Journal of Fractional Calculus and Applications
Vol. 11(1) Jan. 2020, pp. 112-120.
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

NOTE ON GENERALIZATIONS OF A SYMMETRIC q-SERIES IDENTITY

XUE-FANG WANG, JIAN CAO*

Dedicated to Professor G.E. Andrews on the occasion of his 80th birthday

Abstract. The main object of this paper is to generalize a symmetric identity which is given in a recent work [Discrete Math. **339**(2016), 2994–2997.] by the method of q-difference equation. In addition, we generalize symmetric identity by fractional integral. Moreover, we generalize symmetric identity by moment integrals. Finally, we generalize symmetric identity by generating function for Al-Salam–Carlitz polynomial $\Phi_n^{(a,b)}(x,y|q)$.

1. Introduction

In this paper, we follow the notations and terminology in [16] and suppose that $0 < q < 1$. In this paper, we follow the notations and terminology in [16] and suppose that $0 < q < 1$. We first show a list of various definitions and notations in q-calculus which are useful to understand the subject of this paper. The basic hypergeometric series ${}_r\phi_s$

$${}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \tag{1}$$

converges absolutely for all z if $r \le s$ and for $|z| < 1$ if $r = s + 1$ and for terminating. The q-series and its compact factorials are defined respectively by

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=o}^{n-1}\left(1 - aq^k\right), \quad (a;q)_\infty = \prod_{k=o}^{\infty}\left(1 - aq^k\right), \tag{2}$$

where a is a complex variable. For convenience, we always assume $0 < q < 1$ in the paper, $(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$, where m is a positive integer and n is a non-negative integer or ∞.

In [9, 10], Chen and Liu introduced two q-exponential operators

$$\mathbb{T}(bD_a) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} (bD_a)^n, \qquad \mathbb{E}(b\theta_a) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} (b\theta_a)^n.$$

2010 *Mathematics Subject Classification.* 05A30, 11B65, 33D15, 33D45, 33D60, 39A13, 39B32.

Key words and phrases. Symmetric q-series identities; Fractional q-integral; Moment integrals; Generating function.

Submitted Dec. 12, 2018.

The Rogers–Szegö polynomials [1] are given by

$$h_n(b,c|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}b^kc^{n-k}\quad\text{and}\quad g_n(b,c|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}q^{k(k-n)}b^kc^{n-k}. \tag{3}$$

The Al-Salam–Carlitz polynomials [6, Eq. (4.4)]

$$\Phi_n^{(a)}(b,c|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(a;q)_kb^kc^{n-k},\quad\text{and}\quad \Psi_n^{(a)}(b,c|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^kq^{\binom{k+1}{2}-nk}\Big(\frac{1}{a};q\Big)_k(ab)^kc^{n-k}. \tag{4}$$

The Al-Salam–Carlitz polynomials reduce to the Rogers–Szegö polynomials with $a=0$.

The Rogers–Szegö polynomials play important roles in the theory of orthogonal polynomials. Liu [18, 19] obtained several important results by the following q-difference equations. Liu and Zeng [23] studied relations between q-difference equations and q-orthogonal polynomials. For more information, please refer to [3, 12, 13, 14, 15, 17, 20, 21, 22, 27, 29, 30, 31, 32, 33].

Proposition 1. *Let $f(a,b)$ be a two-variable analytic function at $(0,0)\in\mathbb{C}^2$. Then*

(A) *f can be expanded in terms of $h_n(a,b|q)$ if and only if f satisfies the functional equation*

$$bf(aq,b)-af(a,bq)=(b-a)f(a,b). \tag{5}$$

(B) *f can be expanded in terms of $g_n(a,b|q)$ if and only if f satisfies the functional equation*

$$af(aq,b)-bf(a,bq)=(a-b)f(aq,bq). \tag{6}$$

In [4], Andrews gave a wonderful introduction of Ramanujan's lost" notebook, and listed some interesting identities contained therein. One of which is the following beautiful symmetric identity. Where if

$$f(\alpha,\beta):=\frac{1}{1-\alpha}+\sum_{n\ge1}\frac{\beta^n}{(1-\alpha x^n)(1-\alpha x^{n-1}y)(1-\alpha x^{n-2}y^2)\cdots(1-\alpha y^n)}.$$

Then

$$f(\alpha,\beta)=f(\beta,\alpha).$$

The identity we present here is a refinement of the case where $x=q, y=q^2$.

