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ANALYTIC SOLUTION FOR SECOND-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS VIA HPM

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Abstract. Fractional differential equations usually appear perplexing to solve. Therefore, finding comprehensive methods for solving them sounds of high importance. In this article, Homotopy Perturbation Method is used to solve specific second-order fractional differential equations based on conformable fractional derivative. The results obtained demonstrate the efficiency of the proposed method. Some numerical examples are presented to illustrate the proposed approach.

1. Introduction

Many phenomena in our real world are described by fractional differential equations [1-10]. Although having the exact solution of fractional equations in analyzing the phenomena is essential, there are many fractional differential equations which cannot be solved exactly. Due to this fact, finding the desired approximate solutions of fractional differential equations is clearly vital. In recent years, many effective methods have been proposed for finding approximate solution to fractional differential equations, such as Adomian decomposition method [11, 12], homotopy perturbation method [13-15], homotopy analysis method [16], Optimal homotopy asymptotic method [17, 18], variational iteration method [19], generalized differential transform method [20], finite difference method [21], semi-disrete scheme and Chebyshev collocation method [22], Wavelet Operational [23], First integral method [24], Modified Kudryashov and sine-Gordon expansion method [25, 26], some numerical methods [27, 28], and other methods [29- 36]. In this paper, homotopy perturbation method is utilized to obtain an approximate solution of linear and nonlinear of specific second-order fractional differential equations, based on conformable fractional derivative.

The organization of the paper is as follows: In Section 2, the basic definitions such as conformable fractional derivative and integral will be described. In Section 3, the HPM method and the fractional general homotopy perturbation method for fractional differential equations will be explained. In Section 4, some examples, as

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illustrative examples, by means of the proposed approach will be solved. Finally, conclusions will be given in Section 5.

2. Basic definitions

The purpose of this section is to recall some preliminaries of the proposed method.

2.1. **Conformable fractional derivative.** Given a function $f : [0, \infty) \to \mathbb{R}$. Then conformable fractional derivative of f of order α is defined by

$$T_\alpha(f)(x) = \lim_{\varepsilon \to 0} \frac{(f(x + \varepsilon x^{1-\alpha}) - f(x))}{\varepsilon} \tag{1}$$

for all $x > 0$, $\alpha \in (0, 1)$. If f is α-differentiable in some $(0, a)$, $a > 0$, and provided that $\lim_{x \to 0^+} T_\alpha(f)(x)$ exists, then define $T_\alpha(f)(0) = \lim_{x \to 0^+} T_\alpha(f)(x)$.
If the conformable derivative of f of order α exists, then we simply say that f is α-differentiable (see [37, 38]).
One can easily show that T_α satisfies all the following properties (see [37]):
Let $\alpha \in (0, 1]$ and f, g be α-differentiable at a point $x > 0$, then

A: For $a, b \in \mathbb{R}$ $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$,
B: For all $p \in \mathbb{R}$ $T_\alpha(x^p) = px^{p-\alpha}$,
C: For all constant functions $f(x) = \lambda$, $T_\alpha(\lambda) = 0$,
D: $T_\alpha(f.g) = g.T_\alpha(f) + f.T_\alpha(g)$,
E: $T_\alpha\left(\frac{f}{g}\right) = \frac{g.T_\alpha(f) - f.T_\alpha(g)}{g^2}$,
F: $T_\alpha(f) = x^{1-\alpha}\frac{df}{dx}$.

2.2. **Conformable fractional integral.** Given a function $f : [a, \infty) \to \mathbb{R}$, $a \geq 0$. Then the conformable fractional integral of f is defined by

$$I_\alpha^a(f)(x) = \int_a^x \frac{f(t)}{t^{1-\alpha}} dt, \tag{2}$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1)$ (see [37, 38]). For the sake of simplicity, lets consider $I_\alpha^0(f)(x) = I_\alpha(f)(x)$. One of the most useful results is the following statement (see [37]):

For all $x \geq a$, and any continuous function in the domain of I_α^a, we have

$$T_\alpha(I_\alpha^a f(x)) = f(x).$$

3. Overviews of the methods

In this section, homotopy perturbation method is remembered, and then this approach is presented for solving specific second-order fractional differential equations, which is called fractional general homotopy perturbation method.

3.1. **Homotopy perturbation method.** To illustrate the basic ideas of this method, consider the following functional equation

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{3}$$

with the following boundary conditions

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma,$$

where A is a functional operator, B is a boundary operator, $f(r)$ is a known function, and Γ is the boundary of the domain Ω. The operator A can be decomposed into a linear part and a non-linear one, designated as L and N respectively.
Therefore, Eq. (3) can be written as follows

$$L(u) + N(u) - f(r) = 0.$$

Using the homotopy technique, a homotopy $v(r,p) : \Omega \times [0,1] \to \mathbb{R}$ can be constructed which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{4}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of the solution of Eq. (3) which satisfies the boundary conditions. Obviously, from Eq. (4) we have

$$H(v,0) = L(v) - L(u_0) = 0,$$
$$H(v,1) = A(v) - f(r) = 0.$$

By changing the value of p from zero to unity, $v(r,p)$ changes from $u_0(r)$ to $u(r)$; in topology this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. Due to the fact that $p \in [0,1]$ can be considered as a small parameter, consequently, we consider the solution of Eq. (4) as a power series in p as the following form

$$v = \sum_{n=0}^{\infty} v_n p^n, \tag{5}$$

setting $p = 1$, yields the solution of Eq. (3) with the boundary conditions as follows

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots .$$

3.2. **Fractional general homotopy perturbation method.** Consider the general second-order fractional differential equations with initial value

$$\begin{aligned} &T_\alpha T_\alpha u + Q(x) T_\alpha u + F(x,u) = g(x), \\ &u(0) = A, \;\; T_\alpha u(0) = B, \end{aligned} \tag{6}$$

where F is a functional operator, and Q, g are known function, and A, B are certain constant, and u is an unknown function. We construct the following fractional general homotopy perturbation,

$$(1-p)(T_\alpha T_\alpha y + Q(x) T_\alpha y) + p\,[T_\alpha T_\alpha y + Q(x) T_\alpha y + F(x,y) - g(x)] = 0,$$
$$T_\alpha T_\alpha y + Q(x) T_\alpha y + p\,[F(x,y) - g(x)] = 0, \tag{7}$$

where $p \in [0,1]$ is the embedding parameter. We assume that solution of (7) is as follows

$$y = \sum_{n=0}^{\infty} y_n p^n = y_0 + y_1 p + y_2 p^2 + y_3 p^3 + \cdots . \tag{8}$$

