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SOLUTION SETS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. The aim of this paper is to study an initial value problem for a fractional differential inclusions using the Riemann-Liouville fractional derivative. We apply appropriate fixed point theorems for multivalued maps to obtain the existence results for the given problems covering convex as well as non-convex cases for multivalued maps. We also obtain some topological properties of the solution sets.

1. Introduction

In this paper, we are concerned with the solutions sets for the initial value problems (IVP for short), for fractional order differential inclusions of the form

$$D^{\alpha}y(t) \in F(t, y(t)), \text{ a.e. } t \in J' = (0, T], \quad 0 < \alpha \leq 1, \tag{1}$$

$$\lim_{t \longrightarrow 0^+} t^{1-\alpha}y(t) = c, \tag{2}$$

where D^{α} is the standard Riemann-Liouville fractional derivative, $J = [0, T]$, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a Carathéodory multivalued function ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$) and $c \in \mathbb{R}$.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. see ([12]). There has been a significant development in fractional differential and partial differential equations in recent years; see the monographs of Kilbas *et al* [22], Benchohra *et al*[1, 2], Zhou *al* [32, 33] and the papers of Delbosco and Rodino [13], Diethelm *et al* [12], El-Sayed [15, 16], Kilbas and Marzan [20] and the references therein.

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $y(0)$, $y'(0)$, etc.

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the same requirements of boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types see [27].

In 1890, Peano [26] proved that the Cauchy problem for ordinary differential equations has local solutions although the uniqueness property does not hold in general. For the case where the uniqueness does not hold, Kneser [21] proved in 1923 that the solution set is a continuum, i.e. closed and connected. In 1942, Aronszajn [5] improved this result for differential inclusions in the sense that he showed that the solution set is compact and acyclic, and he specified this continuum to be an R_δ-set. An analogous result has been obtained for differential inclusions with u.s.c. convex valued nonlinearities by De Blasi and Myjak in 1985 (see [20]). In the case of differential inclusions on unbounded domains, some existence results together with topological structures of solution sets have also been obtained in [4], the monograph of Dragoni *et al* [14] are an excellent references to study the properties of structure topological of this kind of inclusions.

Let us also mention some study of fractional differential inclusions. It was started in early 2000's see for instance [9, 16]. Note that we improve some recent results in this topic (see [33], for instance.)

We are focused in the continuity of the state y only on J' and the existence of the above value without nullity, hence this hardness impose us a choice of a special Banach space $\mathcal{C}_\alpha([0,T],\mathbb{R})$ that will be specified later. We show that this constructed space is in a natural way, in the sense that, one recover the characterization of the relatively compact subset in the space $C(J,R)$ when J is compact.

The paper is organized as follows, in section 2 we give some general results and preliminaries, in section 3 we give the first result when the nonlinearity is upper semi-continuous and takes convex values, we prove then that solution sets is nonempty and compact, our main tool is the Leray-Schauder alternative, in section 4 we give the second one when the nonlinearity takes non-convex values, by usefulness of Covitz Nadler contraction we prove that (1)-(2) has one solution, the compactness, contractibility and acyclicity of solution sets is also proved and in the last section we give an example which illustrate our results.

2. Preliminaries

This section presents the notations and definitions used throughout this paper, and give some preliminary facts from multivalued analysis. Let $[a,b]$ be an interval in $\mathbb{R}$, $C([a,b],\mathbb{R})$ the Banach space of all continuous functions from $[a,b]$ into $\mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in [a,b]\},$$

and $L^1([a,b],\mathbb{R})$ the Banach space of all functions $y : [a,b] \to \mathbb{R}$ which are Lebesgue integrable with the norm

$$\|y\|_{L^1} = \int_a^b |y(t)|dt.$$

Let X be a metric space. Define $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$. Consider the

Hausdorff pseudo-metric distance

$$H_d : \mathcal{P}(X) \to \mathbb{R}^+ \cup \{\infty\},$$

defined by

$$H_d(A, B) := \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\},$$

where $d(a, B) = \inf_{b\in B} d(a, b)$ and $d(A, b) = \inf_{a\in A} d(a, b)$. From this definitions, it's clear that $(\mathcal{P}_{cl,b}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space.

