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COEFFICIENT PROBLEM CONCERNING SOME NEW SUBCLASSES OF ANALYTIC AND UNIVALENT FUNCTIONS

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Abstract. The primary focus of the present paper is to study few properties of fractional analytic function $f_\alpha(z)$ belonging to certain new subclasses of analytic functions $S^*_{A,\,B,\,m}(\alpha,\ \beta,\ \lambda)$ and $K_{A,\,B,\,m}(\alpha,\ \beta,\ \lambda)$ defined in the open unit disk.

1. Introduction

Suppose that A denote the class of all analytic functions f with series expansion

$$f(z) = z + a_2z^2 + a_3z^3 + ... \tag{1}$$

normalized with $f'(0) - 1 = 0 = f(0)$ in the open unit disk $D = \{z \in \mathrm{C} : |\mathrm{z}| < 1\}$. Let S denote the class of univalent function $f \in A$ in D. Also suppose that S^* denote the subclass of S consisting of the functions $f(z)$which are starlike in D. A function $f(z) \in K$ is said to be convex in D if $f(z) \in S$ satisfies $zf'(z) \in S^*$. See ([2]), ([3]), ([4]) and ([8]) for details. In view of the above definitions, we can write that

$$K \subset S^* \subset S \subset A \tag{2}$$

and $f(z) \in S^*$ if and only if

$$\int_0^z \frac{f(t)}{t}\,dt \in K. \tag{3}$$

Further, the function $f(z)$ having the series expansion

$$f(z) = \frac{z}{1-z^2} = z + z^3 + z^5 ... = \sum_{k=0}^{\infty} z^{2k+1} \qquad (z \in D) \tag{4}$$

belong to the analytic class S^* while the function $f(z)$ of the form

$$f(z) = \frac{z}{1-z} = z + z^2 + z^3 ... = \sum_{k=0}^{\infty} z^{k+1} \qquad (z \in D) \tag{5}$$

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is in the class K. In 2017, ([2]) introduced and studied the fractional analytic function $f_\alpha(z)$ of the form

$$f_\alpha(z) = \frac{z}{1-z^\alpha} = z + \sum_{k=1}^{\infty} z^{1+k\alpha} \qquad (z \in D) \tag{6}$$

for some real α $(0 < \alpha \leq 2)$ in the open unit disk. For more details on the kind of analytic function $f_\alpha(z)$ defined in (6), interested reader can refer to ([1]), ([6]) and ([10]). Now, for the purpose of the present investigation we shall consider a more generalized form of (6) whereby $f_\alpha(z)$ is given by

$$f_\alpha(z) = \frac{A(z-\omega)}{A+B(z-\omega)^\alpha} = (z-\omega) + \sum_{k=1}^{\infty} (-1)^k \frac{B^k}{A^k} (z-\omega)^{1+k\alpha} \qquad (z \in D) \tag{7}$$

for some real α $(0 < \alpha \leq 2)$, $-1 \leq B < A \leq 1$ where ω is a fixed point in D . Now using (7), a new class A_α of analytic functions $f_\alpha(z)$ is given in D such that

$$f_\alpha(z) = \frac{A(z-\omega)}{A+B(z-\omega)^\alpha} = (z-\omega) + \sum_{k=1}^{\infty} (-1)^k \frac{B^k}{A^k}\, a_{k+1}(z-\omega)^{1+k\alpha} \qquad (z \in D) \tag{8}$$

for some real α $(0 < \alpha \leq 2)$, $-1 \leq B < A \leq 1$ where ω is a fixed point in D.

Also for $-1 \leq B < A \leq 1$, $\lambda \geq 0$, $\alpha > 0$, $m \in N_0 = N \bigcup \{0\}$ and fixed ω in D, we introduce the following linear differential operator such that

$$\begin{aligned} &D^0 f_\alpha(z) = f_\alpha(z) \\ &D^1 f_\alpha(z) + \frac{\lambda}{2}\left(D^0 f_\alpha(z)\right) = (z-\omega)\left(\frac{2+\lambda}{2}\right)\left(D^{m-1} f_\alpha(z)\right) \\ &\vdots \\ &D^m f_\alpha(z) + \tfrac{\lambda}{2}\left(D^{m-1} f_\alpha(z)\right) = (z-\omega)\left(\tfrac{2+\lambda}{2}\right)\left(D^{m-1} f_\alpha(z)\right)' . \end{aligned} \tag{9}$$

