Journal of Fractional Calculus and Applications
Vol. 11(1) Jan. 2020, pp. 82-90.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

ON CERTAIN INTEGRAL TRANSFORM INVOLVING GENERALIZED BESSEL-MAITLAND FUNCTION AND APPLICATIONS

MUSHARRAF ALI, WASEEM A. KHAN AND IDREES A. KHAN

Abstract. In this article, we establish a new integral formula involving the generalized Bessel-Maitland function defined by Khan et al. [9], which is expressed in terms of generalized (Wright) hypergeometric function. Some interesting and special cases of our main result are also considered.

1. Introduction

In recent years, many authors (see, e.g., [1-4] have developed numerous integral formulas involving a variety of special functions. Also many integral formulas associated with the Bessel functions of several kinds have been presented (see, e.g., [6-9]). Those integrals involving Bessel-Maitland functions are not only of great interest to the pure mathematics, but they are often of extreme importance in many branches of theoretical and applied physics and engineering (see [20]). Several methods for evaluating infinite or finite integrals involving Bessel-Maitland functions have been known (see, e.g.,[4] and [19]). However, these methods usually work on a case-by-case basis.

Currently, Ghayasuddin and Khan [4], Khan et al. [6-9] gave certain interesting new class of integral formulas involving the generalized Bessel-Maitland function, which are expressed in terms of the generalized (Wright) hypergeometric function. In the present sequel to the aforementioned investigations, we present two generalized integral formulas involving generalized Bessel-Maitland functions, which are expressed in terms of the generalized (Wright) hypergeometric function. Some special cases and the (potential) usefulness of our main results are also considered and remarked, respectively.

2010 *Mathematics Subject Classification.* 33C45, 33C60, 33E12.
Key words and phrases. Generalized Bessel-Maitland function, Wright hypergeometric function and integrals, Mittag-Leffler function.
Submitted December 09, 2018.

The Bessel-Maitland function $J_\nu^\mu(z)$ [12;Eq.(8.3)] defined by the following series representation:

$$J_\nu^\mu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(\mu n + \nu + 1)} = \phi(\mu, \nu + 1; -z). \tag{1.1}$$

Singh et al. [19] introduced the following generalization of Bessel-Maitland function as:

$$J_{\nu,q}^{\mu,\gamma} = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma(\mu n + \nu + 1)n!}, \tag{1.2}$$

where $\mu, \nu, \gamma \in \mathbb{C}, Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0$ and $q \in (0,1) \bigcup \mathbb{N}$ and $(\gamma)_0 = 1, (\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$ denotes the generalized Pochhammer symbol.

Recently, Ghayasuddin and Khan [4] introduced and investigated generalized Bessel-Maitland function defined as

$$J_{\nu,\gamma,\delta}^{\mu,q,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma(\mu n + \nu + 1)(\delta)_{pn}}, \tag{1.3}$$

where $\mu, \nu, \gamma, \delta \in \mathbb{C}$, $\Re(\mu) \geq 0, \Re(\nu) \geq -1, \Re(\gamma) \geq 0, \Re(\delta) \geq 0; p, q > 0$ and $q < \Re(\alpha) + p$.

In particular Khan et al. [9] introduced and investigated a new extension of Bessel-Maitland function as follows:

$$J_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}(\gamma)_{qn}(-z)^n}{\Gamma(n\beta + \alpha + 1)(\delta)_{pn}(\nu)_{n\sigma}}, \tag{1.4}$$

where $\alpha, \beta, \mu, \rho, \nu, \gamma, \sigma, \delta \in \mathbb{C}; \Re(\beta) > 0, \Re(\rho) > 0, \Re(\mu) > 0, \Re(\nu) > 0, \Re(\alpha) \geq -1, \Re(\gamma) > 0, \Re(\delta) > 0, \Re(\sigma) > 0; p, q > 0$, and $q < \Re(\alpha) + p$.

Relation with Mittag-Leffler functions:

(i) On replacing α by $\alpha - 1$ in (1.4), we get the following interesting relation:

$$J_{\alpha-1,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(-z) = E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z), \tag{1.5}$$

where $E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)$ is the Mittag-Leffler function defined by Khan and Ahmed [10].