Then A.E. Patkowski [25] obtained the following symmetric q-series identity.

Proposition 2 ([25, Eq. (1.3)])**.** *We have, for arbitrary a, and $|b|<1$, $|t|<1$,*

$$\sum_{n=0}^{\infty}\frac{(-abq^{n+1};q)_nt^n}{(bq^n;q)_{n+1}}=\sum_{n=0}^{\infty}\frac{(-atq^{n+1};q)_nb^n}{(tq^n;q)_{n+1}}. \tag{7}$$

In this paper, we first generalize this symmetric q-series identity by the method of q-difference equation.

Theorem 3. *For arbitrary $|a| < 1$, $|b| < 1$ and $|t| < 1$, we have*

$$\sum_{n=0}^{\infty}\frac{(-atq^{n+1};q)_n}{(tq^n;q)_{n+1}}h_n(c,b|q)=\sum_{n=0}^{\infty}\frac{(-abq^{n+1};q)_nt^n}{(bq^n;q)_{n+1}}\sum_{k=0}^{n}\frac{(q^{-n},bq^n;q)_k(-acq^{2n+1})^k}{(q,-abq^{n+1},bq^{2n+1};q)_k}{}_2\phi_1\left[\begin{matrix}q^{n+1},0\\bq^{2n+1+k}\end{matrix};q,cq^n\right],\tag{8}$$

$$\sum_{n=0}^{\infty}\frac{(-atq^{n+1};q)_n}{(tq^n;q)_{n+1}}g_n(c,b|q)=\sum_{n=0}^{\infty}\frac{(-abq^{n+1};q)_nt^n}{(bq^n;q)_{n+1}}\sum_{k=0}^{\infty}\frac{(-aq,1/(bq^{2n});q)_k}{(q,1/(-abq^{2n}),1/(bq^{n-1});q)_k}\left(\frac{cq^{n+1}}{b}\right)^k$$
$$\times\sum_{n=0}^{\infty}\frac{(q^{n+1};q)_n}{(q^{k+1-n}/b;q)_n}q^{(n+k+1)(n+k)}(c/b)^n.\tag{9}$$

Proof of Theorems 3 . Denoting the LHS of equation (8) can be written by

$$\begin{aligned}f(b,c)&=\sum_{n=0}^{\infty}\frac{(-abq^{n+1};q)_nt^n}{(bq^n;q)_{n+1}}\sum_{k=0}^{n}\frac{(q^{-n},bq^n;q)_k(-acq^{2n+1})^k}{(q,-abq^{n+1},bq^{2n+1};q)_k}{}_2\phi_1\left[\begin{matrix}q^{n+1},0\\bq^{2n+1+k}\end{matrix};q,cq^n\right]\\&=\sum_{n=0}^{\infty}\frac{(-abq^{n+1},bq^{2n+1};q)_\infty t^n}{(-abq^{2n+1},bq^n;q)_\infty}\sum_{k=0}^{n}\frac{(q^{-n},bq^n;q)_k(-acq^{2n+1})^k}{(q,-abq^{n+1},bq^{2n+1};q)_k}{}_2\phi_1\left[\begin{matrix}q^{n+1},0\\bq^{2n+1+k}\end{matrix};q,cq^n\right]\\&=\sum_{n=0}^{\infty}t^n\sum_{k=0}^{\infty}\frac{c^k}{(q;q)_k}D_b^k\left\{\frac{(-abq^{n+1},bq^{2n+1};q)_\infty}{(-abq^{2n+1},bq^n;q)_\infty}\right\}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{c^k}{(q;q)_k}D_b^k\left\{\frac{(-abq^{n+1};q)_nt^n}{(bq^n;q)_{n+1}}\right\}.\end{aligned}$$

By using equation (7), we have

$$\begin{aligned}f(b,c)&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{c^k}{(q;q)_k}D_b^k\left\{\frac{(-atq^{n+1};q)_nb^n}{(tq^n;q)_{n+1}}\right\}\\&=\sum_{n=0}^{\infty}\frac{(-atq^{n+1};q)_n}{(tq^n;q)_{n+1}}\sum_{k=0}^{\infty}\frac{c^k}{(q;q)_k}D_b^k\{b^n\}.\end{aligned}$$