Using (8) and (9), we obtain

$$D^m f_\alpha(z) = (z-\omega) + \sum_{k=1}^{\infty} \left(\frac{2+k\alpha(2+\lambda)}{2}\right)^m (-1)^k \frac{B^k}{A^k} a_{k+1}(z-\omega)^k \tag{10}$$

and

$$D^{m+1} f_\alpha(z) = (z-\omega) + \sum_{k=1}^{\infty} \left(\frac{2+k\alpha(2+\lambda)}{2}\right)^{m+1} (-1)^k \frac{B^k}{A^k} a_{k+1}(z-\omega)^k. \tag{11}$$

Now for the purpose of the present investigation, we propose the following definitions.

Remark A: In particular, when $\omega = \lambda = 0$, $A = \alpha = 1$ and $B = -1$ in (10), we immediately obtain the celebrated Salagean differential operator ([9]).

Definition 1: Let $f_\alpha \in A_\alpha$ satisfies the analytic condition

$$\Re e\left(\frac{D^{m+1} f_\alpha(z)}{D^m f_\alpha(z)}\right) > \beta \tag{12}$$

for real β $(0 \leq \beta < 1)$ where ω is a fixed point in D, then we say that $f_\alpha(z) \in S^*_{A,\,B,\,m}(\alpha,\,\beta,\,\lambda)$ where $S^*_{A,\,B,\,m}(\alpha,\,\beta,\,\lambda)$ denote the class of starlike functions of order β.

Definition 2: Let $f_\alpha \in A_\alpha$ satisfies the analytic condition

$$\Re e\left(\frac{(z-\omega)\left(D^{m+1}f_\alpha(z)\right)'}{D^{m+1}f_\alpha(z)}\right) > \beta \tag{13}$$

for real β $(0 \le \beta < 1)$ where ω is a fixed point in D, then we say that $f_\alpha(z) \in K_{A,\,B,\,m}(\alpha,\,\beta,\,\lambda)$ where $K_{A,\,B,\,m}(\alpha,\,\beta,\,\lambda)$ denote the class of convex functions of order β.

2. Coefficient Inequalities

The results presented here include the coefficient inequalities for functions $f_\alpha(z)$ belonging to the classes $S^*_{A,\,B,\,m}(\alpha,\,\beta,\,\lambda)$ and $K_{A,\,B,\,m}(\alpha,\,\beta,\,\lambda)$ in the open unit disk D.

Theorem 2.1: Let $f_\alpha \in A_\alpha$ satisfies the inequality

$$\sum_{k=1}^{\infty}\frac{(2+k\alpha(2+\lambda))^m}{2^{m+1}}\left(k\alpha(2+\lambda)+2(1-\beta)\right)\frac{|B^k|}{A^k}|a_{k+1}| \le 1-\beta. \tag{14}$$

Then $f_\alpha(z) \in S^*_{A,\,B,\,m}(\alpha,\,\beta,\,\lambda)$ where $\alpha > 0$,$\lambda \ge 0$,$-1 \le B < A \le 1$,$0 \le \beta < 1$,$m \in N\bigcup\{0\}$ and ω is a fixed point in D. The equality is attained for function $f_\alpha(z)$ given by'

$$f_\alpha(z) = (z-\omega)+\sum_{k=1}^{\infty}\frac{2^{m+1}A^k\,(1-\beta)\,e^{i\,\pi}}{k(k+1)\,(2+k\alpha(2+\lambda))^m\,(k\alpha(2+\lambda)+2(1-\beta))\,|B^k|}(z-\omega)^{k\alpha+1}. \tag{15}$$

Proof: Suppose that $f_\alpha \in A_\alpha$ is having the form (1). If $f_\alpha(z)$ satisfies the inequality (14), then

$$\left|\left(\frac{D^{m+1}f(z)}{D^m f(z)}\right)-1\right| = \left|\frac{\sum_{k=1}^{\infty}\left(\frac{2+k\alpha(2+\lambda)}{2}\right)^m\left(\frac{k\alpha(2+\lambda)}{2}\right)(-1)^k\frac{B^k}{A^k}a_{k+1}(z-\omega)^{k\alpha}}{1+\sum_{k=1}^{\infty}\left(\frac{2+k\alpha(2+\lambda)}{2}\right)^m(-1)^k\frac{B^k}{A^k}a_{k+1}(z-\omega)^{k\alpha}}\right|$$