(ii) On setting $\mu = \nu = \sigma = \rho = 1$ and replacing α by $\alpha - 1$ in (1.4), we get

$$J_{\alpha-1,\beta,1,1,\delta,p}^{1,1,\gamma,q}(-z) = E_{\alpha,\beta,p}^{\gamma,\delta,q}(z), \tag{1.6}$$

where $E_{\alpha,\beta,p}^{\gamma,\delta,q}(z)$ is the Mittag-Leffler function defined by Salim and Faraz [17].

(iii) On setting $\mu = \nu = \sigma = \rho = \delta = p = 1$ and replacing α by $\alpha - 1$ in (1.4), we get

$$J_{\alpha-1,\beta,1,1,1,1}^{1,1,\gamma,q}(-z) = E_{\alpha,\beta}^{\gamma,q}(z), \tag{1.7}$$

where $E_{\alpha,\beta}^{\gamma,q}(z)$ is the Mittag-Leffler function defined by Shukla and Prajapati [18].

(iv) On setting $\mu = \nu = \sigma = \rho = \delta = p = q = 1$ and replacing α by $\alpha - 1$ in (1.4), we get

$$J^{1,1,\gamma,1}_{\alpha-1,\beta,1,1,1,1}(-z) = E^{\gamma}_{\alpha,\beta}(z), \tag{1.8}$$

where $E^{\gamma}_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined by Prabhakar [14].

(v) On setting $\mu = \nu = \sigma = \rho = \delta = \gamma = p = q = 1$ and replacing α by $\alpha - 1$ in (1.4), we get

$$J^{1,1,1,1}_{\alpha-1,\beta,1,1,1,1}(-z) = E_{\alpha,\beta}(z), \tag{1.9}$$

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined by Wiman [21].

(vi) On setting $\mu = \nu = \sigma = \rho = \delta = \gamma = p = q = 1$, $\alpha = 0$ and replacing α by $\alpha - 1$ in (1.4), we get

$$J^{1,1,1,1}_{0,\beta,1,1,1,1}(-z) = E_{\beta}(z), \tag{1.10}$$

where $E_{\beta}(z)$ is the Mittag-Leffler function defined by Ghosta Mittag-Leffler [11].

The generalization of the generalized hypergeometric series ${}_pF_q$ is due to Fox [5] and Wright ([21], [22], [23]) who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [17, p.21]; see also [15]):

$${}_p\Psi_q\left[\begin{array}{cccc}(\alpha_1\ ,\ A_1), & ....., & (\alpha_p\ ,\ A_p); & \\ & & & z\\ (\beta_1\ ,\ B_1), & ....., & (\beta_q\ ,\ B_q); & \end{array}\right] = \sum_{k=0}^{\infty} \frac{\prod\limits_{j=1}^{p}\ \Gamma(\alpha_j + A_j k)}{\prod\limits_{j=1}^{q}\ \Gamma(\beta_j + B_j k)}\ \frac{z^k}{k!}\ , \tag{1.11}$$

where the coefficients $A_1, \cdots, A_p$ and $B_1, \cdots, B_q$ are positive real numbers such as

$$\text{(i) } 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0 \text{ and } 0 < |z| < \infty;\ z \neq 0. \tag{1.12}$$

$$\text{(ii) } 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j = 0 \text{ and } 0 < |z| < {A_1}^{-A_1} \ldots {A_p}^{-A_p} {B_1}^{B_1} \ldots {B_q}^{B_q}. \tag{1.13}$$