We can verify that $f(a,b,c)$ satisfies equation (5). Then, we have

$$f(b,c)=\sum_{n=0}^{\infty}u_nh_n(c,b|q),$$

then, we have

$$f(b,0)=\sum_{n=0}^{\infty}u_nb^n=\sum_{n=0}^{\infty}\frac{(-atq^{n+1};q)_nb^n}{(tq^n;q)_{n+1}}.$$

Hence

$$f(b,c)=\sum_{n=0}^{\infty}\frac{(-atq^{n+1};q)_n}{(tq^n;q)_{n+1}}h_n(c,b|q).$$

Using the same way, we gain the equation (9). The proof is complete. □

2. Fractional q-integrals for a symmetric q-series identity

In this section, we use the fractional q-integrals to deduce a new identity for a symmetric q-series. For more information, please refer to [2, 8, 26].

The q-gamma function is defined by [16]

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \in \mathbb{R}\backslash\{0,-1,-2,\ldots\}. \tag{10}$$

The Thomae–Jackson q-integral is defined by [16, 11, 28]

$$\int_a^b f(x)\,\mathrm{d}_q x = (1-q)\sum_{n=0}^{\infty}\Big[bf(bq^n) - af(aq^n)\Big]q^n. \tag{11}$$

The Riemann–Liouville fractional q-integral operator is introduced in [2]

$$\Big(I_q^\alpha f\Big)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} f(t)\,\mathrm{d}_q t. \tag{12}$$

The generalized Riemann–Liouville fractional q-integral operator for $\alpha \in \mathbb{R}^+$ is given by [26]

$$\Big(I_{q,a}^\alpha f\Big)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_a^x (qt/x;q)_{\alpha-1} f(t)\,\mathrm{d}_q t. \tag{13}$$

Proposition 4. *For $\alpha \in \mathbb{R}^+$, $0 < a < x < 1$, we have*

$$I_{q,a}^\alpha\{x^n\} = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{[k]_q!a^{n-k}}{\Gamma_q(\alpha+k+1)}x^{\alpha+k}(a/x;q)_{\alpha+k}. \tag{14}$$

Theorem 5. *For $\alpha \in \mathbb{R}^+$, $0 < c < b < 1$, we have we have*

$$\sum_{n=0}^{\infty}\frac{(-acq^{n+1};q)_n t^n}{(cq^n;q)_{n+1}}\sum_{k=0}^{\infty}\frac{b^{\alpha+k}(c/b;q)_{\alpha+k}}{c^k(q;q)_{\alpha+k}}{}_3\phi_2\left[\begin{matrix} q^{-k}, -acq^{2n+1}, cq^n \\ cq^{2n+1}, -acq^{n+1} \end{matrix}; q, q\right]$$
$$= \sum_{n=0}^{\infty}\frac{(-atq^{n+1};q)_n}{(tq^n;q)_{n+1}}\sum_{k=0}^{n}\frac{(q;q)_n c^{n-k}}{(q;q)_{n-k}}\cdot\frac{b^{\alpha+k}(c/b;q)_{\alpha+k}}{(q;q)_{\alpha+k}}. \tag{15}$$

Proof of Theorems 5 . Multiply $(1-q)^\alpha$ on the both sides of equation (15), the LHS of equation (15) become to

$$(1-q)^\alpha\sum_{n=0}^{\infty}\frac{(-acq^{n+1};q)_n t^n}{(cq^n;q)_{n+1}}\sum_{k=0}^{\infty}\frac{b^{\alpha+k}(c/b;q)_{\alpha+k}}{c^k(q;q)_{\alpha+k}}{}_3\phi_2\left[\begin{matrix} q^{-k}, -acq^{2n+1}, cq^n \\ cq^{2n+1}, -acq^{n+1} \end{matrix}; q, q\right]$$
$$= \sum_{n=0}^{\infty}\frac{(1-q)^\alpha(-acq^{n+1}, cq^{2n+1};q)_\infty t^n}{(-acq^{n+1}, cq^n;q)_\infty}\sum_{k=0}^{\infty}\frac{b^{\alpha+k}(c/b;q)_{\alpha+k}}{c^k(q;q)_{\alpha+k}}{}_3\phi_2\left[\begin{matrix} q^{-k}, -acq^{2n+1}, cq^n \\ cq^{2n+1}, -acq^{n+1} \end{matrix}; q, q\right]$$
$$= \sum_{n=0}^{\infty} I_{q,c}^\alpha\left\{\frac{(-abq^{n+1}, bq^{2n+1};q)_\infty t^n}{(-abq^{n+1}, bq^n;q)_\infty}\right\}$$
$$= \sum_{n=0}^{\infty} I_{q,c}^\alpha\left\{\frac{(-abq^{n+1};q)_n t^n}{(bq^n;q)_{n+1}}\right\}.$$