Substitution of (8) into Eq. (7), we drive

$$\sum_{n=0}^{\infty} T_\alpha T_\alpha y_n p^n + \sum_{n=0}^{\infty} Q(x) T_\alpha y_n p^n + p\left(F\left(x, \sum_{n=0}^{\infty} y_n p^n\right) - g(x)\right) = 0. \quad (9)$$

Collecting terms of like powers p in (9), we obtain

$$\begin{aligned}
&p^0 : T_\alpha T_\alpha y_0 + Q(x) T_\alpha y_0 = 0, y_0(0) = A,\ \ (T_\alpha y_0)(0) = B\\
&p^1 : T_\alpha T_\alpha y_1 + Q(x) T_\alpha y_1 + F(x, y_0) = g(x),\ \ y_1(0) = 0,\ \ (T_\alpha y_1)(0) = 0, \quad (10)\\
&p^2 : T_\alpha T_\alpha y_2 + Q(x) T_\alpha y_2 + y_1 \frac{\partial}{\partial y_0} F(x, y_0) = 0,\ \ y_2(0) = 0,\ \ (T_\alpha y_2)(0) = 0,\\
&p^3 : T_\alpha T_\alpha y_3 + Q(x) T_\alpha y_3 + y_2 \frac{\partial}{\partial y_0} F(x, y_0) + \frac{1}{2} y_1^2 \frac{\partial^2}{\partial y_0^2} F(x, y_0) = 0,\ \ y_3(0) = 0,\\
&\quad (T_\alpha y_3)(0) = 0,\\
&p^4 : T_\alpha T_\alpha y_4 + Q(x) T_\alpha y_4 + y_3 \frac{\partial}{\partial y_0} F(x, y_0) + y_1 y_2 \frac{\partial^2}{\partial y_0^2} F(x, y_0) + \frac{1}{6} y_1^3 \frac{\partial^3}{\partial y_0^3} F(x, y_0) = 0,\\
&\quad y_4(0) = 0,\ \ (T_\alpha y_4)(0) = 0,\\
&\quad \vdots
\end{aligned}$$

Solving Eqs. (10) leads to, solution of fractional differential equation (7) as the following

$$u(x) = \lim_{p \to 1} y = y_0 + y_1 + y_2 + \cdots .$$

4. Examples

In this section, to illustrate the proposed method, five examples will be presented.

4.1. **Example.** Consider the following linear fractional differential equation with initial value

$$T_\alpha T_\alpha u - 3 T_\alpha u + 2u = 2\left(\frac{1}{\alpha} x^\alpha\right)^2 + \left(\frac{1}{\alpha} x^\alpha\right) + 1,\ \ u(0) = (T_\alpha u)(0) = 1. \quad (11)$$

The exact solution of Eq. (11), is $u(x) = \frac{5}{4}\exp(\frac{2}{\alpha} x^\alpha) - 5\exp\left(\frac{1}{\alpha} x^\alpha\right) + \left(\frac{1}{\alpha} x^\alpha\right)^2 + \frac{7}{2}\left(\frac{1}{\alpha} x^\alpha\right) + \frac{19}{4}$.

According to the proposed fractional general homotopy perturbation method, we have

$$\begin{aligned}
&p^0 : T_\alpha T_\alpha y_0 - 3 T_\alpha y_0 = 0, y_0(0) = 1, (T_\alpha y_0)(0) = 1,\\
&p^1 : T_\alpha T_\alpha y_1 - 3 T_\alpha y_1 + 2y_0 = 2(\frac{1}{\alpha} x^\alpha)^2 + (\frac{1}{\alpha} x^\alpha) + 1, \qquad y_1(0) = 0, (T_\alpha y_1)(0) = 0,\\
&p^2 : T_\alpha T_\alpha y_2 - 3 T_\alpha y_2 + 2y_1 = 0, \qquad y_2(0) = 0, \qquad (T_\alpha y_2)(0) = 0,\\
&p^3 : T_\alpha T_\alpha y_3 - 3 T_\alpha y_3 + 2y_2 = 0, \qquad y_3(0) = 0, \qquad (T_\alpha y_3)(0) = 0,\\
&p^4 : T_\alpha T_\alpha y_4 - 3 T_\alpha y_4 + 2y_3 = 0, \qquad y_4(0) = 0, \qquad (T_\alpha y_4)(0) = 0,\\
&\quad \vdots
\end{aligned}$$

Corresponding solution of this system equations are

$$y_0 = \frac{1}{3}\exp\left(\frac{3}{\alpha}x^\alpha\right) + \frac{2}{3},$$

$$y_1 = \left(-\frac{2}{9}\left(\frac{1}{\alpha}x^\alpha\right) + \frac{10}{81}\right)\exp\left(\frac{3}{\alpha}x^\alpha\right) - \frac{2}{9}\left(\frac{1}{\alpha}x^\alpha\right)^3 - \frac{7}{18}\left(\frac{1}{\alpha}x^\alpha\right)^2 - \frac{4}{27}\left(\frac{1}{\alpha}x^\alpha\right) - \frac{10}{81},$$

$$y_2 = \left(\frac{2}{27}\left(\frac{1}{\alpha}x^\alpha\right)^2 - \frac{32}{243}\left(\frac{1}{\alpha}x^\alpha\right) + \frac{82}{729}\right)\exp\left(\frac{3}{\alpha}x^\alpha\right) - \frac{1}{4}\left(\frac{1}{\alpha}x^\alpha\right)^4 - \frac{11}{81}\left(\frac{1}{\alpha}x^\alpha\right)^3$$
$$- \frac{5}{27}\left(\frac{1}{\alpha}x^\alpha\right)^2 - \frac{50}{243}\left(\frac{1}{\alpha}x^\alpha\right) - \frac{82}{729},$$