A special case of (1.11) is

$${}_p\Psi_q\left[\begin{array}{cccc}(\alpha_1\ ,\ 1), & ....., & (\alpha_p\ ,\ 1); & \\ & & & z\\ (\beta_1\ ,\ 1), & ....., & (\beta_q\ ,\ 1); & \end{array}\right] = \frac{\prod\limits_{j=1}^{p}\ \Gamma(\alpha_j)}{\prod\limits_{j=1}^{q}\ \Gamma(\beta_j)}\ {}_pF_q\left[\begin{array}{cccc}\alpha_1, & ....., & \alpha_p\ ; & \\ & & & z\\ \beta_1, & ....., & \beta_q\ ; & \end{array}\right], \tag{1.14}$$

where ${}_pF_q$ is the generalized hypergeometric series defined by [15]

$$\begin{aligned}{}_pF_q\left[\begin{array}{cccc}\alpha_1, & ....., & \alpha_p\ ; & \\ & & & z\\ \beta_1, & ....., & \beta_q\ ; & \end{array}\right] &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n}\ \frac{z^n}{n!}\\ &= {}_pF_q(\alpha_1, \cdots, \alpha_p;\ \beta_1, \cdots \beta_q;\ z),\end{aligned} \tag{1.15}$$

where $(\lambda)_n$ is the Pochhammer's symbol (see [15]).

For our present investigation the following interesting and useful result due to Obhettinger [13] will be required:

$$\int_0^\infty x^{\mu-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}dx = 2\lambda a^{-\lambda}\left(\frac{a}{2}\right)^{\mu}\frac{\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)}, \tag{1.16}$$

provided $0 < \Re(\mu) < \Re(\lambda)$.

2. Main Result

Theorem 2.1. If $\alpha,\beta,\gamma,\mu,\eta,\nu,\sigma,\rho,\delta,\lambda \in \mathbb{C}$, $\sigma+\eta+p-\rho-q>0$, $\Re(\alpha)>0, \Re(\beta)>0, \Re(\gamma)>0, \Re(\mu)>0, \Re(\lambda)\geq -1, \Re(\nu)>0, \Re(\delta)>0, \Re(\sigma)>0, \Re(\rho)>0, \Re(\eta)>0$, $p,q>0$ and $q<\Re(\alpha)+p$, then

$$\int_0^\infty x^{\alpha-1}(x+a+\sqrt{x^2+2ax})^{-\beta} J^{\mu,\rho,\gamma,q}_{\eta,\lambda,\nu,\sigma,\delta,p}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\alpha}a^{\alpha-\beta}\frac{\Gamma(2\alpha)\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)}\,{}_5\Psi_5\left[\begin{array}{l}(\mu,\rho),(\gamma,q),(\beta-\alpha,1),(\beta+1,1),(1,1);\\ (\nu,\sigma),(\lambda+1,\eta),(\beta,1),(1+\beta+\alpha,1),(\delta,p);\end{array}\ \frac{-y}{a}\right]. \tag{2.1}$$

Proof. In order to establish our main integral (2.1), we denote the left hand side of (2.1) by I and then by using (1.4), to get:

$$I = \int_0^\infty x^{\alpha-1}(x+a+\sqrt{x^2+2ax})^{-\beta}$$
$$\times\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}}{\Gamma(\eta n+\lambda+1)(\nu)_{\sigma n}(\delta)_{pn}}\left(\frac{-y}{x+a+\sqrt{x^2+2ax}}\right)^n dx. \tag{2.2}$$

Evaluating the above integral with the help of (1.16) and after little simplification, we found:

$$I = 2^{1-\alpha}a^{\alpha-\beta}\frac{\Gamma(2\alpha)\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma(\mu+\rho n)\Gamma(\gamma+qn)\Gamma(\beta+n-\alpha)\Gamma(\beta+n+1)\Gamma(n+1)}{\Gamma(\nu+\sigma n)\Gamma(\eta n+\lambda+1)\Gamma(\beta+n)\Gamma(1+\beta+n+\alpha)\Gamma(\delta+pn)n!}\left(\frac{-y}{a}\right)^n. \tag{2.3}$$

Which upon using (1.11) yields (2.1). This completes the proof of our main result.

