

NEW TRAVELLING WAVE SOLUTIONS FOR TIME-SPACE FRACTIONAL EQUATIONS ARISING IN NONLINEAR OPTICS

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ABSTRACT. In this paper, the authors proposed new wave solutions of time-space fractional Dodd-Bullough-Mikhailov, Tzitzica-Dodd-Bullough and Tzitzica equations by using a reliable analytical method called sub-equation method. The fractional derivatives considered in this study are handled in conformable sense. Conformable derivative is an easy and applicable type of fractional derivative which satisfies the basic properties of known derivative with integer order such as Leibniz rule, quotient rule, chain rule. With these properties, conformable derivatives superior to other famous fractional derivatives such as Caputo and Riemann-Liouville.

1. INTRODUCTION

In the last decades, the number of studies on partial differential equations have increased dramatically since they can be used in many areas of science including physical, engineering, social, biological and chemical sciences [1-3]. Most of these studies mainly focused on obtaining the exact solutions of fractional partial differential equations. However, the known fractional derivative definitions such as Riemann-Liouville and Caputo do not always have capabilities to achieve the exact solutions. Since they do not satisfy some basic principles of known integer order derivatives, it is not possible to solve some fractional derivatives using these definitions. For instance

- Riemann-Liouville derivative definition does not satisfy $D^\mu c = 0$ where c is a real constant is and μ is not a natural number. (Caputo derivative definition satisfies this property).
- Riemann-Liouville and Caputo derivatives do not satisfy the derivative of the product of two functions.

$$D^\mu(fg) = gD^\mu(f) + fD^\mu(g).$$

2010 *Mathematics Subject Classification.* 35R11.

Key words and phrases. Conformable fractional derivative, Sub-Equation method, Dodd-Bullough-Mikhailov equation, Tzitzica-Dodd-Bullough equation, Tzitzica Equation.

Submitted Jan. 30, 2019.

- Riemann-Liouville and Caputo derivatives do not satisfy the derivative of the quotient of two functions.

$$D^\mu \left(\frac{f}{g} \right) = \frac{gD^\mu(f) - fD^\mu(g)}{g^2}.$$

- Riemann-Liouville and Caputo derivatives do not satisfy the chain rule.

$$D^\mu(fog) = f^{(\mu)}(g(t))g^{(\mu)}(t).$$

- Riemann-Liouville and Caputo derivatives do not satisfy $D^\mu D^\beta(f) = D^{\mu+\beta}(f)$ generally.
- Caputo derivative definition assumes that the function f is differentiable.

Due to the limitations of known definitions indicated above, the scientists studied to find a new definition for fractional derivative which can satisfy all the basic principles. Recently, the conformable fractional derivative suggested by Khalil et al. which is a simple and efficient fractional derivative definition [4].

Definition 1.1. Let $f : (0, \infty) \rightarrow \mathbb{R}$, be a function. Then, the conformable fractional derivative of f of order μ is defined in [4] as follows:

$$D_t^\mu f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > 0, \mu \in (0, 1)$.

Definition 1.2. Let $a \geq 0$ and $t \geq a$. Suppose f be a function defined on $(a, t]$ and $\mu \in \mathbb{R}$. In this case, the μ - fractional integral of f is defined in [4] by,