Definition 2.1. *A multivalued map $N : X \to \mathcal{P}(X)$ is called*
(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(N(x), N(y)) \leq \gamma d(x, y) \ \ \forall x, \ \ y \in X,$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Notice that if N is γ-Lipschitz and X is a Banach space, then for every $\gamma' > \gamma$

$$N(x) \subset N(y) + \gamma' d(x, y)B(0, 1),$$

where $B(0, 1)$ refers to the unit ball in X.

Lemma 2.1. [10] *Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Let X, Y be two metric spaces. We denote by $P(Y)$ the family of all nonempty subsets of Y and by $K(X)$ (resp. $K_v(X)$) we denote the collection of all nonempty compact (resp. nonempty compact convex) subsets of X.
A multivalued map $F : X \to P(Y)$ is said to be

- (i) upper semi-continuous (u.s.c. for short)if the set $F^{-1}(V) = \{x \in X, \ F(x) \subset V\}$ is open subset of X for any open $V \subset Y$,
- (ii) closed (resp. convex) if its graph $\Gamma_F = \{(x, y) \in X \times Y : y \in F(x)\}$ is a closed (resp. convex) subset of $X \times Y$,
- (iii) F is bounded on bounded sets if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in X for all $B \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in B}\{\sup\{\|y\| : y \in F(x)\}\} < \infty$).

For each $x \in X$, define the set of selections of F by

$$S_{F,x} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, x(t)) \text{ a.e. } t \in J\}.$$

A multifunction $F : [0; T] \to K(X)$ is said to be:

- (i) strongly measurable if there exists a sequence $\{F_n\}_{n=1}^{+\infty}$ of step multifunctions such that $H_d(F_n(t) - F(t)) \to 0$ as $n \to +\infty$ for $\mu - a.e.\ t \in [0, T]$,

where μ denotes a Lebesgue measure on $[0, T]$ and H_d is the Hausdorff metric on $K(X)$.

By the symbol $L^1([0, T], X)$ we denote the space of all Bochner summable functions,

- (ii) integrable provided it has a Bochner summable selection $f \in L^1([0, T], X)$, i.e. $f(t) \in F(t)$ for a.e. $t \in [0, T]$,
- (iii) integrably bounded if there exists a summable function $q(.) \in L^1([0, T], X)$ such that $\|F(t)\| = \sup\{\|y\| : y \in F(t)\} \leq q(t)$ for a.e. $t \in [0, T]$.

Every strongly measurable multivalued map F admits a strongly measurable selection $f : [0,T] \to E$, i.e.,

$$f(t) \in F(t) \text{ for } a.e.\ t \in [0,T].$$

When the nonlinearity takes convex values, Mazur's Lemma may be useful:

Lemma 2.2. *Let X be a normed space and $(x_k)_{k\in\mathbb{N}} \subset X$ a sequence weakly converging to a limit $x \in E$. Then there exists a sequence of convex combinations $y_m = \sum_{k=1}^{k=m} \alpha_{mk} x_k$ with $\alpha_{mk} \geq 0$ for $k = 1, \ldots, m$ and $\sum_{k=0}^{k=m} \alpha_{mk} = 1$ which converges strongly to x.*

Lemma 2.3. [11] *If $F : X \to \mathcal{P}_{cl}(Y)$ is u.s.c. then $\mathfrak{gr}(F)$ is a closed subset of $X \times Y$, i.e. for every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ and $(y_n)_{n\in\mathbb{N}} \subset Y$, if when $n \longrightarrow \infty$, $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$ and $y_n \in F(x_n)$, then $y_* \in F(x_*)$. Conversely, if F is completely continuous and has a closed graph, then it is u.s.c.*

Finally, the following results are easily deduced from the theoretical limit set properties.

Lemma 2.4. [7] *Let $(K_n)_{n\in\mathbb{N}} \subset K \subset X$ be a sequence of subsets where K is a compact subset of a separable Banach space X. Then*

$$\overline{co}(\lim_{n\longrightarrow\infty} \sup K_n) = \bigcap_{N>0} \overline{co}(\bigcup_{n\geq N} K_N),$$

where $\overline{co}A$ refers to the closure of the convex hull of A.