Variation of (2.1): It the conditions of our main result be satisfied, then the following integral formula holds true:

$$\int_0^\infty x^{\alpha-1}(x+a+\sqrt{x^2+2ax})^{-\beta} J^{\mu,\rho,\gamma,q}_{\eta,\lambda,\nu,\sigma,\delta,p}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\alpha}a^{\alpha-\beta}\frac{\Gamma(2\alpha)\Gamma(\beta+1)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\lambda+1)\Gamma(1+\beta+\alpha)}$$
$$\times{}_{\rho+q+3}F_{\sigma+\eta+p+2}\left[\begin{array}{l}\Delta(\rho,\mu),\Delta(q,\gamma),\beta+1,\beta-\alpha,\ 1;\\ \Delta(\sigma,\nu),\Delta(p;\delta),\Delta(\eta,\lambda+1),\beta,1+\beta+\alpha;\end{array}\ \frac{-\rho^\rho q^q y}{\sigma^\sigma\eta^\eta p^p a}\right], \tag{2.4}$$

where $\Delta(m;l)$ abbreviates the array of m parameters $\frac{l}{m}, \frac{l+m}{m}, ..., \frac{l+m-1}{m}$, $m \geq 1$.

Proof: By writing the right hand side of (2.4) in the original summation and using the result

$$\Gamma(\alpha+n) = \Gamma(\alpha)(\alpha)_n$$

and

$$(l)_{kn} = k^{kn}(\frac{l}{k})_n(\frac{l+1}{k})_n...(\frac{l+k-1}{k})_n,$$

(Gauss multiplication theorem) in (2.3) and assuming up the given series with the help of (1.15), we easily arrive at our required result (2.4).

3. Special Cases

(i). On replacing λ by $\lambda - 1$ in (2.1) and then by using (1.5), we obtain:

$$\int_0^\infty x^{\alpha-1}(x+a+\sqrt{x^2+2ax})^{-\beta} E^{\mu,\rho,\gamma,q}_{\eta,\lambda,\nu,\sigma,\delta,p}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= 2^{1-\alpha}a^{\alpha-\beta}\frac{\Gamma(2\alpha)\Gamma(\nu)\Gamma(\delta)}{\Gamma(\mu)\Gamma(\gamma)}{}_5\Psi_5\left[\begin{array}{ll}(\mu,\rho),(\gamma,q),(\beta-\alpha,1),(\beta+1,1),(1,1); & \\ & \frac{y}{a}\\ (\nu,\sigma),(\lambda,\eta),(\beta,1),(1+\beta+\alpha,1),(\delta,p); & \end{array}\right], \quad (3.1)$$

where $\alpha,\beta,\gamma,\mu,\eta,\nu,\sigma,\rho,\delta,\lambda \in \mathbb{C}$, $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\mu) > 0, \Re(\nu) > 0,, \Re(\lambda) > 0, \Re(\delta) > 0, \Re(\sigma) > 0, \Re(\rho) > 0, \Re(\eta) > 0$, $p, q > 0$ and $q < \Re(\alpha) + p$.

(ii). On replacing λ by $\lambda - 1$ in (2.4) and then by using (1.5), we attain:

$$\int_0^\infty x^{\alpha-1}(x+a+\sqrt{x^2+2ax})^{-\beta} E^{\mu,\rho,\gamma,q}_{\eta,\lambda,\nu,\sigma,\delta,p}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= 2^{1-\alpha}a^{\alpha-\beta}\frac{\Gamma(2\alpha)\Gamma(\beta+1)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\lambda)\Gamma(1+\beta+\alpha)}$$

$$\times{}_{\rho+q+3}F_{\sigma+\eta+p+2}\left[\begin{array}{ll}\Delta(\rho,\mu),\Delta(q,\gamma),\beta+1,\beta-\alpha,\ 1; & \\ & \frac{\rho^\rho q^q y}{\sigma^\sigma \eta^\eta p^p a}\\ \Delta(\sigma,\nu),\Delta(p,\delta),\Delta(\eta,\lambda),\beta,1+\beta+\alpha; & \end{array}\right], \quad (3.2)$$

where $\alpha,\beta,\gamma,\mu,\eta,\nu,\sigma,\rho,\delta,\lambda \in \mathbb{C}$, $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \Re(\mu) > 0, \Re(\nu) > 0,, \Re(\lambda) > 0, \Re(\delta) > 0, \Re(\sigma) > 0, \Re(\rho) > 0, \Re(\eta) > 0$, $p, q > 0$ and $q < \Re(\alpha) + p$.