$${}_t I_a^\mu f(t) = \int_a^t \frac{f(x)}{x^{1-\mu}} dx.$$

With the definitions above, the theorem that the conformable fractional derivative definition satisfies the properties is given in Ref. [4]. Since the conformable fractional derivative is an easy and efficient way to be applied on many fractional derivative equations, it has been used by many scientists to provide new solutions for existing differential equations. Ilie et al. [5] presented general solutions of Bernoulli and Riccati fractional differential equations using conformable fractional derivative definition. Eslami et al. [6] applied the first integral method to provide the exact solutions for the nonlinear conformable fractional Bogoyavlenskii equations. Rezazadeh et al. [7] used conformable fractional derivative and obtained exact solutions for nonlinear time fractional Sharma-Tasso-Olver equation. Ilie et al. [8] applied adomian decomposition method to get analytical solutions for Bruto type equations with the aim of conformable fractional derivative. Aminikhah et al. [9] used sub-equation method to find the exact solutions for the time fractional long-wave equations with the help of conformable derivatives. Similarly, Rezazadeh and Ziabary [10] used the sub-equation method in constructing the travelling wave solutions of the conformable fractional generalized Kuramoto-Sivanshinsky equation. In this paper, the sub-equation method will be used to obtain the exact solutions of Fractional Tzitzica-Dodd-Bullough equation, Fractional Dodd-Bullough-Mikhailov equation and Tzitzica Equation, respectively.

2. DESCRIPTION OF THE SUB-EQUATION METHOD

This section presents a brief description of fractional sub-equation method [11]. Consider the nonlinear fractional partial differential equation below

$$P(u, D_t^\alpha u, D_x^\beta u, D_t^{(2\alpha)} u, D_x^{(2\beta)} u, \dots) = 0 \tag{1}$$

where all the fractional derivatives are in conformable sense. $u(x, t)$ is any function and $D_t^{(n\alpha)}$ means n times conformable fractional derivative $u(x, t)$. The sub-equation method will be explained step by step as follows:

Step 1. Using the wave transformation, we obtain the following equalities.

$$u(x, t) = U(\xi), \quad \xi = m \frac{x^\beta}{\beta} + k \frac{t^\alpha}{\alpha}. \tag{2}$$

Here, m, k are arbitrary constants to be examined later and $\alpha, \beta \in (0, 1)$.

Now, Eq. (1) can be rewritten in the form of the following nonlinear ODE by using chain rule [12]:

$$G(U, U', U'', \dots) = 0 \tag{3}$$

where prime shows the known derivative with respect to ξ .

Step 2. Assume that Eq. (3) has a solution in the following form

$$U(\xi) = \sum_{i=0}^N a_i \varphi^i(\xi), \quad a_N \neq 0 \tag{4}$$

where $a_i (0 \leq i \leq N)$ are constant coefficients to be determined later. N is a positive integer which can be found by balancing procedure in Eq. (3) and $\varphi(\xi)$ satisfies the following ordinary differential equation

$$\varphi'(\xi) = \sigma + (\varphi(\xi))^2 \tag{5}$$

where σ is a constant. Some special solutions for the Eq. (5) are given in the following formulas.

$$\varphi(\xi) = \begin{cases} -\sqrt{-\sigma} \tanh(\sqrt{-\sigma}\xi), & \sigma < 0 \\ -\sqrt{-\sigma} \coth(\sqrt{-\sigma}\xi), & \sigma < 0 \\ \sqrt{\sigma} \tan(\sqrt{\sigma}\xi), & \sigma > 0 \\ \sqrt{\sigma} \cot(\sqrt{\sigma}\xi), & \sigma > 0 \\ -\frac{1}{\xi + \varpi}, \varpi \text{ is a cons.}, & \sigma = 0 \end{cases} \tag{6}$$

Step 3. Eqs. (4) and (5) are substituted into Eq. (3) and the coefficients of $\varphi^i(\xi)$ are set to zero. This procedure gives a nonlinear algebraic system in $a_i (i = 0, 1, \dots, N)$.

Step 4. Finally, the obtained constants from the nonlinear algebraic system at previous step and the solutions of Eq. (5) are placed into Eq. (4) with the help of formulas given in (6). This provide the exact solutions for Eq. (1).