Lemma 2.5. [7] *Let X, Y be two metric spaces.*
If $F : X \to \mathcal{P}_{cp}(Y)$ is u.s.c. then for each $x_0 \in X$,

$$\lim_{x\longrightarrow x_0} \sup F(x) = F(x_0).$$

We end these ingredients of multivalued analysis with some definitions and a result regarding the measurability of multivalued maps.

Lemma 2.6. [17] *Let $(\Sigma, \mathcal{A})$ be a measurable space, (X, d) a separable, complete metric space (Polish space) and $F : \Sigma \to \mathcal{P}(X)$ a multivalued map with nonempty closed values. If F is measurable, then it has a measurable selection.*

Definition 2.2. *A multivalued map $G : J\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if*

(a) *$t \to G(t,u)$ is measurable for each $u \in \mathbb{R}$,*
(b) *$t \to G(t,u)$ is upper semi-continuous for almost all $t \in J$,*
(c) *for each $q > 0$, there exists $\varphi_q \in L^1(J, \mathbb{R}_+)$ such that*

$$\|G(t,u)\| = \sup\{|v| : v \in G(t,u)\} \leq \varphi_q(t) \text{ for all } |u| < q \text{ and for a.e. } t \in J.$$

Lemma 2.7. [23] *Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ be an L^1-Carathéodory multivalued map and let Γ be a linear continuous mapping from $L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$, then the operator*

$$\begin{aligned} \Gamma \circ S_F \quad : \quad & C(J,\mathbb{R}) \to \mathcal{P}_{pc,c}(C(J,\mathbb{R})) \\ & x \longmapsto \Gamma \circ S_{F,x} \end{aligned}$$

is a closed graph operator in $C(J,\mathbb{R}) \times C(J,\mathbb{R})$.

For further reading and details on multivalued analysis, we refer the reader to the books of Andres and Górniewicz [3], Aubin and Cellina [6], Aubin and Frankowska [7], Deimling [11], Górniewicz [17], Kamenskii *et al* [19], Hu and Papageorgiou [28, 29].

We begin with some elementary notions from geometric topology. For details, we recommend [3, 24]. Let X be a Banach space and $\mathcal{P}_{cv,cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex, closed}\}$.

Definition 2.3. *Let $A \in \mathcal{P}(X)$, the set A is called a contractible space provided there exists a continuous homotopy $H : A \times [0,1] \to A$ and $x_0 \in A$ such that*

(a) $H(x,0) = x,$ *for all* $x \in A$,
(b) $H(x,1) = x_0,$ *for all* $x \in A$,

i.e. if the identity map is homotopic to a constant map (A is homotopically equivalent to a point).

Note that if $A \in \mathcal{P}_{cv,cl}(X)$, then A is contractible, but the class of contractible sets is much larger than the class of closed convex sets.

We begin with some definitions from the theory of fractional calculus.

Definition 2.4. [15, 16]. *The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by:*

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = h(t) * \varphi_\alpha(t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2.5. [16]. *For a function h given on the interval $[0,b]$, the Riemann-Liouville fractional derivative of h of order $\alpha \in \mathbb{R}_+$ is defined by:*

$$D_{0^+}^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left(\int_0^t (t-s)^{n-\alpha-1} h(s)ds \right).$$

Lemma 2.8. *Let $v : [0,b] \to [0,+\infty)$ be a real function and $w(.)$ is a nonnegative, locally integrable function on $[0,b]$. Assume that there are constants $a > 0$ and $0 < \beta < 1$ such that*

$$v(t) \leq w(t) + a \int_0^t \frac{v(s)}{(t-s)^\beta} ds,$$

then there exists a constant $K = K(\beta)$ such that

$$v(t) \leq w(t) + ka \int_0^t \frac{w(s)}{(t-s)^\beta} ds \text{ for every } t \in (0,b].$$

3. Existence Results

3.1. **The upper semi-continuous case.**

In this section, we present a global existence result and prove the compactness of the solution set for problem $(1)-(2)$ by using a nonlinear alternative for multivalued maps combined with a compactness argument.