$$\le \frac{\sum_{k=1}^{\infty}\left(\frac{2+k\alpha(2+\lambda)}{2}\right)^m\left(\frac{k\alpha(2+\lambda)}{2}\right)\frac{|B^k|}{A^k}|a_{k+1}||z-\omega|^{k\alpha}}{1-\sum_{k=1}^{\infty}\left(\frac{2+k\alpha(2+\lambda)}{2}\right)^m\frac{|B^k|}{A^k}|a_{k+1}||z-\omega|^{k\alpha}}$$
$$< \frac{\sum_{k=1}^{\infty}\left(\frac{2+k\alpha(2+\lambda)}{2}\right)^m\left(\frac{k\alpha(2+\lambda)}{2}\right)\frac{|B^k|}{A^k}|a_{k+1}|}{1-\sum_{k=1}^{\infty}\left(\frac{2+k\alpha(2+\lambda)}{2}\right)^m\frac{|B^k|}{A^k}|a_{k+1}|} \le 1-\beta\,.$$

This shows that $f_\alpha(z) \in S^*_{A,\,B,\,m}(\alpha,\,\beta,\,\lambda)$. Now suppose that $f_\alpha(z)$ is given by (8), then we have that

$$\sum_{k=1}^{\infty}\frac{(2+k\alpha(2+\lambda))^m}{2^{m+1}}\left(k\alpha(2+\lambda)+2(1-\beta)\right)\frac{|B^k|}{A^k}|a_{k+1}| = \sum_{k=1}^{\infty}\frac{1-\beta}{k(k-1)}$$

$$= (1-\beta)\sum_{k=1}^{\infty}\frac{1}{k(k-1)} = 1-\beta$$

and this obviously ends the proof of theorem 2.1.

If $m = 0$ in theorem 2.1, then the following corollary is immediate.

Corollary 2.2: Let $f_\alpha \in A_\alpha$ satisfies the inequality

$$\sum_{k=1}^{\infty} \frac{1}{2} \left(k\alpha(2+\lambda) + 2(1-\beta)\right) \frac{|B^k|}{A^k} |a_{k+1}| \leq 1-\beta.$$

Then $f_\alpha(z) \in S^*_{A,\,B,\,0}(\alpha,\,\beta,\,\lambda)$. The equality is attained for function $f_\alpha(z)$ given by'

$$f_\alpha(z) = (z-\omega) + \sum_{k=1}^{\infty} \frac{2A^k\,(1-\beta)}{k(k+1)\left(k\alpha(2+\lambda)+2(1-\beta)\right)|B^k|}(z-\omega)^{k\alpha+1}.$$

Corollary 2.3: Let $f_\alpha \in A_\alpha$ satisfies the inequality

$$\sum_{k=1}^{\infty} \frac{1}{2} \left(k\alpha(2+\lambda) + 2\right) \frac{|B^k|}{A^k} |a_{k+1}| \leq 1.$$

Then $f_\alpha(z) \in S^*_{A,\,B,\,0}(\alpha,\,0,\,\lambda)$. The equality is attained for function $f_\alpha(z)$ given by'

$$f_\alpha(z) = (z-\omega) + \sum_{k=1}^{\infty} \frac{2A^k}{k(k+1)\left(k\alpha(2+\lambda)+2\right)|B^k|}(z-\omega)^{k\alpha+1}.$$

Corollary 2.4: Let $f_\alpha \in A_\alpha$ satisfies the inequality

$$\sum_{k=1}^{\infty} \frac{1}{2} \left(k\alpha(2+\lambda) + 2\right) |a_{k+1}| \leq 1.$$

Then $f_\alpha(z) \in S^*_{1,\,-1,\,0}(\alpha,\,0,\,\lambda)$. The equality is attained for function $f_\alpha(z)$ given by'

$$f_\alpha(z) = (z-\omega) + \sum_{k=1}^{\infty} \frac{2}{k(k+1)\left(k\alpha(2+\lambda)+2\right)}(z-\omega)^{k\alpha+1}.$$

Corollary 2.5: Let $f_\alpha \in A_\alpha$ satisfies the inequality

$$\sum_{k=1}^{\infty} \frac{1}{2} \left(3k\alpha + 2\right) |a_{k+1}| \leq 1.$$

Then $f_\alpha(z) \in S^*_{1,\,-1,\,0}(\alpha,\,0,\,1)$. The equality is attained for function $f_\alpha(z)$ given by'

$$f_\alpha(z) = (z-\omega) + \sum_{k=1}^{\infty} \frac{2}{k(k+1)\left(3k\alpha+2\right)}(z-\omega)^{k\alpha+1}.$$

Remark B:
In a special situation when $m = 0,\ \lambda = 0,\ A = 1$ and $B = -1$. Then the inequality (14) readily yield the result obtained by ([2]).