Similarly, the RHS of equation (15) become to

$$\sum_{n=0}^{\infty}\frac{(-atq^{n+1};q)_n}{(tq^n;q)_{n+1}}(1-q)^\alpha\sum_{k=0}^{n}\frac{(q;q)_n c^{n-k}}{(q;q)_{n-k}}\cdot\frac{b^{\alpha+k}(c/b;q)_{\alpha+k}}{(q;q)_{\alpha+k}} \tag{16}$$
$$= \sum_{n=0}^{\infty}\frac{(-atq^{n+1};q)_n}{(tq^n;q)_{n+1}} I_{q,c}^\alpha\{b^n\} = I_{q,c}^\alpha\left\{\frac{(-atq^{n+1};q)_n b^n}{(tq^n;q)_{n+1}}\right\}, \tag{17}$$

then, we use Proposition 2 can obtain the equation (15). The proof is complete. □

3. Moment integrals for a symmetric q-series identity

In this section, we use the moment integrals to deduce a new identity for a symmetric q-series.

Al-Salam and Carlitz [1] defined moments of two discrete distributions $d\alpha^{(a)}(x)$ and $d\beta^{(a)}(x)$ by Rogers-Szego polynomials as follow

$$\int_{-\infty}^{\infty} x^n \,\mathrm{d}\alpha^{(a)}(x) = h_n(a|q) \quad \text{and} \quad \int_{-\infty}^{\infty} x^n \,\mathrm{d}\beta^{(a)}(x) = g_n(a|q), \tag{18}$$

where $\alpha^{(a)}(x)$ is a step function whose jumps occur at the points q^k and aq^k for $k \in \mathbb{N}$, while the jumps of $\beta^{(a)}(x)$ occur at the points q^{-k} for $k \in \mathbb{N}$. These jumps are given by

$$d\alpha^{(a)}(q^k) = \frac{q^k}{(a;q)_\infty (q, q/a;q)_k} \quad \text{and} \quad d\alpha^{(a)}(aq^k) = \frac{q^k}{(1/a;q)_\infty (q, aq;q)_k}, \tag{19}$$

$$d\beta^{(a)}(q^{-k}) = \frac{a^k q^{k^2} (aq^{k+1})_\infty}{(q;q)_k}. \tag{20}$$

Liu gained the following expression of bivariate Rogers-Szegö polynomials by the technique of partial fraction [18, Eq. (4.20)].

$$h_n(a,b|q) = \frac{a^n}{(b/a;q)_\infty} \sum_{k=0}^{\infty} \frac{q^{(n+1)}k}{(q, aq/b;q)_k} + \frac{b^n}{(a/b;q)_\infty} \sum_{k=0}^{\infty} \frac{q^{(n+1)}k}{(q, qb/a;q)_k}. \tag{21}$$

So it's natural to defined the generalized discrete probability measure $\alpha^{(a,b)}$ by

$$\alpha^{(a,b)} = \sum_{k=0}^{\infty} \left[\frac{q^k}{(a/b;q)_\infty (q, qb/a;q)_k} \varepsilon_{bq^k} + \frac{q^k}{(b/a;q)_\infty (q, aq/b;q)_k} \varepsilon_{aq^k} \right], \tag{22}$$

where the bivariate Rogers–Szegö polynomials expressed by

$$h_n(a,b|q) = \int_{-\infty}^{+\infty} x^n \,\mathrm{d}\alpha^{(a,b)}(x), \tag{23}$$

and their generating function are given [18, Eq. (2.3)]

$$\sum_{n=0}^{\infty} h_n(a,b|q) \frac{t^n}{(q;q)_n} = \frac{1}{(at, bt;q)_\infty} = \int_{-\infty}^{+\infty} \frac{1}{(xt;q)_\infty} \,\mathrm{d}\alpha^{(a,b)}(x). \tag{24}$$

Cao [5] generalized equation (24) by the method of transformation.