Consider the Banach space

$$\mathcal{C}_\alpha([0,T],\mathbb{R}) := \{y \in C((0,T],\mathbb{R}) : \lim_{t\longrightarrow 0} t^{1-\alpha}y(t) \text{ exists } \}.$$

Endowed with the norm

$$\|y\|_\alpha := \sup\{t^{1-\alpha}|y(t)| : t \in [0,T]\},$$

$\mathcal{C}_\alpha$ is a Banach space. For A a subset of the space $\mathcal{C}_\alpha([0,T],R)$, define A_α by $A_\alpha := \{y_\alpha : y \in A\}$, where

$$y_\alpha(t) := \begin{cases} t^{1-\alpha}y(t), & t \in (0,T], \\ \lim\limits_{t\longrightarrow 0} t^{1-\alpha}y(t), & t = 0. \end{cases}$$

Lemma 3.1. *Let A be a bounded set in $\mathcal{C}_\alpha([0,T],\mathbb{R})$. Assume that A_α is equicontinuous on $C([0,T],\mathbb{R})$. Then A is relatively compact in $\mathcal{C}_\alpha([0,T],\mathbb{R})$.*

Proof. Let $\{y_n\}_{n=1}^\infty \subset A$, then $\{(y_\alpha)_n\}_{n=1}^\infty \subset C([0,T],R)$. From Arzelá-Ascoli theorem, the set $K_0 = \{(y_\alpha)_n : n \in \mathbb{N}^*\}$ is relatively compact in $C([0,T],\mathbb{R})$, thus there exists a subsequence of $\{(y_\alpha)_n\}_{n\in\mathbb{N}}$, still denoted by $\{(y_\alpha)_n\}_{n=1}^\infty$, which converges to y where $y \in (C([0,T],\mathbb{R}), \|.\|_\infty)$.
Hence

$$\|(y_\alpha)_n - y\|_\alpha = \sup\{|t^{1-\alpha}y_{\alpha n}(t) - y(t)|, t \in [0,T]\} \longrightarrow 0.$$

Therefore

$$\{y_n\}_{n=1}^\infty \longrightarrow y \text{ on } \mathcal{C}_\alpha([0,T],\mathbb{R}).$$

□

Let us define what we mean by a solution of problem $(1)-(2)$.

Definition 3.1. *A function $y \in \mathcal{C}_\alpha$ is said to be a solution of problem $(1)-(2)$ if there exists $v \in L^1(J,\mathbb{R})$ such that $v(t) \in F(t,y(t))$ a.e. $t \in J$ satisfies the differential inclusion $D^\alpha y(t) \in F(t,y(t))$ on J' and condition*

$$\lim_{t\longrightarrow 0} t^{1-\alpha}y(t) = c,$$

is satisfied.

For the existence of solutions for the problem (1)-(2), we need the following auxiliary lemmas.

Lemma 3.2. [31] *Let $\alpha > 0$, then the differential equation*

$$^{RL}D^\alpha_{a^+}h(t) = 0,$$

has solutions $h(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \ldots + c_n(t-a)^{\alpha-n}$ for some $c_i \in R$, $i = 1\ldots n$, where $n = [\alpha]+1$.

Lemma 3.3. [31] *Let $\alpha > 0$, then*

$$I^{\alpha RL}D^\alpha_{a^+}h(t) = h(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \ldots + c_n(t-a)^{\alpha-n}$$

for some $c_i \in R$, $i = 0,\ldots,n$, where $n = [\alpha]+1$.

As a consequence of Lemma (3.2) and Lemma (3.3) we have the following result which is useful in what follows.