(iii). On setting $\mu = \nu = \rho = \sigma = 1$ and replacing λ by $\lambda - 1$ in (2.1) and then by using (1.6), we find:

$$\int_0^\infty x^{\alpha-1}(x+a+\sqrt{x^2+2ax})^{-\beta}E^{\gamma,\delta,q}_{\eta,\lambda,p}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$=2^{1-\alpha}a^{\alpha-\beta}\frac{\Gamma(2\alpha)\Gamma(\delta)}{\Gamma(\gamma)}{}_4\Psi_4\left[\begin{array}{ll}(\gamma,q),(\beta-\alpha,1),(\beta+1,1),(1,1); & \\ & \frac{y}{a}\\ (\lambda,\eta),(\beta,1),(1+\beta+\alpha,1),(\delta,p); & \end{array}\right], \tag{3.3}$$

where $\alpha,\beta,\gamma,\eta,\delta,\lambda\in\mathbb{C}$, $\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\ ,\Re(\lambda)>0,\Re(\delta)>0,\Re(\eta)>0$, $p,q>0$ and $q<\Re(\alpha)+p$.

(iv). On setting $\mu=\nu=\rho=\sigma=1$ and replacing λ by $\lambda-1$ in (2.4) and then by using (1.6), we get:

$$\int_0^\infty x^{\alpha-1}(x+a+\sqrt{x^2+2ax})^{-\beta}E^{\gamma,\delta,q}_{\eta,\lambda,p}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$=2^{1-\alpha}a^{\alpha-\beta}\frac{\Gamma(2\alpha)\Gamma(\beta+1)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\lambda)\Gamma(1+\beta+\alpha)}$$
$$\times{}_{q+3}F_{\eta+p+2}\left[\begin{array}{ll}\Delta(q,\gamma),\beta+1,\beta-\alpha,\ 1; & \\ & \frac{q^q y}{\eta^\eta p^p a}\\ \Delta(p,\delta),\Delta(\eta,\lambda),\beta,1+\beta+\alpha; & \end{array}\right], \tag{3.4}$$

where $\alpha,\beta,\gamma,\eta,\delta,\lambda\in\mathbb{C}$, $\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\ ,\Re(\lambda)>0,\Re(\delta)>0,\Re(\eta)>0$, $p,q>0$ and $q<\Re(\alpha)+p$.

(v). On setting $\mu=\nu=\rho=\sigma=\delta=p=1$ and replacing λ by $\lambda-1$ in (2.1) and then by using (1.7), we find:

$$\int_0^\infty x^{\alpha-1}(x+a+\sqrt{x^2+2ax})^{-\beta}E^{\gamma,q}_{\eta,\lambda}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$=2^{1-\alpha}a^{\alpha-\beta}\frac{\Gamma(2\alpha)}{\Gamma(\gamma)}{}_4\Psi_3\left[\begin{array}{ll}(\gamma,q),(\beta-\alpha,1),(\beta+1,1),(1,1); & \\ & \frac{y}{a}\\ (\beta,1),(1+\beta+\alpha,1),(\lambda,\eta); & \end{array}\right], \tag{3.5}$$

where $\alpha,\beta,\gamma,\eta,\lambda\in\mathbb{C}$, $\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\ ,\Re(\lambda)>0,\Re(\eta)>0$, $q\in(0,1)\bigcup\mathbb{N}$.

(vi). On setting $\mu=\nu=\rho=\sigma=\delta=p=1$ and replacing λ by $\lambda-1$ in (2.4) and then by using (1.7), we acquire:

$$\int_0^\infty x^{\alpha-1}(x+a+\sqrt{x^2+2ax})^{-\beta}E^{\gamma,q}_{\eta,\lambda}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$=2^{1-\alpha}a^{\alpha-\beta}\frac{\Gamma(2\alpha)\Gamma(\beta+1)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\lambda)\Gamma(1+\beta+\alpha)}$$
$$\times{}_{q+3}F_{\eta+2}\left[\begin{array}{ll}\Delta(q;\gamma),\beta+1,\beta-\alpha,\ 1; & \\ & \frac{q^q y}{\eta^\eta a}\\ \Delta(\eta,\lambda),\beta,1+\beta+\alpha; & \end{array}\right], \tag{3.6}$$

where $\alpha,\beta,\gamma,\eta,\lambda\in\mathbb{C}$, $\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\ ,\Re(\lambda)>0,\Re(\eta)>0$, $q\in(0,1)\bigcup\mathbb{N}$.