Proposition 6 ([5, Eq. (1.11)]). *For $x \in \mathbb{N}$ and $d/c = q^{-x}$, if* $\max\{|cs|, |as|, |at|, |bs|, |bt|\} < 1$, *we have*

$$\int_{-\infty}^{\infty} \frac{(dx;q)_\infty}{(ax, bx, cx;q)_\infty} \,\mathrm{d}\alpha^{(s,t)}(x) = \frac{(ds, abst;q)_\infty}{(cs, as, at, bs, bt;q)_\infty} {}_3\phi_2 \left[\begin{matrix} d/c, as, bs \\ ds, abst \end{matrix} ; q, ct \right]. \tag{25}$$

Corollary 7. *For $x \in \mathbb{N}$, if* $\max\{|as|, |at|, |bs|, |bt|, |abst|\} < 1$, *we have*

$$\int_{-\infty}^{\infty} \frac{(cx;q)_\infty}{(ax, bx;q)_\infty} \,\mathrm{d}\alpha^{(s,t)}(x) = \frac{(cs, abst;q)_\infty}{(as, at, bs, bt;q)_\infty} {}_2\phi_2 \left[\begin{matrix} as, bs \\ cs, abst \end{matrix} ; q, ct \right]. \tag{26}$$

Proposition 8 ([7, Eq. (2.10)]). *For $n \in \mathbb{N}$, we have*

$$\mathbb{E}(b\theta_a)\{(at;q)_\infty\} = (at, bt;q)_\infty, \tag{27}$$

$$\mathbb{E}(b\theta_a)\{a^n (at;q)_\infty\} = a^n (at, bt;q)_\infty {}_2\phi_1 \left[\begin{matrix} q^{-n}, q/(at) \\ 0 \end{matrix} ; q, bt \right]. \tag{28}$$

Proposition 9. *For $x \in \mathbb{N}$, if* $\max\{|as|, |at|, |bs|, |bt|, |abst|\} < 1$*, we have*

$$\int_{-\infty}^{\infty} \frac{(cx, dx; q)_\infty}{(ax, bx; q)_\infty} \, \mathrm{d}\alpha^{(s,t)}(x) = \frac{(cs, ds, abst; q)_\infty}{(as, at, bs, bt; q)_\infty} \sum_{j=0}^{\infty} \frac{q^{\binom{j}{2}}(-ct)^j (bs, as; q)_j}{(q, cs, ds, abst; q)_j} {}_2\phi_1 \left[\begin{matrix} q^{-j}, 1/(csq^{j-1}) \\ 0 \end{matrix} ; q, dsq^j \right]. \tag{29}$$

Proof of Proposition 9 . By using the equation (26), we have

$$\mathbb{E}(s\theta_t) \left\{ \int_{-\infty}^{\infty} \frac{(cx; q)_\infty}{(ax, bx; q)_\infty} \, \mathrm{d}\alpha^{(s,t)}(x) \right\} = \frac{(ds, abst; q)_\infty}{(as, at, bs, bt; q)_\infty} \sum_{j=0}^{\infty} \frac{q^{\binom{j}{2}}(-1)^j (bs, as; q)_j}{(q, ds, abst; q)_j} \mathbb{E}(s\theta_t) \left\{ (ct)^j (csq^j; q)_\infty \right\}. \tag{30}$$

Then the LHS of the equation (30) can be written by

$$\begin{aligned} \mathbb{E}(s\theta_t) \left\{ \int_{-\infty}^{\infty} \frac{(cx; q)_\infty}{(ax, bx; q)_\infty} \, \mathrm{d}\alpha^{(s,t)}(x) \right\} &= \int_{-\infty}^{\infty} \frac{1}{(ax, bx; q)_\infty} \mathbb{E}(s\theta_t) \{(cx; q)_\infty\} \, \mathrm{d}\alpha^{(s,t)}(x) \\ &= \int_{-\infty}^{\infty} \frac{(cx, dx; q)_\infty}{(ax, bx; q)_\infty} \, \mathrm{d}\alpha^{(s,t)}(x). \end{aligned} \tag{31}$$