Theorem 2.6: Let the function $f_\alpha(z) \in A_\alpha$given by (8) satisfies the inequality

$$\sum_{k=1}^{\infty} \frac{1}{2^{m+1}} \left(k\,\alpha - \beta + 1\right) \left(2 + k\,\alpha(2+\lambda)\right)^{m+1} (-1)^k \frac{B^k}{A^k} |a_{k+1}| \leq 1-\beta \qquad (16)$$

for some $\beta\ (0 \leq \beta < 1),\ \alpha > 0,\ \lambda \geq 0,\ -1 \leq B < A \leq 1$ and ω is a fixed point in D. Then

$$f_\alpha(z) \in K_{A,\,B,\,m}(\alpha,\ \beta).$$

Equality is attained for function $f_\alpha(z)$given by

$$f_\alpha(z) = (z-\omega) + \sum_{k=1}^{\infty} \frac{2^{m+1}(1-\beta)}{k(k-1)(k\,\alpha-\beta+1)(2+k\,\alpha(2+\lambda))^{m+1}}\,(z-\omega)^{k\,\alpha+1}. \quad (17)$$

Proof: Let $f_\alpha(z) \in A_\alpha$ be given by (8). If $f_\alpha(z)$ satisfies the (16), then

$$\left|\frac{z\left(D^{m+1}f_\alpha(z)\right)'}{D^{m+1}f_\alpha(z)} - 1\right| = \left|\frac{\sum_{k=1}^{\infty} k\alpha\left(\frac{2+k\,\alpha(2+\lambda)}{2}\right)^{m+1}(-1)^k\frac{B^k}{A^k}\,a_{k+1}(z-\omega)^{k\alpha}}{1+\sum_{k=1}^{\infty}\left(\frac{2+k\,\alpha(2+\lambda)}{2}\right)^{m+1}(-1)^k\frac{B^k}{A^k}\,a_{k+1}(z-\omega)^{k\alpha}}\right|$$

$$\leq \frac{\sum_{k=1}^{\infty} k\alpha\left(\frac{2+k\,\alpha(2+\lambda)}{2}\right)^{m+1}\frac{|B|^k}{A^k}\,|a_{k+1}||z-\omega|^{k\alpha}}{1-\sum_{k=1}^{\infty}\left(\frac{2+k\,\alpha(2+\lambda)}{2}\right)^{m+1}\frac{|B|^k}{A^k}\,|a_{k+1}||z-\omega|^{k\alpha}}$$

$$< \frac{\sum_{k=1}^{\infty} k\alpha\left(\frac{2+k\,\alpha(2+\lambda)}{2}\right)^{m+1}\frac{|B|^k}{A^k}\,|a_{k+1}|}{1-\sum_{k=1}^{\infty}\left(\frac{2+k\,\alpha(2+\lambda)}{2}\right)^{m+1}\frac{|B|^k}{A^k}\,|a_{k+1}|} \;\leq\; 1-\beta$$

showing that $f_\alpha(z) \in K_{A,\,B,\,m}(\alpha,\ \beta)$and this completes the proof of theorem 2.6.

3. Partial Sums

In this section, we discuss the partial sums of the analytic function $f_\alpha(z)$ defined by (8). Here, we propose the following definition.

$$f_\alpha(z) = (z-\omega) + (-1)^k\frac{B^k}{A^k}\,a_{k+1}(z-\omega)^{k\,\alpha+1} \qquad k = 1,\ 2,\ 3,\ \ldots \quad (18)$$

for some real $\alpha\ (0 < \alpha \leq 2)$ *and* $-1 \leq B < A \leq 1$.