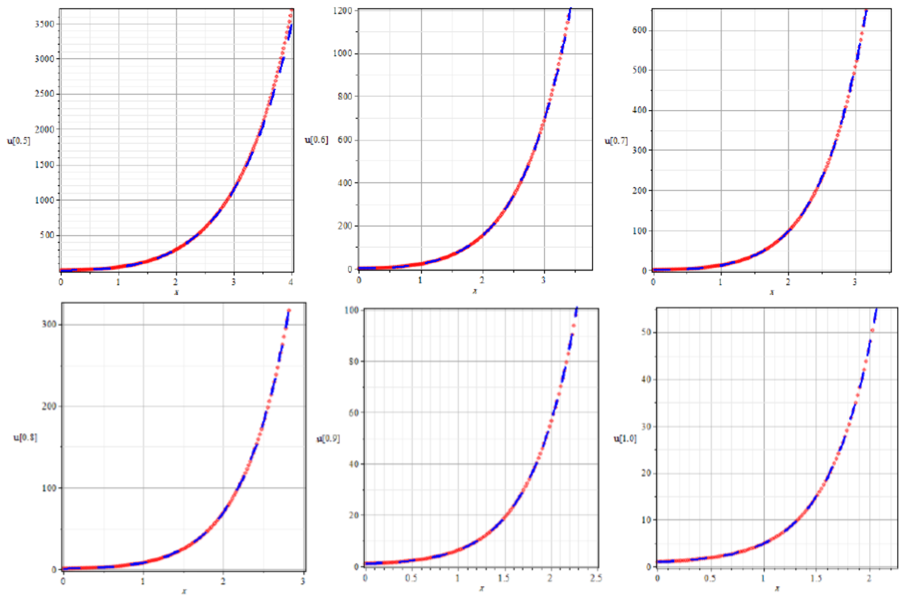
$$y_3 = \left(-\frac{4}{243}\left(\frac{1}{\alpha}x^\alpha\right)^3 + \frac{44}{729}\left(\frac{1}{\alpha}x^\alpha\right)^2 - \frac{28}{243}\left(\frac{1}{\alpha}x^\alpha\right) + \frac{212}{2187}\right)\exp\left(\frac{3}{\alpha}x^\alpha\right) - \frac{2}{405}\left(\frac{1}{\alpha}x^\alpha\right)^5$$
$$- \frac{5}{162}\left(\frac{1}{\alpha}x^\alpha\right)^4 - \frac{20}{243}\left(\frac{1}{\alpha}x^\alpha\right)^3 - \frac{110}{729}\left(\frac{1}{\alpha}x^\alpha\right)^2 - \frac{128}{729}\left(\frac{1}{\alpha}x^\alpha\right) - \frac{212}{2187},$$

$$\vdots$$

Therefore, seven-terms approximation to the solution of Eq. (11), will be obtained as the following form

$$\begin{aligned} u(x) = \Bigg(&0.8784173596 - 0.7369359910\left(\frac{1}{\alpha}x^\alpha\right) + 0.2865981360\left(\frac{1}{\alpha}x^\alpha\right)^2 \\ &- 0.06672424597\left(\frac{1}{\alpha}x^\alpha\right)^3 + 0.009923961456\left(\frac{1}{\alpha}x^\alpha\right)^4 \\ &- 0.0009077207065\left(\frac{1}{\alpha}x^\alpha\right)^5 + 0.00004064421074\left(\frac{1}{\alpha}x^\alpha\right)^6\Bigg)\exp\left(\frac{3}{\alpha}x^\alpha\right) \\ &+ 0.1215826404 - 0.8983160878\left(\frac{1}{\alpha}x^\alpha\right) - 1.028668281\left(\frac{1}{\alpha}x^\alpha\right)^2 \\ &- 0.5964029873(\frac{1}{\alpha}x^\alpha)^3 - 0.1228894647\left(\frac{1}{\alpha}x^\alpha\right)^4 - 0.01826618571\left(\frac{1}{\alpha}x^\alpha\right)^5 \\ &- 0.001920438957\left(\frac{1}{\alpha}x^\alpha\right)^6 - 0.0001306421060\left(\frac{1}{\alpha}x^\alpha\right)^7 \\ &- 0.000004354736865\left(\frac{1}{\alpha}x^\alpha\right)^8. \end{aligned}$$

In Figures 1, the exact and approximate solutions of fractional equation for $\alpha = 0.5$, up to 1.0, is plotted.



(Exact solution (space dash) and approximate solution (point))

Figure 1.a : The comparison 7th-order approximation of HPM and exact solution for Example 4.1

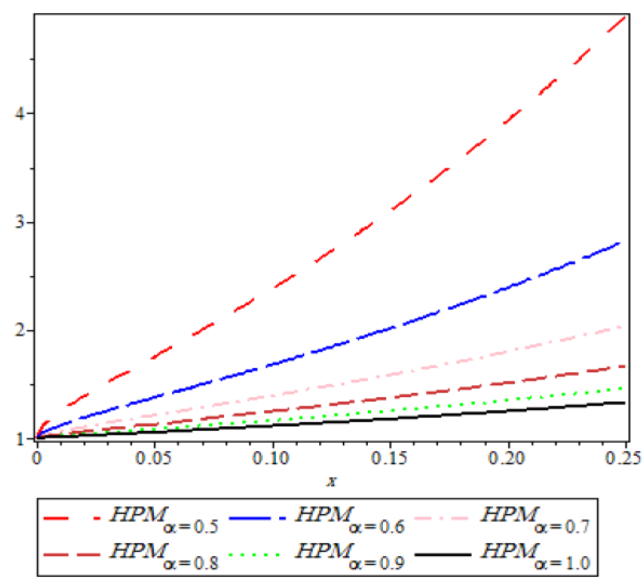


Figure 1.b : The 7th-order approximation of HPM for different values α versus exact solution

4.2. **Example.** Consider the following linear fractional differential equation with initial value

$$T_\alpha T_\alpha u + 2T_\alpha u + u = \exp\left(-\frac{1}{\alpha}x^\alpha\right), \quad u(0)=0, \quad (T_\alpha u)(0) = 1. \tag{12}$$

