{6}u^3 = 0. \tag{14}$$

Eq.(14) can be converted to the ordinary differential equation by the aid of wave transformation, $\xi = x - m\frac{t^\alpha}{\alpha}$ and chain rule [5] as

$$(m^2 - 1)U^{''} + U - \frac{1}{6}U^3 = 0. \tag{15}$$

Using Eq.(3) and Eq.(4), we have

$$X(\xi) = Y(\xi), Y_\xi(\xi) = \frac{X^3(\xi) - 6X(\xi)}{6(m^2 - 1)}. \tag{16}$$

According to the first integral method, we suppose that $X(\xi) = X$ and $Y(\xi) = Y$ are non-trivial solutions of Eq.(13) and

$$P(X,Y) = \sum_{i=0}^{r} a_i(X)Y^i$$

is an irreducible function in the complex domain $\mathbb{C}(X,Y)$, such that

$$P(X(\xi),Y(\xi))=\sum_{i=0}^{r}a_i(X(\xi))Y(\xi)^i \tag{17}$$

where $a_i(X)(i=0,1,2,?,r)$ are polynomials of X and $a_r(X)\neq 0$. Using Eq.(17) is the first integral for system Eq.(16), owing to the division theorem, there $g(X)+h(X)Y$ in $\mathbb{C}(X,Y)$ such that

$$\frac{\partial P}{\partial \xi}=\frac{\partial P}{\partial X}\frac{\partial X}{\partial \xi}+\frac{\partial P}{\partial Y}\frac{\partial Y}{\partial \xi}=(g(X)+h(X)Y)\left(\sum_{i=0}^{r}a_i(X)Y^i\right). \tag{18}$$

We consider $r=1$ in Eq.(18). By equating with the coefficients of $Y^i(i=0,1,2)$ in both sides of Eq.(18), then we have

$$a_1^{'}(X)=a_1(X)h(X), \tag{19}$$

$$a_0^{'}(X)=a_1(X)g(X)+a_0(X)h(X), \tag{20}$$

$$a_1(X)\frac{X^3-6X}{6(m^2-1)}=a_0(X)g(X). \tag{21}$$

From Eq.(19), if we choose $a_1(X)$ as a constant, then we find $h(X)=0$. Then we take $a_1(X)=1$ for convenience. From the Eq.(20) and Eq.(21), the degrees of polynomials $g(X)$ and $a_0(X)$ can be found. So we have $degg(g(X))=1$ and $degg(a_0(X))=2$. Suppose that $g(X)=AX+B$, then then we find

$$a_0(X)=\frac{A}{2}X^2+BX+C$$

where C is arbitrary integration constant. Replacing $a_0(X)$, $a_1(X)$ and $g(X)$ in Eq.(21) and equating the coefficients of $X^i(i=0,1,2,3)$ to zero, we have the following equation system

$$\frac{1}{6(m^2-1)}-\frac{A^2}{2}=0,$$

$$\frac{3AB}{2}=0,$$

$$B^2+AC+\frac{1}{m^2-1}=0,$$

$$BC=0.$$

Solving the system led to

$$A=\frac{1}{\sqrt{3(m^2-1)}},C=-\frac{\sqrt{3}}{\sqrt{(m^2-1)}},B=0. \tag{22}$$

So the exact solution of conformable time fractional Sine-Gordon equation is

$$u(x,t)=\sqrt{6}\tanh\left(\frac{x-m\frac{t^\alpha}{\alpha}}{\sqrt{2(m^2-1)}}-d\sqrt{6}\right)$$

where d is an arbitrary integral constant.

4. Conclusion

First integral method which includes division theorem in its solution procedure is employed to get the exact solutions of time fractional Generalized Zakharov-Kuznetsov-Benjamin-Bona-Mahony Equation and Sine-Gordon Equation. The results show that implemented method is efficient, accurate and reliable for obtaining the analytical solutions of fractional partial differential equations in conformable sense. Also the obtained results are new and firstly seen in the literature.

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