Theorem 3.1: Let the function $f_\alpha(z)$ be of the form (18) with $|a_{k+1}| \leq 1,\ 0 < \alpha \leq 2,\ \lambda \geq 0$
$-1 \leq B < A \leq 1$ and $m \in N_0 = N\bigcup\{0\}$. Then for $|z-\omega| = (r+d)$

$$\Re e\left(\frac{D^{m+1}f_\alpha(z)}{D^m f_\alpha(z)}\right) > \frac{1-\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^{m+1}\frac{|B|^k}{A^k}\,|a_{k+1}|}{1-\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^{m}\frac{|B|^k}{A^k}\,|a_{k+1}|} \quad (19)$$

and

$$\Re e\left(\frac{D^{m+1}f_\alpha(z)}{D^m f_\alpha(z)}\right) \geq \frac{1-\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^{m+1}\frac{|B|^k}{A^k}\,(r+d)^{k\alpha}}{1-\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^{m}\frac{|B|^k}{A^k}\,(r+d)^{k\alpha}}. \quad (20)$$

Proof: Suppose that $f_\alpha(z)$ be of the form (18), then

$$\Re e\left(\frac{D^{m+1}f_\alpha(z)}{D^m f_\alpha(z)}\right) = \Re e\left(1+\frac{\left(\frac{k\,\alpha(2+\lambda)}{2}\right)\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^m(-1)^k\frac{B^k}{A^k}\,a_{K+1}\,(z-\omega)^{k\alpha}}{1+\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^m(-1)^k\frac{B^k}{A^k}\,a_{K+1}\,(z-\omega)^{k\alpha}}\right)$$

$$= 1+\Re e\left(\frac{\left(\frac{k\,\alpha(2+\lambda)}{2}\right)\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^m\frac{|B|^k}{A^k}\,|a_{K+1}|\,(r+d)^{k\alpha}\begin{pmatrix}\cos(k\alpha\theta+\rho)+\\ i\,\sin(k\alpha\theta+\rho)\end{pmatrix}}{1+\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^m\frac{|B|^k}{A^k}\,|a_{K+1}|\,(r+d)^{k\alpha}\begin{pmatrix}\cos(k\alpha\theta+\rho)+\\ i\,\sin(k\alpha\theta+\rho)\end{pmatrix}}\right)$$

where $a_{k+1} = |a_{k+1}|\,e^{i\,\theta}$. It implies that

$$\Re e\left(\frac{D^{m+1}f_\alpha(z)}{D^m f_\alpha(z)}\right)$$
$$= 1 + \frac{\left(\frac{k\,\alpha(2+\lambda)}{2}\right)\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^m\left[\cos(k\alpha\theta+\rho)+\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^m\frac{|B|^k}{A^k}\,|a_{K+1}|\,(r+d)^{k\alpha}\right]}{1+2\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^m\frac{|B|^k}{A^k}\,|a_{K+1}|\,(r+d)^{k\alpha}\cos(k\alpha\theta+\rho)+\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^{2m}\frac{|B|^{2k}}{A^{2k}}\,|a_{K+1}|^2\,(r+d)^{2k\alpha}}.$$

Suppose that $h(t)$ is defined by

$$h(t) = \frac{\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^m\frac{|B|^k}{A^k}\,|a_{K+1}|\,(r+d)^{k\alpha}+t}{1+2\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^m\frac{|B|^k}{A^k}\,|a_{K+1}|\,(r+d)^{k\alpha}t+\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^{2m}\frac{|B|^{2k}}{A^{2k}}\,|a_{K+1}|^2\,(r+d)^{2k\alpha}}$$
$$(t=\cos(k\alpha\theta+\rho)).$$

Then, we have that $h'(t) > 0$ and $|a_{k+1}| \leq 1$. Finally, we obtain

$$\Re e\left(\frac{D^{m+1}f_\alpha(z)}{D^m f_\alpha(z)}\right) > 1 - \frac{\left(\frac{k\,\alpha(2+\lambda)}{2}\right)\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^m\frac{|B|^k}{A^k}\,|a_{K+1}|\,(r+d)^{k\alpha}}{1-\left(\frac{k\,\alpha(2+\lambda)+2}{2}\right)^m\frac{|B|^k}{A^k}\,|a_{K+1}|\,(r+d)^{k\alpha}}. \quad (21)$$

By letting $r \to 1$and $d = 0$ in (21), we obtain the desired result as contained in (19) while we have the inequality in (20) by setting $|a_{k+1}| = 1$ in (21).

Remark C: In a special case when $m = 0$ and $\lambda = 0$ in (19) and (20), then we obtain the results due to ([2]). For recent work on partial sums, interested reader can refer to ([1]), ([2]) and ([5]).

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