The exact solution of Eq. (12), is $u(x) = \left(\frac{1}{2}\left(\frac{1}{\alpha}x^\alpha\right)^2 + \frac{1}{\alpha}x^\alpha\right)\exp\left(-\frac{1}{\alpha}x^\alpha\right)$. Conforming to the proposed fractional general homotopy, results in

$$\begin{aligned}
&p^0 : T_\alpha T_\alpha y_0 + 2T_\alpha y_0 = 0, \quad y_0(0)=0, \quad (T_\alpha y_0)(0) = 1,\\
&p^1 : T_\alpha T_\alpha y_1 + 2T_\alpha y_1 + y_0 = \exp(-\frac{1}{\alpha}x^\alpha), \quad y_1(0)=0, (T_\alpha y_1)(0) = 0,\\
&p^2 : T_\alpha T_\alpha y_2 + 2T_\alpha y_2 + y_1 = 0, \quad y_2(0)=0, \quad (T_\alpha y_2)(0) = 0,\\
&p^3 : T_\alpha T_\alpha y_3 + 2T_\alpha y_3 + y_2 = 0, \quad y_3(0)=0, (T_\alpha y_3)(0) = 0,\\
&p^4 : T_\alpha T_\alpha y_4 + 2T_\alpha y_4 + y_3 = 0, \quad y_4(0)=0, \quad (T_\alpha y_4)(0) = 0,\\
&\quad\vdots
\end{aligned}$$

Matching solution of this system equations are a follows

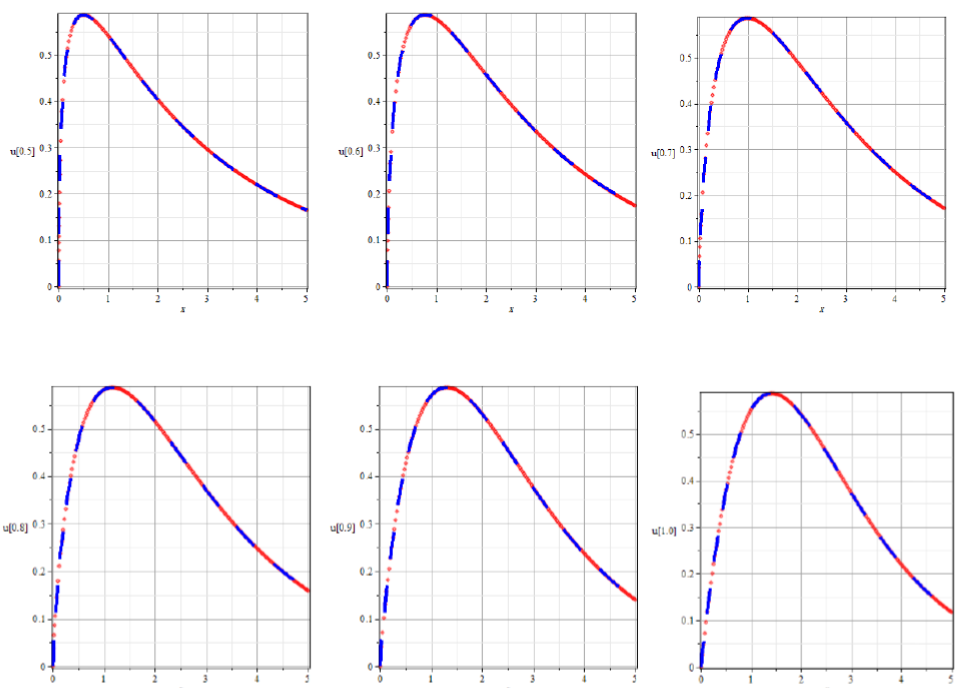
$$\begin{aligned}
y_0 =& -\frac{1}{2}\exp\left(-\frac{2}{\alpha}x^\alpha\right) + \frac{1}{2},\\
y_1 =& -\exp\left(-\frac{1}{\alpha}x^\alpha\right) - \frac{1}{4}\left(\frac{1}{\alpha}x^\alpha\right) + \frac{3}{4},\\
y_2 =& -\exp\left(-\frac{1}{\alpha}x^\alpha\right) + \left(-\frac{1}{16}\left(\frac{1}{\alpha}x^\alpha\right)^2 + \frac{1}{16}\left(\frac{1}{\alpha}x^\alpha\right) + \frac{5}{16}\right)\exp\left(-\frac{2}{\alpha}x^\alpha\right)\\
&+ \frac{1}{16}\left(\frac{1}{\alpha}x^\alpha\right)^2 - \frac{7}{16}\left(\frac{1}{\alpha}x^\alpha\right) + \frac{11}{16},\\
y_3 =& -\exp\left(-\frac{1}{\alpha}x^\alpha\right) + \left(-\frac{1}{96}\left(\frac{1}{\alpha}x^\alpha\right)^3 + \frac{5}{32}\left(\frac{1}{\alpha}x^\alpha\right) + \frac{11}{32}\right)\exp\left(-\frac{2}{\alpha}x^\alpha\right)\\
&- \frac{1}{96}\left(\frac{1}{\alpha}x^\alpha\right)^3 + \frac{1}{8}\left(\frac{1}{\alpha}x^\alpha\right)^2 - \frac{15}{32}\left(\frac{1}{\alpha}x^\alpha\right) + \frac{21}{32},\\
&\vdots
\end{aligned}$$

Thus, seven-terms approximation to the solution of Eq. (12), will be obtained as the following form

$$u(x) = \Bigg(1.533691406 + 0.6801757812\left(\frac{1}{\alpha}x^\alpha\right) + 0.1069335938\left(\frac{1}{\alpha}x^\alpha\right)^2$$

$$
\begin{aligned}
&+ 0.001953125000\left(\frac{1}{\alpha}x^{\alpha}\right)^{3} - 0.001627604167\left(\frac{1}{\alpha}x^{\alpha}\right)^{4} \\
&- 0.0002278645833\left(\frac{1}{\alpha}x^{\alpha}\right)^{5} - 0.00001085069444\left(\frac{1}{\alpha}x^{\alpha}\right)^{6}\Bigg) \\
&\exp\left(-\frac{2}{\alpha}x^{\alpha}\right) + 4.466308594 - 2.612792969\left(\frac{1}{\alpha}x^{\alpha}\right) \\
&+ 0.6860351562\left(\frac{1}{\alpha}x^{\alpha}\right)^{2} - 0.1035156250\left(\frac{1}{\alpha}x^{\alpha}\right)^{3} \\
&+ 0.009440104167\left(\frac{1}{\alpha}x^{\alpha}\right)^{4} - 0.0004882812500\left(\frac{1}{\alpha}x^{\alpha}\right)^{5} \\
&+ 0.00001085069444\left(\frac{1}{\alpha}x^{\alpha}\right)^{6} - 6\exp\left(-\frac{1}{\alpha}x^{\alpha}\right).
\end{aligned}
$$

In Figures 2, the exact and approximate solutions of fractional equation for $\alpha = 0.5$, up to 1.0 , is plotted



(Exact solution (space dash) and approximate solution (point))

Figure 2.a : The comparison 7th-order approximation of CGHPM and exact solution for Example 4.2.