Lemma 3.4. *Let $0 < \alpha \leq 1$ and let $\rho \in \mathcal{C}_\alpha$. A function y is a solution of the fractional integral equation*

$$y(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\rho(s)ds, \text{ a.e. } t \in J,$$

if and only if y is a solution of the fractional initial-value problem

$$D^\alpha y(t) = \rho(t) \text{ for each } t \in J',$$

$$\lim_{t\longrightarrow 0} t^{1-\alpha}y(t) = c.$$

Our first result is based on the nonlinear alternative of Leray-Schauder type for multivalued maps [18]. We assume that F is a compact and convex valued multivalued map which satisfies following hypotheses:

(H1) $F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is a Carathéodory multi-valued map,
(H2) there exist nonnegative constants $a,\ b \in R$ such that

$$\|F(t,x)\|_{\mathcal{P}} \leq a|x| + b, \text{ for a.e. } t \in J \text{ and each } x \in \mathbb{R}.$$

Theorem 3.1. *Under Assumptions $(H_1)-(H_2)$, the initial-value problem $(1)-(2)$ has at least one solution. Moreover, the solution set $S_F(c)$ is compact.*

Consider the operator $N : \mathcal{C}_\alpha \to \mathcal{P}(\mathcal{C}_\alpha)$ defined for $y \in \mathcal{C}_\alpha$ by

$$N(y) := \{h \in \mathcal{C}_\alpha : h(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds, \text{ a.e. } t \in J\},$$

where $v \in S_{F,y} = \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,y(t)) \text{ a.e. } t \in J\}$.

Note that from [30, Theorem 5.10], the set $S_{F,y}$ is nonempty if and only if the mapping $t \to \inf\{\|v\| : v \in F(t,y(t))\}$ belongs to $L^1(J)$.

It is further bounded if and only if the mapping

$$t \to \|F(t,y(t))\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t,y(t))\}$$

belongs to $L^1(J)$, this particularly holds true when F satisfies (H_1).

Clearly, from Lemma (3.4) N fixed points are solutions to $(1)-(2)$. We shall prove that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type [18]. The proof will be given in several steps.

The proof is given in several steps,

Step 1:
$N(y)$ is convex for each $y \in \mathcal{C}_\alpha$.

Indeed, if $h_1,\ h_2$ belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$h_i(t) \quad = \quad t^{\alpha-1}c + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_i(s)ds, \quad i = 1, 2.$$

Let $0 \leq d \leq 1$. Then, for each $t \in J$, we have

$$\begin{aligned}(dh_1 + (1-d)h_2)(t) &= t^{\alpha-1}c \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[dv_1(s) + (1-d)v_2(s)]ds.\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have

$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2:
N maps bounded sets into bounded sets in $\mathcal{C}_\alpha(J,\mathbb{R})$.

Indeed, it is enough to show that there exists a positive constant l such that for each $y \in B_r = \{y \in \mathcal{C}_\alpha(J,\mathbb{R}) : \|y\|_\alpha \leq r\}$ one has $\|N(y)\|_\alpha \leq l$. Let $y \in B_r$. Then for each $h \in N(y)$, there exists $v \in S_{F,y}$ such that

$$h(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds.$$

By (H_2) we have

$$\begin{aligned}|t^{1-\alpha}h(t)| &\leq |c| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v(s)|ds \\ &\leq |c| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(a|y(s)|+b)ds \\ &\leq |c| + \frac{at^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}|s^{1-\alpha}y(s)|ds + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}bds \\ &\leq |c| + \frac{raT^\alpha\Gamma(\alpha)}{\Gamma(2\alpha)} + \frac{bT}{\Gamma(1+\alpha)} = l.\end{aligned}$$

Step 3: *N maps bounded sets into equicontinuous sets of $\mathcal{C}_\alpha([0,T],\mathbb{R})$.*
Let $\tau_1, \tau_2 \in (0,T]$, $\tau_1 < \tau_2$ and B_r be a bounded set of $\mathcal{C}_\alpha([0,T],\mathbb{R})$ as Claim 1, let $y \in B_r$ and $h \in N(y)$,
then