(vii). On setting $\mu=\nu=\rho=\sigma=\delta=p=q=1$ and replacing λ by $\lambda-1$ in (2.1) and then by using (1.8), we obtain:

$$\int_0^\infty x^{\alpha-1}(x+a+\sqrt{x^2+2ax})^{-\beta}E^{\gamma}_{\eta,\lambda}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=2^{1-\alpha}a^{\alpha-\beta}\frac{\Gamma(2\alpha)}{\Gamma(\gamma)}{}_4\Psi_3\left[\begin{array}{c}(\gamma,1),(\beta-\alpha,1),(\beta+1,1),(1,1);\\ \\(\beta,1),(1+\beta+\alpha,1),(\lambda,\eta);\end{array}\ \frac{y}{a}\right],\tag{3.7}$$

where $\alpha,\beta,\gamma,\eta,\lambda\in\mathbb{C}$, $\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\ ,\Re(\lambda)>0,\Re(\eta)>0$.

(viii). On setting $\mu=\nu=\rho=\sigma=\delta=p=q=1$ and replacing ν by $\nu-1$ in (2.1) and then by using (1.7), we get:

$$\int_0^\infty x^{\alpha-1}(x+a+\sqrt{x^2+2ax})^{-\beta}E^{\gamma}_{\eta,\lambda}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=2^{1-\alpha}a^{\alpha-\beta}\frac{\Gamma(2\alpha)\Gamma(\beta+1)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\lambda)\Gamma(1+\beta+\alpha)}$$

$$\times{}_4F_{\eta+2}\left[\begin{array}{c}\gamma,\beta+1,\beta-\alpha,\ 1;\\ \\ \Delta(\eta,\lambda),1+\beta+\alpha,\beta;\end{array}\ \frac{y}{\eta^{\eta}a}\right],\tag{3.8}$$

where $\alpha,\beta,\gamma,\eta,\lambda\in\mathbb{C}$, $\Re(\alpha)>0,\Re(\beta)>0,\Re(\gamma)>0,\ ,\Re(\lambda)>0,\Re(\eta)>0$.

(ix). On setting $\mu=\nu=\rho=\eta=p=q=\delta=\gamma=1$ in (2.1) and then by using (1.8), we attain:

$$\int_0^\infty x^{\alpha-1}(x+a+\sqrt{x^2+2ax})^{-\beta}E_{\eta,\lambda}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=2^{1-\alpha}a^{\alpha-\beta}\Gamma(2\alpha){}_3\Psi_3\left[\begin{array}{c}(\beta-\alpha,1),(\beta+1,1),(1,1);\\ \\(\beta,1),(1+\beta+\alpha,1),(\lambda,\eta);\end{array}\ \frac{y}{a}\right],\tag{3.9}$$

where $\alpha,\beta,\eta,\lambda\in\mathbb{C}$, $\Re(\alpha)>0,\Re(\beta)>0,\ ,\Re(\lambda)>0,\Re(\eta)>0$.

(x). On setting $\mu=\nu=\rho=\eta=p=q=\delta=\gamma=1$ in (2.1) and then by using (1.8), we obtain:

$$\int_0^\infty x^{\alpha-1}(x+a+\sqrt{x^2+2ax})^{-\beta}E_{\eta,\lambda}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$=2^{1-\alpha}a^{\alpha-\beta}\frac{\Gamma(2\alpha)\Gamma(\beta+1)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\lambda)\Gamma(1+\beta+\alpha)}$$

$$\times {}_3F_{\eta+2}\left[\begin{array}{ll}\beta+1, \beta-\alpha,\ 1; & \\ & \frac{y}{\eta^{\eta}a} \\ \Delta(\eta,\lambda), 1+\beta+\alpha, \beta; & \end{array}\right], \tag{3.10}$$

where $\alpha, \beta, \eta, \lambda \in \mathbb{C}$, $\Re(\alpha) > 0, \Re(\beta) > 0,\ , \Re(\lambda) > 0, \Re(\eta) > 0$.