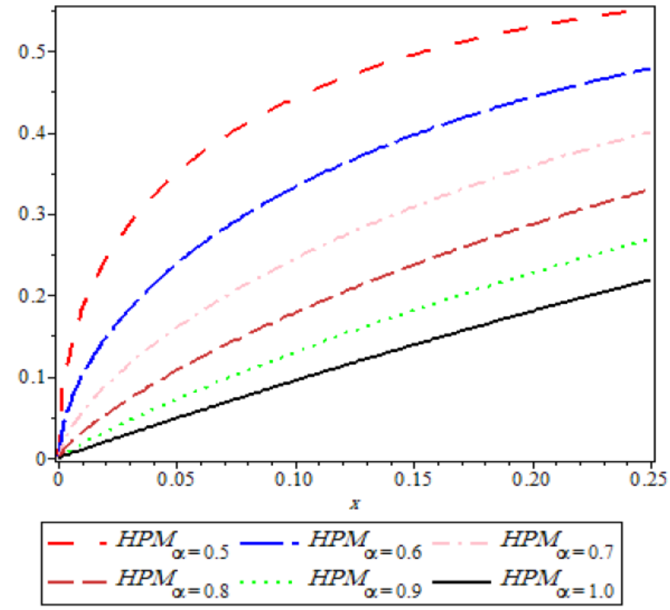


Figure 2.b : The 7th-order approximation of HPM for different values α versus exact solution.

4.3. **Example.** Consider the following linear fractional differential equation with initial value

$$\left(\frac{1}{\alpha}x^{\alpha}\right)T_\alpha T_\alpha u+8T_\alpha u+\left(\frac{1}{\alpha}x^{\alpha}\right)^2u=\left(\frac{1}{\alpha}x^{\alpha}\right)^6-\left(\frac{1}{\alpha}x^{\alpha}\right)^5+44\left(\frac{1}{\alpha}x^{\alpha}\right)^3-30\left(\frac{1}{\alpha}x^{\alpha}\right)^2,$$
$$u(0)=(T_\alpha u)(0)=0. \tag{13}$$

The exact solution of Eq. (13), is $u(x)=\left(\frac{1}{\alpha}x^{\alpha}\right)^4-\left(\frac{1}{\alpha}x^{\alpha}\right)^3.$
Consistent with the fractional general homotopy perturbation method, we obtain

$$p^0:\left(\frac{1}{\alpha}x^{\alpha}\right)T_\alpha T_\alpha y_0+8T_\alpha y_0=0,\quad y_0(0)=(T_\alpha y_0)(0)=0,$$
$$p^1:\left(\frac{1}{\alpha}x^{\alpha}\right)T_\alpha T_\alpha y_1+8T_\alpha y_1+\left(\frac{1}{\alpha}x^{\alpha}\right)^2y_0=\left(\frac{1}{\alpha}x^{\alpha}\right)^6-\left(\frac{1}{\alpha}x^{\alpha}\right)^5+44\left(\frac{1}{\alpha}x^{\alpha}\right)^3$$
$$-30\left(\frac{1}{\alpha}x^{\alpha}\right)^2,y_1(0)=(T_\alpha y_1)(0)=0$$
$$p^2:\left(\frac{1}{\alpha}x^{\alpha}\right)T_\alpha T_\alpha y_2+8T_\alpha y_2+\left(\frac{1}{\alpha}x^{\alpha}\right)^2y_1=0,\qquad y_2(0)=(T_\alpha y_2)(0)=0,$$
$$p^3:\left(\frac{1}{\alpha}x^{\alpha}\right)T_\alpha T_\alpha y_3+8T_\alpha y_3+\left(\frac{1}{\alpha}x^{\alpha}\right)^2y_2=0,\qquad y_3(0)=(T_\alpha y_3)(0)=0,$$
$$\vdots$$

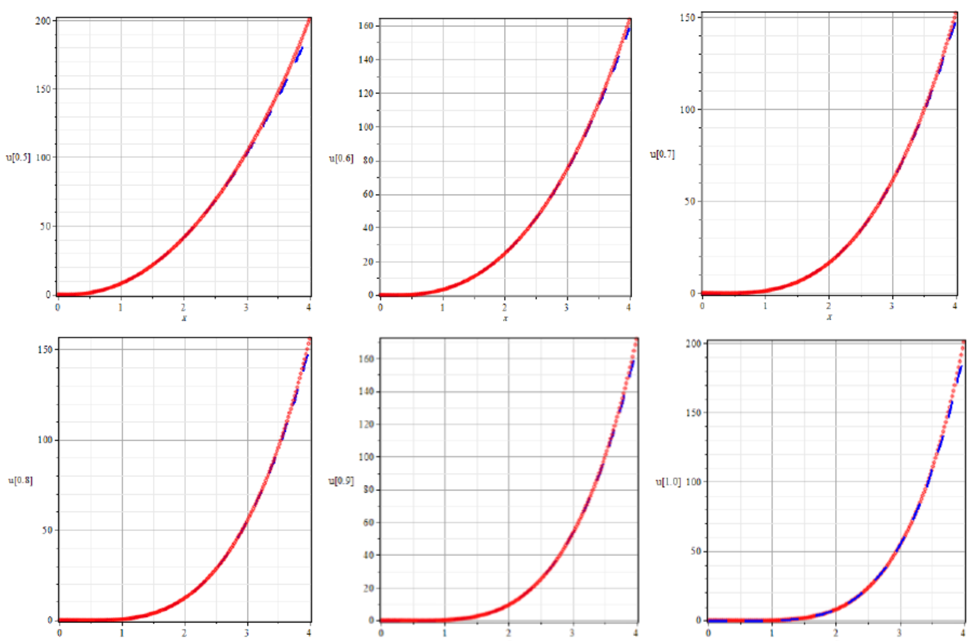
Corresponding solution of this system equations are