$$\begin{aligned}&|\tau_2^{1-\alpha}h(\tau_2) - \tau_1^{1-\alpha}h(\tau_1)| \\ \leq\; & \frac{|(\tau_2^{1-\alpha}-\tau_1^{1-\alpha}|}{\Gamma(\alpha)}\int_0^{\tau_1}|(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}||v(s)|ds \\ +\; & \frac{\tau_2^{1-\alpha}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}|v(s)|ds, \\ \leq\; & \frac{(\tau_2^{1-\alpha}-\tau_1^{1-\alpha})}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_2-s)^{\alpha-1}s^{\alpha-1}ar\,ds + \frac{(\tau_2^{1-\alpha}-\tau_1^{1-\alpha})}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_2-s)^{\alpha-1}b\,ds \\ +\; & \frac{(\tau_2^{1-\alpha}-\tau_1^{1-\alpha})}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_1-s)^{\alpha-1}s^{\alpha-1}ar\,ds + \frac{(\tau_2^{1-\alpha}-\tau_1^{1-\alpha})}{\Gamma(\alpha)}\int_0^{\tau_1}(\tau_1-s)^{\alpha-1}b\,ds \\ +\; & \frac{\tau_2^{1-\alpha}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}s^{\alpha-1}ar\,ds + \frac{\tau_2^{1-\alpha}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}b\,ds,\end{aligned}$$

which yields

$$\begin{aligned}|\tau_2^{1-\alpha}N(y)(\tau_2) - \tau_1^{1-\alpha}N(y)(\tau_1)| &\leq \frac{ra(\tau_2^{2\alpha-1}+\tau_1^{2\alpha-1})\mathcal{B}(\alpha,\alpha)}{\Gamma(\alpha)}(\tau_2^{1-\alpha} - \tau_1^{1-\alpha}) \\ &+ \frac{b[(\tau_2^\alpha+\tau_1^\alpha)-(\tau_2-\tau_1)^\alpha]}{\Gamma(1+\alpha)}(\tau_2^{1-\alpha} - \tau_1^{1-\alpha}) \\ &+ [\frac{ra\tau_2^{1-\alpha}\tau_1^{\alpha-1}}{\Gamma(1+\alpha)} + \frac{b\tau_2^{1-\alpha}}{\Gamma(1+\alpha)}](\tau_2-\tau_1)^\alpha.\end{aligned}$$

As $\tau_2 \longrightarrow \tau_1$ the right-hand side of the above inequality tends to zero. Then $N(B_r)$ is equicontinuous.

As a consequence of Steps 1 to 3 together with lemma (3.1), we can conclude that $N : \mathcal{C}_\alpha \to \mathcal{P}(\mathcal{C}_\alpha)$ is completely continuous.

Step 4: *N is u.s.c.*

To this end, it is sufficient to show that N has a closed graph. Let $h_n \in N(y_n)$ be such that $h_n \longrightarrow h$ and $y_n \longrightarrow y$ as $n \longrightarrow +\infty$.

Then there exists $M > 0$ such that $\|y_n\|_\alpha \leq M$. We shall prove that $h \in N(y)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that, for a.e. $t \in J$, we have

$$h_n(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s)ds,$$

(H_2) implies that $v_n(t) \in at^{\alpha-1}M + bB(0,1)$. Then $(v_n)_{n\in\mathbb{N}}$ is integrably bounded in $L^1(J,\mathbb{R})$. It follows that $(v_n)_{n\in\mathbb{N}}$ is weakly compact . There exists a subsequence still denoted $(v_n)_{n\in\mathbb{N}}$, which converges weakly to some limit $v \in L^1(J,\mathbb{R})$. Furthermore, the mapping $\Gamma : L^1(J,\mathbb{R}) \to \mathcal{C}_\alpha(J,\mathbb{R})$ defined by

$$\Gamma(g)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds$$

is a continuous linear operator.

Then it remains continuous if these spaces are endowed with their weak topologies [25]. Moreover, for a.e. $t \in J$ we have

$$h(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds.$$

It remains to prove that $v \in F(t, y(t))$, a.e. $t \in J$. Mazur's Lemma (2.2) yields the existence of $\alpha_i^n \geq 0$, $i = 1, \ldots, k(n)$ such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinations $g_n(.) = \sum_{i=1}^{k(n)} \alpha_i^n v_i(.)$ converges strongly to v in L^1 Using Lemma (2.4), we obtain that

$$\begin{array}{rcl} v(t) & \in & \bigcap_{n\geq 1} \overline{\{g_n(t)\}} \text{ a.e. } t \in J \\ & \subset & \bigcap_{n\geq 1} \overline{co}\{v_k(t),\ k \geq n\} \\ & \subset & \bigcap_{n\geq 1} \overline{co}\left\{\bigcup_{k\geq n} F(t, y_k(t))\right\} \\ & = & \overline{co}\left\{\limsup F(t, y_k(t)))\right\}. \end{array} \tag{3}$$