3. IMPLEMENTATION OF THE SUB-EQUATION METHOD

3.1. Wave Solutions for Fractional Tzitzica-Dodd-Bullough Equation. Consider the following fractional Tzitzica-Dodd-Bullough equation:

$$D_x^\beta D_t^\alpha u - e^{-u} - e^{-2u} = 0, \quad t \geq 0, \quad 0 < \alpha, \beta \leq 1 \tag{7}$$

where all the derivatives are in conformable sense. Using the chain rule defined in [12] and with the aid of the wave transform $\xi = m\frac{x^\beta}{\beta} - k\frac{t^\alpha}{\alpha}$ and $u = U(\xi)$ we get

$$-mkU_{\xi\xi} - e^{-U} - e^{-2U} = 0. \quad (8)$$

Then, with the following transformation

$$V = e^{-U} \quad (9)$$

in the Equation (8), we get the following ordinary differential equation

$$mk(V_{\xi\xi}V - (V_{\xi})^2) - V^3 - V^4 = 0. \quad (10)$$

Applying balancing procedure in Equation (10), we obtain $N = 1$. Thus, we can write

$$V(\xi) = a_0 + a_1\varphi(\xi). \quad (11)$$

We first place the Eq. (11) and Eq. (5) into Eq. (10). Then the coefficients of $\varphi(\xi)$ are collected and equated to zero. Hence, we get the coefficients of $\varphi(\xi)$ as follows:

$$\begin{aligned} \varphi^0(\xi) : & -a_0^3 - a_0^4 - a_1^2 km\sigma^2 = 0, \\ \varphi^1(\xi) : & -3a_0^2 a_1 - 4a_0^3 a_1 + 2a_0 a_1 km\sigma = 0, \\ \varphi^2(\xi) : & -3a_0 a_1^2 - 6a_0^2 a_1^2 = 0, \\ \varphi^3(\xi) : & -a_1^3 - 4a_0 a_1^3 + 2a_0 a_1 km = 0, \\ \varphi^4(\xi) : & -a_1^4 + a_1^2 km = 0. \end{aligned}$$

After solving the equations above in Mathematica, we get,

$$a_0 = -\frac{1}{2}, a_1 = \pm \frac{i}{2\sqrt{\sigma}}, k = -\frac{1}{4m\sigma}. \quad (12)$$

Using the equalities in (12) and the wave transform, we obtain the following travelling wave solutions

$$\begin{aligned} u_{1,2}(x,t) &= -\ln\left(-\frac{1}{2} \pm \frac{1}{2} \tanh\left(\left(\frac{mx^\beta}{\beta} + \frac{t^\alpha}{4m\alpha\sigma}\right)\sqrt{-\sigma}\right)\right), \\ u_{3,4}(x,t) &= -\ln\left(-\frac{1}{2} \pm \frac{1}{2} \coth\left(\left(\frac{mx^\beta}{\beta} + \frac{t^\alpha}{4m\alpha\sigma}\right)\sqrt{-\sigma}\right)\right), \\ u_{5,6}(x,t) &= -\ln\left(-\frac{1}{2} \pm \frac{i}{2} \tan\left(\left(\frac{mx^\beta}{\beta} + \frac{t^\alpha}{4m\alpha\sigma}\right)\sqrt{\sigma}\right)\right), \\ u_{7,8}(x,t) &= -\ln\left(-\frac{1}{2} \pm \frac{i}{2} \cot\left(\left(\frac{mx^\beta}{\beta} + \frac{t^\alpha}{4m\alpha\sigma}\right)\sqrt{\sigma}\right)\right). \end{aligned}$$

3.2. Wave Solutions for Fractional Dodd-Bullough-Mikhailov Equation.

As another equation, consider the following fractional Dodd-Bullough-Mikhailov equation:

$$D_x^\beta D_t^\alpha u + e^u + e^{-2u} = 0, \quad t \geq 0, \quad 0 < \alpha, \beta \leq 1. \quad (13)$$