$$
\begin{aligned}
y_0 =& 0,\\
y_1 =& \frac{1}{98}\left(\frac{1}{\alpha}x^\alpha\right)^7 - \frac{1}{78}\left(\frac{1}{\alpha}x^\alpha\right)^6 + \left(\frac{1}{\alpha}x^\alpha\right)^4 - \left(\frac{1}{\alpha}x^\alpha\right)^3,\\
y_2 =& -\frac{1}{16660}\left(\frac{1}{\alpha}x^\alpha\right)^1 0 + \frac{1}{11232}\left(\frac{1}{\alpha}x^\alpha\right)^9 - \frac{1}{98}\left(\frac{1}{\alpha}x^\alpha\right)^7 + \frac{1}{78}\left(\frac{1}{\alpha}x^\alpha\right)^6,\\
y_3 =& \frac{1}{4331600}\left(\frac{1}{\alpha}x^\alpha\right)^1 3 - \frac{1}{2560896}\left(\frac{1}{\alpha}x^\alpha\right)^1 2 + \frac{1}{16660}\left(\frac{1}{\alpha}x^\alpha\right)^1 0 - \frac{1}{11232}\left(\frac{1}{\alpha}x^\alpha\right)^9,\\
&\vdots
\end{aligned}
$$

Then, four-terms approximation to the solution of Eq. (13), will be obtained as the following form

$$
u(x) = \left(\frac{1}{\alpha}x^\alpha\right)^4 - \left(\frac{1}{\alpha}x^\alpha\right)^3 + \frac{1}{4331600}\left(\frac{1}{\alpha}x^\alpha\right)^{13} - \frac{1}{2560896}\left(\frac{1}{\alpha}x^\alpha\right)^{12}.
$$

In Figures 3, the exact and approximate solutions of fractional equation for $\alpha = 0.5$, up to 1.0, is plotted.



(Exact solution (space dash) and approximate solution (point))

Figure 3.a : The comparison 7th-order approximation of CGHPM and exact solution for Example 4.3.

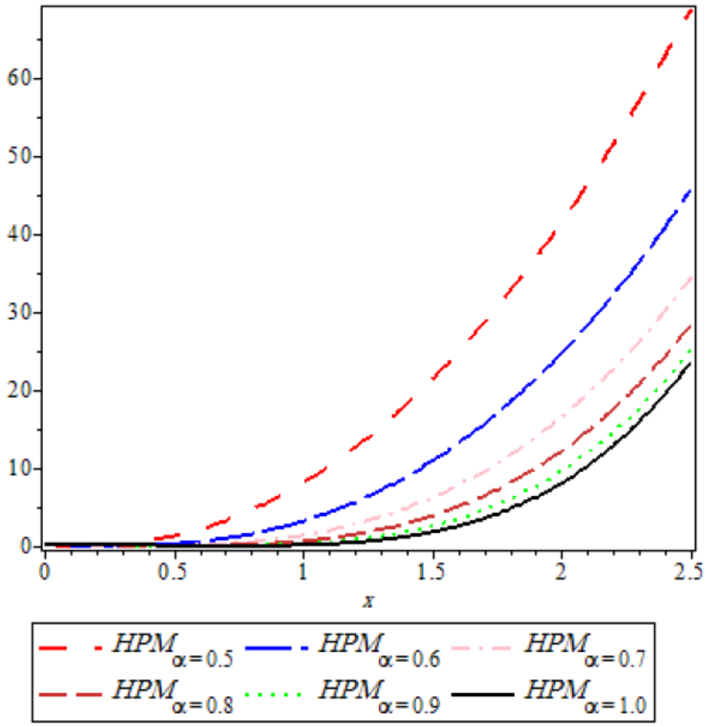


Figure 3.b :The 4th-order approximation of HPM for different values α versus exact solution.

4.4. **Example.** Consider the following nonlinear fractional differential equation with initial value

$$T_\alpha T_\alpha u - 2T_\alpha u + u^2 = 0, \quad u(0) = \frac{1}{2}, \quad (T_\alpha u)(0) = 1. \tag{14}$$

The approximate solution of Eq. (14), is $u_{app}(x) = \frac{1}{2} + \left(\frac{1}{\alpha}x^\alpha\right) + \frac{7}{8}\left(\frac{1}{\alpha}x^\alpha\right)^2 + \frac{5}{12}\left(\frac{1}{\alpha}x^\alpha\right)^3 + \frac{5}{96}\left(\frac{1}{\alpha}x^\alpha\right)^4 - \frac{7}{80}\left(\frac{1}{\alpha}x^\alpha\right)^5.$

By the proposed fractional general HPM approach,

$$\begin{aligned}
&p^0 : T_\alpha T_\alpha y_0 - 2T_\alpha y_0 = 0, \quad y_0(0) = \frac{1}{2}, \quad (T_\alpha y_0)(0) = 1,\\
&p^1 : T_\alpha T_\alpha y_1 - 2T_\alpha y_1 + y_0^2 = 0, \quad y_1(0) = 0, \quad (T_\alpha y_1)(0) = 0,\\
&p^2 : T_\alpha T_\alpha y_2 - 2T_\alpha y_2 + 2y_0 y_1 = 0, y_2(0) = 0, \quad (T_\alpha y_2)(0) = 0,\\
&p^3 : T_\alpha T_\alpha y_3 - 2T_\alpha y_3 + y_1^2 + 2y_0 y_2 = 0, \quad y_3(0) = 0, (T_\alpha y_3)(0) = 0,\\
&p^4 : T_\alpha T_\alpha y_3 - 2T_\alpha y_3 + 2y_1 y_2 + 2y_0 y_3 = 0, \quad y_4(0) = 0, (T_\alpha y_4)(0) = 0,\\
&\quad \vdots
\end{aligned}$$