Using the equation (28), the RHS of the equation (30) becomes

$$\begin{aligned} &\frac{(ds, abst; q)_\infty}{(as, at, bs, bt; q)_\infty} \sum_{j=0}^{\infty} \frac{q^{\binom{j}{2}}(-1)^j (bs, as; q)_j}{(q, ds, abst; q)_j} \mathbb{E}(s\theta_t) \left\{ (ct)^j (csq^j; q)_\infty \right\} \\ &= \frac{(cs, ds, abst; q)_\infty}{(as, at, bs, bt; q)_\infty} \sum_{j=0}^{\infty} \frac{q^{\binom{j}{2}}(-ct)^j (bs, as; q)_j}{(q, cs, ds, abst; q)_j} {}_2\phi_1 \left[\begin{matrix} q^{-j}, 1/(csq^{j-1}) \\ 0 \end{matrix} ; q, dsq^j \right]. \end{aligned}$$

The proof is complete. □

Theorem 10. *For $x \in \mathbb{N}$, if* $\max\left\{\left|-axq^{2n+1}\right|, \left|-ayq^{2n+1}\right|, |xq^n|, |yq^n|\right\} < 1$*, we have*

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(-abq^{n+1}; q)_n}{(bq^n; q)_{n+1}} h_n(x, y|q) \\ &= \sum_{n=0}^{\infty} \frac{(-axq^{n+1}; q)_n b^n}{(xq^n; q)_{n+1}} \cdot \frac{(-axyq^{3n+1}; q)_\infty}{(-ayq^{2n+1}, yq^n; q)_\infty} \sum_{j=0}^{\infty} \frac{q^{\binom{j}{2}}(-yq^{2n+1})^j (xq^n, -axq^{2n+1}; q)_j}{(q, -axq^{n+1}, xq^{2n+1}, -axyq^{3n+1}; q)_j} \\ &\qquad \times {}_2\phi_1 \left[\begin{matrix} q^{-j}, 1/(xq^{2n+j}) \\ 0 \end{matrix} ; q, -axq^{2n+1+j} \right]. \end{aligned} \tag{32}$$

Remark 11. *Let $y = 0$ in Theorem 10, equation* (32) *reduces to* (7)*.*

Proof of Theorem 10. From a symmetric q-series identity

$$\sum_{n=0}^{\infty} \frac{(-abq^{n+1}; q)_n t^n}{(bq^n; q)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-atq^{n+1}; q)_n b^n}{(tq^n; q)_{n+1}}. \tag{33}$$

Acting moment integral on both sides of the equation (33), we have

$$\sum_{n=0}^{\infty} \frac{(-abq^{n+1}; q)_n}{(bq^n; q)_{n+1}} \int_{-\infty}^{\infty} t^n \, \mathrm{d}\alpha^{(x,y)}(t) = \sum_{n=0}^{\infty} b^n \int_{-\infty}^{\infty} \frac{(-atq^{n+1}, tq^{2n+1}; q)_\infty}{(-atq^{2n+1}, tq^n; q)_\infty)} \, \mathrm{d}\alpha^{(x,y)}(t). \tag{34}$$

Then use the equation (23) and (29), we obtain equation (32). The proof is complete. □

4. Generating functions for a symmetric q-series identity

In this section, motivated by the results of Liu's [24], we use the generating function for Al-Salam–Carlitz polynomial $\Phi_n^{(a,b)}(x,y|q)$ to generalize symmetric q-series identity.