References

[1] Choi, J, Agarwal, P, Mathur, S and Purohit, S.D, Certain new integral formulas involving the generalized Bessel functions, Bull. Korean Math. Soc., **51**4(2014), 995-1003.

[2] Choi, J, Agarwal, P, Certain unified integrals involving a product of Bessel functions of first kind, Honam Mathematical J., **35**4(2013), 667-677.

[3] Choi, J, Agarwal, P, Certain unified integrals associated with Bessel functions, Boundary Value Problem, Vol 2013,95, 2013.

[4] Ghayasuddin, M, Khan, W. A, A new extension of Bessel Maitland function and its properties, Matematicki Vesnik, (2018), 1-11.

[5] Fox, C, The G and H-functions and symmetrical Fourier kernels, Trans. Amer. Math. Soc., 98(1961), 395-429.

[6] Khan, W. A, Nisar, K S, Unified integral operator involving generalized Bessel-Maitland function, Proc. Jangjeon Mathe. Soc. 21(2018), 339 - 346.

[7] Khan, W. A, Nisar, K S, Beta type integral formula associated with Wright generalized Bessel function, Acta Math. Univ. Comenianae, 1(2018), 117-125.

[8] Khan, W. A, Nisar, K S, Choi, J, An integral formula of the Mellin transform type involving the extended Wright Bessel function, Far East J. Math., 102(2017), 2903-2912.

[9] Khan, W. A, Nisar, K S, Ahmad, M, On certain integral transforms involving generalized Bessel-Maitland function, Submitted.

[10] Khan, M. A, Ahmed, S, On some properties of the generalized Mittag-Leffler function, Springer Plus 2013,2:337.

[11] Mittag-Leffler, G.M, Sur la nouvelle fonction $E_\alpha(x)$, C. R. Acad. Sci. Paris, 137(1903), 554-558.

[12] Marichev, O. I, Handbook of integral transform and Higher transcendental functions. Theory and algorithm tables, Ellis Horwood, Chichester[John Wiley and Sons], New York, 1983.

[13] Oberhettinger, F, Tables of Mellin transforms, Springer, New York, 1974.

[14] Prabhakar, T. R, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J., 19(1971), 7?15.

[15] Rainville, E. D, Special functions, Macmillan, New York, 1960.

[16] Samko, S. G, Kilbas, A.A and Marichev, O.I, Fractional Integrals and derivatives. Theory and Applications, Gordan and Breach, Yverdon et al., 1993.

[17] Salim, T. O and Faraj, W, A generalization of Mittag-Leffler function and integral operator associated with fractional calculus, J. Frac. Calc. Appl., 3(5)(2012), 1-13.

[18] Shukla, A. K and Prajapati, J.C, On a generalization of Mittag-Leffler function and its properties, J. Math. Anal. Appl., 336(2007), 797-811.

[19] Singh, M, Khan, M. A, Khan, A. H, On some properties of a generalization of Bessel-Maitland function, Int. J. Math. Trends and Tech., 14(1) (2014), 46-54.

[20] Watson, G. N, A treatise on the theory of Bessel functions, Cambridge University Press, 1962.

[21] Wiman, A, Uber den fundamental Satz in der Theorie der Funktionen $E_\alpha(x)$, Acta Math., 29(1905), 191-201.

[22] Wright, E. M, On the coefficients of power series having exponential singularities., Proc. London Math. Soc., 8(1933), 71-79.

[23] Wright, E. M, The asymptotic expansion of the generalized hypergeometric function, J. London Math. Soc., 10(1935), 286-293.

Musharraf Ali

Department of Mathematics, G.F. College, Shahjahanpur-242001, India

E-mail address: drmusharrafali@gmail.com

Waseem A. Khan
Department of Mathematics, Faculty of Science, Integral University, Lucknow-226026, India
E-mail address: waseem08_khan@rediffmail.com

Idrees A. Khan
Department of Mathematics, Faculty of Science, Integral University, Lucknow-226026, India
E-mail address: khanidrees077@gmail.com