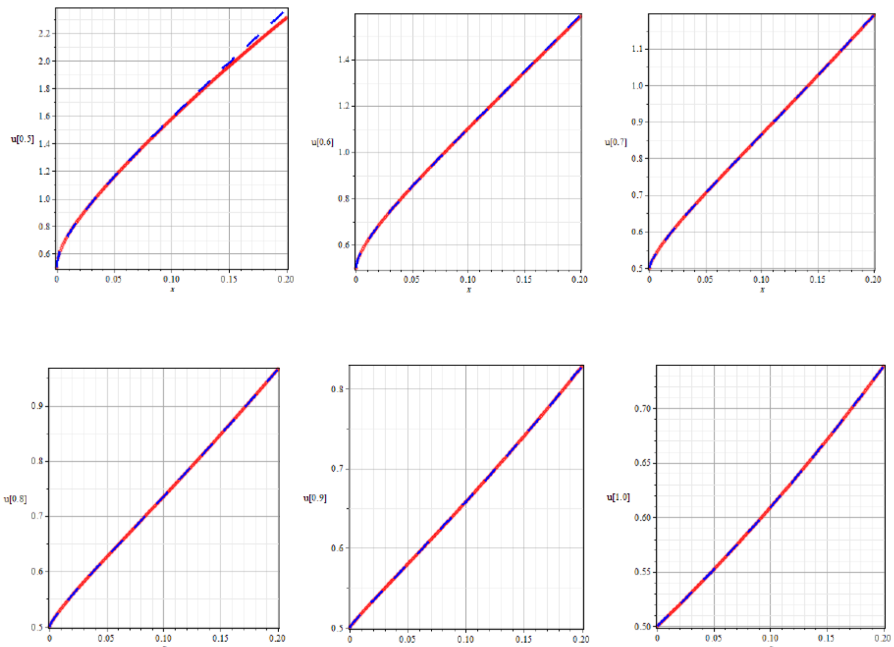
Corresponding solution of this system equations are

$$
\begin{aligned}
y_0 =& \frac{1}{2}\exp\left(\frac{2}{\alpha}x^\alpha\right),\\
y_1 =& -\frac{1}{32}\exp(\left(\frac{4}{\alpha}x^\alpha\right) + \frac{1}{16}\exp\left(\frac{2}{\alpha}x^\alpha\right) - \frac{1}{32},\\
y_2 =& \frac{1}{768}\exp\left(\frac{6}{\alpha}x^\alpha\right) - \frac{1}{128}\exp(\left(\frac{4}{\alpha}x^\alpha\right) + \left(\frac{1}{64}\left(\frac{1}{\alpha}x^\alpha\right) + \frac{1}{256}\right) exp\left(\frac{1}{\alpha}x^\alpha\right) + \frac{1}{384},\\
y_3 =& -\frac{7}{147456}\exp\left(\frac{8}{\alpha}x^\alpha\right) + \frac{1}{2048}\exp\left(\frac{6}{\alpha}x^\alpha\right)\left(-\frac{1}{512}\left(\frac{1}{\alpha}x^\alpha\right) + \frac{1}{4096}\right)\exp\left(\frac{4}{\alpha}x^\alpha\right)\\
&+ \left(\frac{1}{1536}\left(\frac{1}{\alpha}x^\alpha\right) + \frac{25}{18432}\right)\exp\left(\frac{2}{\alpha}x^\alpha\right) + \frac{1}{2048}\left(\frac{1}{\alpha}x^\alpha\right) + \frac{11}{16348},\\
&\vdots
\end{aligned}
$$

Consequently, seven-terms approximation to the solution of Eq. (14), will be obtained as the following form

$$
\begin{aligned}
u(x) = \Bigg(&\Big(05649925850 + 0.1579214555\left(\frac{1}{\alpha}x^\alpha\right)\\
&+ 0.0001109970940\left(\frac{1}{\alpha}x^\alpha\right)^2\Big)\exp\left(\frac{2}{\alpha}x^\alpha\right)\Big(0.00004959106445\left(\frac{1}{\alpha}x^\alpha\right)^2\\
&+ 0.002166027493\left(\frac{1}{\alpha}x^\alpha\right) + 0.03853380450\Big)\exp\left(\frac{4}{\alpha}x^\alpha\right)\\
&+ \Big(0.000004768371582\left(\frac{1}{\alpha}x^\alpha\right)^2 + 0.0001511838701\left(\frac{1}{\alpha}x^\alpha\right)\\
&+ 0.0001754174630\Big)\exp\left(\frac{6}{\alpha}x^\alpha\right)\\
&- \Big(0.0000008406462493\left(\frac{1}{\alpha}x^\alpha\right) + 0.00007003545761\Big)\exp\left(\frac{8}{\alpha}x^\alpha\right)\\
&- \Big(0.000000251664055\left(\frac{1}{\alpha}x^\alpha\right) + 0.000002693622201\Big)\exp\left(\frac{10}{\alpha}x^\alpha\right)\\
&- 0.00000009148209183\exp\left(\frac{12}{\alpha}x^\alpha\right) + 0.00000001646223522\exp\left(\frac{14}{\alpha}x^\alpha\right)\Bigg)\\
&- 0.02814552344 + 0.0003902753194\left(\frac{1}{\alpha}x^\alpha\right) - 0.000005722045898\left(\frac{1}{\alpha}x^\alpha\right)^2.
\end{aligned}
$$

In Figures 4, the approximate solution and solution of general HPM of fractional equation, for $\alpha = 0.5$, up to 1.0, is plotted.



(Exact solution (space dash) and approximate solution (point))

Figure 4.a : The comparison 7th-order approximation of HPM and approximate solution for Example 4.4.

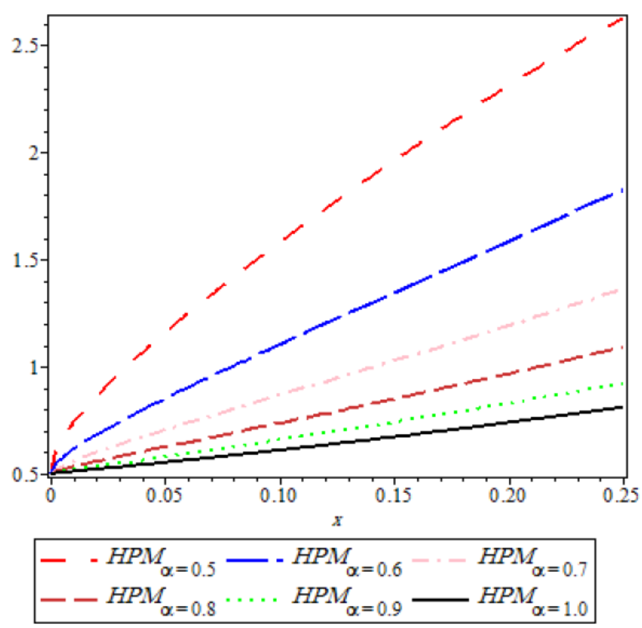


Figure 4.b : The 7th-order approximation of HPM for different values α versus exact solution.