Similar to Section 3.1, applying the chain rule in [12] and wave transform yields

$$-mkU_{\xi\xi} + e^U + e^{-2U} = 0. \quad (14)$$

Considering following transformation

$$V = e^U \quad (15)$$

in Eq. (14) gives,

$$mk(-V_{\xi\xi}V + (V_{\xi})^2) + V^3 + 1 = 0. \tag{16}$$

Balancing the highest order derivative term $V_{\xi\xi}$ with nonlinear term V^3 in Eq. (16), we get $N = 2$. Thus, we can write

$$V(\xi) = a_0 + a_1 \varphi(\xi) + a_2 \varphi^2(\xi). \tag{17}$$

Following the same procedure in Section 3.1, we substitute Eq. (17) using Eq. (5) into Eq. (16). The coefficients of $\varphi(\xi)$ are found as follows:

$$\begin{aligned} \varphi^0(\xi) : & 1 + a_0^3 + a_1^2 km\sigma^2 - 2a_0 a_2 km\sigma^2 = 0, \\ \varphi^1(\xi) : & 3a_0^2 a_1 - 2a_0 a_1 km\sigma + 2a_1 a_2 km\sigma^2 = 0, \\ \varphi^2(\xi) : & 3a_0 a_1^2 + 3a_0^2 a_2 - 8a_0 a_2 km\sigma + 2a_2^2 km\sigma^2 = 0, \\ \varphi^3(\xi) : & a_1^3 + 6a_0 a_1 a_2 - 2a_0 a_1 km - 2a_1 a_2 km\sigma = 0, \\ \varphi^4(\xi) : & 3a_1^2 a_2 + 3a_0 a_2^2 - a_1^2 km - 6a_0 a_2 km = 0, \\ \varphi^5(\xi) : & 3a_1 a_2^2 - 4a_1 a_2 km = 0, \\ \varphi^6(\xi) : & a_2^3 - 2a_2^2 km = 0. \end{aligned}$$

Solving the equation system in computer software Mathematica give us the following solution set:

$$a_0 = \frac{1}{2}, a_1 = 0, a_2 = \frac{3}{2\sigma}, k = \frac{3}{4m\sigma}.$$

Using the solution set above, we found the exact solutions for fractional Dodd-Bullough-Mikhailov equation (13):

$$\begin{aligned} u_1(x, t) &= \ln \left(\frac{1}{2} - \frac{3}{2} \tanh \left(\left(\frac{mx^\beta}{\beta} - \frac{3t^\alpha}{4m\alpha\sigma} \right) \sqrt{-\sigma} \right)^2 \right), \\ u_2(x, t) &= \ln \left(\frac{1}{2} - \frac{3}{2} \coth \left(\left(\frac{mx^\beta}{\beta} - \frac{3t^\alpha}{4m\alpha\sigma} \right) \sqrt{-\sigma} \right)^2 \right), \\ u_3(x, t) &= \ln \left(\frac{1}{2} + \frac{3}{2} \tan \left(\left(\frac{mx^\beta}{\beta} - \frac{3t^\alpha}{4m\alpha\sigma} \right) \sqrt{\sigma} \right)^2 \right), \\ u_4(x, t) &= \ln \left(\frac{1}{2} + \frac{3}{2} \cot \left(\left(\frac{mx^\beta}{\beta} - \frac{3t^\alpha}{4m\alpha\sigma} \right) \sqrt{\sigma} \right)^2 \right). \end{aligned}$$

3.3. Wave Solutions for Fractional Tzitzica Equation. Our third example is fractional Tzitzica Equation. Consider the following fractional differential equation,

$$D_t^{2\alpha} u - D_x^{2\beta} u - e^u + e^{-2u} = 0, \quad t \geq 0, \quad 0 < \alpha, \beta \leq 1. \tag{18}$$