The homogeneous polynomials $\Phi_n^{(\alpha,\beta)}(b,c|q)$ is defined by

$$\Phi_n^{(\alpha,\beta)}(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(\alpha;q)_k(\beta;q)_{n-k}x^ky^{n-k}. \tag{35}$$

Proposition 12 ([24, Proposition3.2])**.** *If* $\max\{|xt|,|yt|\}<1$*, we have*

$$\sum_{n=0}^{\infty}\Phi_n^{(a,b)}(x,y|q)\frac{t^n}{(q;q)_n}=\frac{(axt,byt;q)_\infty}{(ab,tx,ty;q)_\infty}. \tag{36}$$

Theorem 13. *If* $\max\{|c|,|q|,\left|-abq^{2n+1}\right|,|bq^n|,|b|,|t|\}<1$*, we have*

$$\sum_{n=0}^{\infty}\frac{(-atq^{n+1};q)_n(c;q)_kb^n}{(tq^n;q)_{n+1}(q;q)_k}=\sum_{n=0}^{\infty}\frac{(c,-abq^{n+1},bq^{2n+1};q)_\infty t^n}{(q;-abq^{2n+1},bq^n;q)_\infty}{}_3\phi_2\left[\begin{matrix}q/c,-abq^{2n+1},bq^n\\-abq^{n+1},bq^{2n+1}\end{matrix};q,c\right]. \tag{37}$$

Remark 14. *Let* $c=0$ *in Theorem 13, equation* (37) *reduces to* (7)*.*

Proof of Theorem 13. By Using equation (36), let $a=q^{-n},b=q^{n+1},x=-aq^{2n+1},y=q^n,t=b$ and $\max\left\{\left|-abq^{2n+1}\right|,|bq^n|\right\}<1$, then we have

$$\begin{aligned}\sum_{n=0}^{\infty}t^n\sum_{k=0}^{\infty}\Phi_k^{(q^{-n},q^{n+1})}(-aq^{2n+1},q^n|q)\frac{b^k}{(q;q)_k}&=\sum_{n=0}^{\infty}t^n\frac{(-abq^{n+1},bq^{2n+1};q)_\infty}{(-abq^{2n+1},bq^n;q)_\infty}\\&=\sum_{n=0}^{\infty}\frac{(-abq^{n+1};q)_nt^n}{(bq^n;q)_{n+1}}.\end{aligned} \tag{38}$$

Then, the LHS of equation (37) can be written by

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(c,-abq^{n+1},bq^{2n+1};q)_\infty t^n}{(q;-abq^{2n+1},bq^n;q)_\infty}{}_3\phi_2\left[\begin{matrix}q/c,-abq^{2n+1},bq^n\\-abq^{n+1},bq^{2n+1}\end{matrix};q,c\right]\\&=\sum_{n=0}^{\infty}t^n\sum_{k=0}^{\infty}\frac{(c;q)_k\Phi_k^{(q^{-n},q^{n+1})}(-aq^{2n+1},q^n|q)b^k}{(q,q;q)_k}\\&=\sum_{n=0}^{\infty}\frac{(-abq^{n+1};q)_nt^n}{(bq^n;q)_{n+1}}\cdot\frac{(c;q)_k}{(q;q)_k}\\&=\sum_{n=0}^{\infty}\frac{(-atq^{n+1};q)_n(c;q)_kb^n}{(tq^n;q)_{n+1}(q;q)_k}.\end{aligned}$$

The proof is complete. □

Acknowledgments

This work was supported by the National Natural Science Foundation of China (No. 11501155).

References

[1] W.A. Al-Salam and L. Carlitz, *Some orthogonal q-polynomials*, Math. Nachr. **30**(1965), 47–61.

[2] W.A. Al-Salam, *Some fractional q-integral and q-derivatives*, Proc. Edinburgh Math. Soc. **15**(1966), 135–140.

[3] G.E. Andrews, *Applications of basic hypergeometric series*, SIAM Rev. **16**(1974), 441–484.

[4] G.E. Andrews, *An introduction to Ramanujan's "lost" notebook*, Amer. Math. Monthly **2**(1979), 89–108.

[5] J. Cao, *A note on moment integrals and some applications*, J. Math. Anal. Appl. **410**(2014), 348–360.

[6] J. Cao, *A note on generalized q-difference equations for q-beta and Andrews-Askey integral*, J. Math. Anal. Appl. **412**(2014), 841–851.

[7] J. Cao, *New proof of generating function for Rogers–Szegö polynomials*, Appl. Math. Comput. **207**(2009), 486–492.

[8] J. Cao, H.M. Srivastava, *Some iterated fractional q-integrals and their applications*, Fract. Calc. Appl. Anal. **21**(2018), 672–695.