4.5. **Example.** Consider the conformable fractional Bratu-type equation with initial value

$$T_\alpha T_\alpha u + \pi^2 \exp(-u) = 0, \qquad u(0) = 0, \qquad (T_\alpha u)(0) = \pi. \tag{15}$$

The exact solution of Eq. (15), is $u(x) = \ln\left(1 + \sin\left(\frac{\pi}{\alpha}x^\alpha\right)\right)$.
According to the proposed fractional general HPM approach, reads

$p^0 : T_\alpha T_\alpha y_0 = 0, \quad y_0(0) = 0, \quad (T_\alpha y_0)(0) = \pi,$

$p^1 : T_\alpha T_\alpha y_1 + \pi^2 \exp(-y_0) = 0, \quad y_1(0) = 0, \quad (T_\alpha y_1)(0) = 0,$

$p^2 : T_\alpha T_\alpha y_2 - \pi^2 y_1 \exp(-y_0) = 0, \quad y_2(0) = 0, (T_\alpha y_2)(0) = 0,$

$p^3 : T_\alpha T_\alpha y_3 - \pi^2 y_2 \exp(-y_0) + \frac{1}{2}\pi^2 {y_1}^2 \exp(-y_0) = 0, \quad y_3(0) = 0, \quad (T_\alpha y_3)(0) = 0,$

$\vdots$

Corresponding solution of this system equations are

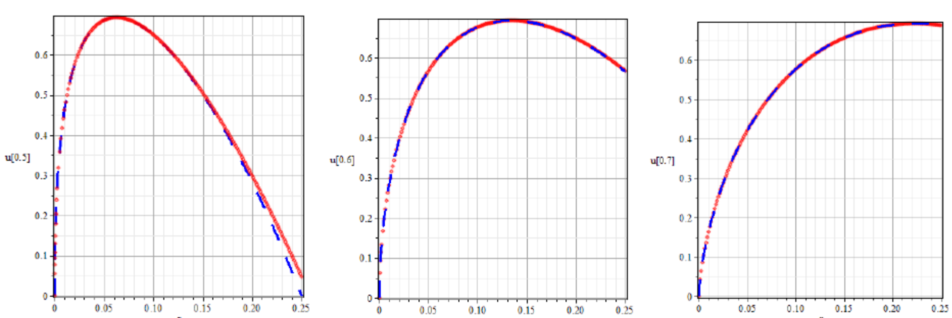
$$y_0 = \frac{\pi}{\alpha}x^\alpha$$

$$y_1 = -\exp\left(-\frac{\pi}{\alpha}x^\alpha\right) - \left(\frac{\pi}{\alpha}x^\alpha\right) + 1,$$

$$y_2 = \left(-\frac{\pi}{\alpha}x^\alpha\right)\exp\left(-\frac{\pi}{\alpha}x^\alpha\right) - \exp\left(-\frac{\pi}{\alpha}x^\alpha\right) - \frac{1}{4}\exp\left(\frac{-2\pi}{\alpha}x^\alpha\right) - \frac{\pi}{2\alpha}x^\alpha + \frac{5}{4},$$

$$\vdots$$

In Figures 5, the exact and seven-terms approximate solutions of fractional equation for $\alpha = 0.5$, up to 1.0, are plotted.



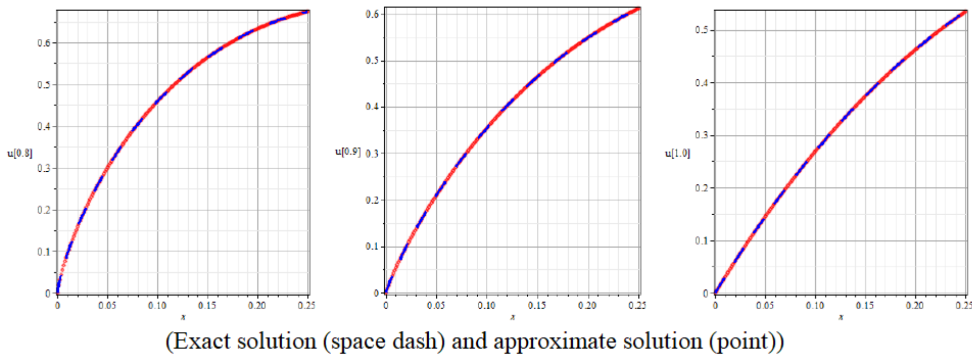


Figure 5.a : The comparison 7th-order approximation of CGHPM and exact solution for Example 4.5.

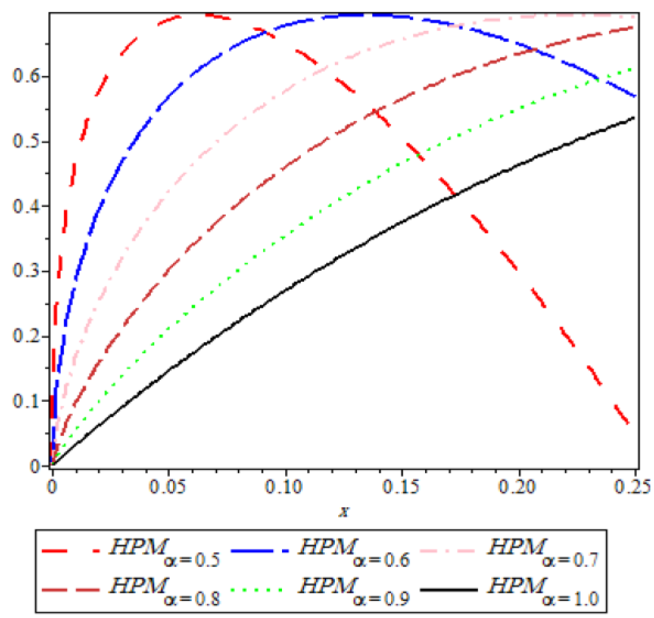


Figure 5.b : The 7th-order approximation of HPM for different values α versus exact solution.

5. Conclusion

In this paper, Homotopy Perturbation method has been applied to obtain the solutions of fractional differential equations. To this aim, a conformable fractional derivative has been used to find the solution. The results showed that the definition is the simplest tool to obtain the approximation solutions of linear and nonlinear specific second-order fractional differential equations in comparison to the other definitions. To show the effectiveness and simplicity of the method, some second-order fractional differential equations as an example have been solved with form conformable fractional derivative and the fractional general homotopy perturbation method. It can be concluded from the result that, the convergence as well as the accuracy of approximate solution of general HPM approach for fractional differential equation is similar to HPM method for ordinary differential equation.

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