We apply the chain rule described in [12] and the wave transform, then we get the equation below,

$$(k^2 - m^2)U_{\xi\xi} - e^U + e^{-2U} = 0. \tag{19}$$

Applying the following transform

$$V = e^U \tag{20}$$

in Eq. (19) yields,

$$(k^2 - m^2)(V_{\xi\xi}V - (V_{\xi})^2) - V^3 + 1 = 0. \tag{21}$$

Balancing the highest order derivative term $V_{\xi\xi}$ with nonlinear term V^3 in Eq. (21), we get $N = 2$, thus we can write

$$V(\xi) = a_0 + a_1 \varphi(\xi) + a_2 \varphi(\xi)^2. \quad (22)$$

Substituting Eq. (22) using (5) into Eq. (21), equating the coefficients of power of φ to zero gives:

$$\begin{aligned} \varphi^0(\xi) : & 1 - a_0^3 - a_1^2(k^2 - m^2)\sigma^2 + 2a_0a_2(k^2 - m^2)\sigma^2 = 0, \\ \varphi^1(\xi) : & -3a_0^2a_1 + 2a_0a_1(k^2 - m^2)\sigma - 2a_1a_2(k^2 - m^2)\sigma^2 = 0, \\ \varphi^2(\xi) : & -3a_0a_1^2 - 3a_0^2a_2 + 8a_0a_2(k^2 - m^2)\sigma - 2a_2^2(k^2 - m^2)\sigma^2 = 0, \\ \varphi^3(\xi) : & -a_1^3 - 6a_0a_1a_2 + 2a_0a_1(k^2 - m^2) + 2a_1a_2(k^2 - m^2)\sigma = 0, \\ \varphi^4(\xi) : & -3a_1^2a_2 - 3a_0a_2^2 + a_1^2(k^2 - m^2) + 6a_0a_2(k^2 - m^2) = 0, \\ \varphi^5(\xi) : & -3a_1a_2^2 + 4a_1a_2(k^2 - m^2) = 0, \\ \varphi^6(\xi) : & -a_2^3 + 2a_2^2(k^2 - m^2) = 0. \end{aligned} \quad (23)$$

With the aid of Mathematica, we obtain the following solution set.

$$a_0 = -\frac{1}{2}, a_1 = 0, a_2 = -\frac{3}{2\sigma}, k = \pm \frac{\sqrt{-3 + 4m^2\sigma}}{2\sqrt{\sigma}}.$$

Finally, the exact solutions for time-space fractional Tzitzica equation (18) are found as follows:

$$\begin{aligned} u_{1,2}(x, t) &= \ln \left(-\frac{1}{2} + \frac{3}{2} \tanh \left(\sqrt{-\sigma} \left(\frac{mx^\beta}{\beta} \pm \frac{t^\alpha \sqrt{-3 - 4m^2\sigma}}{2\alpha\sqrt{-\sigma}} \right) \right)^2 \right), \\ u_{3,4}(x, t) &= \ln \left(-\frac{1}{2} + \frac{3}{2} \coth \left(\sqrt{-\sigma} \left(\frac{mx^\beta}{\beta} \pm \frac{t^\alpha \sqrt{-3 - 4m^2\sigma}}{2\alpha\sqrt{-\sigma}} \right) \right)^2 \right), \\ u_{5,6}(x, t) &= \ln \left(-\frac{1}{2} - \frac{3}{2} \tan \left(\sqrt{\sigma} \left(\frac{mx^\beta}{\beta} \pm \frac{t^\alpha \sqrt{-3 + 4m^2\sigma}}{2\alpha\sqrt{\sigma}} \right) \right)^2 \right), \\ u_{7,8}(x, t) &= \ln \left(-\frac{1}{2} - \frac{3}{2} \cot \left(\sqrt{\sigma} \left(\frac{mx^\beta}{\beta} \pm \frac{t^\alpha \sqrt{-3 + 4m^2\sigma}}{2\alpha\sqrt{\sigma}} \right) \right)^2 \right). \end{aligned}$$

Note: All the obtained results have been checked with Mathematica by putting them back into the original equations and seen that all the solutions are correct.

4. CONCLUSION

In this paper, authors give the exact solutions for conformable space-time fractional Dodd-Bullough-Mikhailov Equation, Tzitzica-Dodd-Bullough Equation and Tzitzica Equation using the sub-equation method. To best of our knowledge all the obtained solutions are firstly presented in the literature. The solutions show that the method is applicable, reliable and accurate for conformable fractional partial differential equations.

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