[9] W.Y.C. Chen, Z.G. Liu, *Parameter augmentation for basic hypergeometric series*, J. Combridge. Theory Ser. A**80**(1997), 175–195.

[10] W.Y.C. Chen, Z.G. Liu, *Parameter augmentation for basic hypergeometric series*, B.E. Sagan, R.P. Stanley, Mathmatical Essays in honor of Gian-Carlo Rota, Birkäuser, Basel, 1998, pp. 111–129.

[11] F.H. Jackson, *On q-difinite integrals*, Quat.J.Pure.Appl.Math. **50**(1910), 101–112.

[12] J.-P. Fang, *Note on a q-contour intergal formula*, Appl. Math. Comput. **233**(2014), 292–297.

[13] J.-P. Fang, *Remarks on generalized q-difference equation*, J. Difference. Equ. Appl. **21**(2015), 934–953.

[14] J.-P. Fang, *Applications of a generalized q-difference equation*, Adv. Difference. Equ. 2014.

[15] J.-P. Fang, *q-difference equation and q-polynomials*, Appl. Math. Comput. **248**(2014), 550–561.

[16] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, second edition, Encyclopedia Math. Appl., vol. 96, Cambridge Univ. Press, Cambridge, 2004.

[17] D.-Q. Lu, *q-difference equation and the Cauchy operator identites*, J. Math. Anal. Appl. **359**(2009), 265–274.

[18] Z.-G. Liu, *Two q-difference equations and q-operator identities*, J. Difference Equ. Appl. **16**(2010), 1293–1307.

[19] Z.-G. Liu, *An extension of the non-terminating* ${}_6\phi_5$ *summation and the Askey–Wilson polynomials*, J. Difference Equ. Appl. **17**(2011), 1401–1411.

[20] Z.-G. Liu, *On the q-partial differential equations and q-series*, The legacy of Srinivasa Ramanujan, 213C250, Ramanujan Math. Soc. Lect. Notes Ser., 20, Ramanujan Math. Soc., Mysore, 2013.

[21] Z.-G. Liu, *Some operator identities and q-series traansformation formulas*, Discret.Math. **265**(2003), 119–139.

[22] Z.-G. Liu, *A q-extension of a partial differential equation and the Hahn polynomials*, Ramanujan J. **38**(2015), 481–501.

[23] Z.-G. Liu and J. Zeng, *Two expansion formulas involving the Rogers-Szegö polynomials with applications*, Int. J. Number Theory **11**(2015), 507–525.

[24] Z.-G. Liu, *On a system of q-partial differential equations with application to q-series*, Analytic number theory, modular forms and q-hypergeometric series, 445–461, Springer Proc. Math. Stat. 221, Springer, Cham, 2017.

[25] A.E. Patkowski, *On a symmetric q-series identity*, Discrete Math. **339**(2016), 2994–2997.

[26] P.M. Rajković, S.D. Marinković and M.S. Stanković, *Fractional integrals and derivatives in q-calculus*, Appl. Anal. Discrete Math. **1**(2007), 311–323.

[27] R.M. Range, *Complex analysis: A brief tour into higher dimensions*, Amer. Math. Monthly **110**(2003), 89–108.

[28] J. Thomae, *Beitraäge zur Theorie der durch die Heinesche Reihe: Darstellbaren Function*, J. Reine Angew. Math. **70**(1869), 258–281.

[29] M. Wang, *q-integral representation of the Al-Salam–Carlitz polynomials*, Appl. Math. Lett. **22**(2009), 943–945.

[30] M. Wang, *A recurring q-integral formula*, Appl. Math. Lett. **23**(2010), 256–260.

[31] M. Wang, *A new probability distribution with applications*, Pacfic J. Math. **247**(2010), 241–255.

[32] H.S. Wilf, *generatingfunctionology*, Second Edition, Academic Press, 1994.

[33] Z.-Z. Zhang, M. Liu, *Applications of operator identities to the multiple q-binomial theorem and q-Gauss summation theorem*, Discrete Math. **306**(2006), 1424–1437.

Department of Mathematics, Hangzhou Normal University, Hangzhou City, Zhejiang Province, 311121, PR China.
E-mail address: 21caojian@gmail.com; 21caojian@163.com; 188357922542@163.com.