However, the fact that the multivalued $x \to F(., x)$ is u.s.c. and has compact values, together with Lemma (2.5), implies that $\lim_{n\longrightarrow\infty} \sup F(t, y_n(t)) = F(t, y(t))$, a.e. $t \in J$, combining with (3) yields that $v(t) \in \overline{co}F(t, y(t))$, from the convexity and closedness of F it follows that $v(t) \in F(t, y(t))$, a.e. $t \in J$. Thus $h \in N(y)$, proving that N has a closed graph. Finally, with Lemma (2.3) and the compactness of N, we conclude that N is u.s.c.

Step 5 : *A priori bounds on solutions.*

Let $y \in \mathcal{C}_\alpha(J,\mathbb{R})$ be such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. Then there exists $v \in L^1(J,\mathbb{R})$ with $v \in S_{F,y}$ such that, for each $t \in J$.

From (H_2) we have

$$\begin{array}{rcl} |t^{1-\alpha}y(t)| & \leq & |c| + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|v(s)|ds \\ & \leq & |c| + \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(a|y(s)| + b)ds \\ & \leq & |c| + \frac{bT}{\Gamma(1+\alpha)} + \frac{aT^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}s^{\alpha-1}|s^{1-\alpha}y(s)|ds. \end{array}$$

From Lemma (2.8) there exists $k(\alpha) > 0$ such that

$$|t^{1-\alpha}y(t)| \leq L + \frac{ak(\alpha)T^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Lds,$$

where $L = |c| + \frac{bT}{\Gamma(1+\alpha)}$. Therefore

$$\|y\|_\alpha \leq L + \frac{ak(\alpha)T}{\Gamma(1+\alpha)} = \widetilde{M}.$$

Let

$$U := \{y \in \mathcal{C}_\alpha([0,T],\mathbb{R}) : \|y(t)\|_{\Omega_c} < \widetilde{M} + 1\},$$

and consider the operator $N : U \longrightarrow \mathcal{P}_{cv,cp}(\mathcal{C}_\alpha)$. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$.

As a consequence of the nonlinear alternative of Leray-Schauder type [18], we deduce that N has a fixed point y in $\overline{U}$ which is a solution of the problem (1)-(2).

3.2. Compactness of the solution set.

For each $c \in \mathbb{R}$, let $S(F,c) := \{y \in \mathcal{C}_\alpha(J,\mathbb{R}) : y$ is a solution of problem (1)-(2) $\}$. From the previous consideration, there exists $\widetilde{M}$ such that for every $y \in S(F,c)$, $\|y\|_\alpha \leq \widetilde{M}$. Since N is completely continuous, $N(S_F(c))$ is relatively compact in $\mathcal{C}_\alpha$. Let $y \in S(F,c)$, then $y \in N(y)$ and hence $S(F,c) \subset N(S(F,c))$. It remains to prove that $S(F,c)$ is a closed subset in $\mathcal{C}_\alpha$. Let $\{y_n : n \in \mathbb{N}\} \subset S(F,c)$ be such that the sequence $(y_n)_{n\in\mathbb{N}}$ converges to y. For every $n \in \mathbb{N}$, there exists v_n such that $v_n(t) \in F(t, y_n(t))$, a.e. $t \in J$ and

$$y_n(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_n(s)ds.$$

Arguing as in **Step 4**, we can prove that there exists v such that $v(t) \in F(t, y(t))$ and

$$y(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds.$$

Therefore $y \in S(F,c)$ which yields that $S(F,c)$ is closed, and hence compact in $\mathcal{C}_\alpha$. □

4. Second existence result

4.1. Covitz Nadler approach.

We present now a result for the problem (1)-(2) with a nonconvex valued right hand side. Our considerations are based on the fixed point theorem for contraction multivalued maps given by Covitz and Nadler [10].

Theorem 4.1. *Assume that the following hypothesis holds :*

(H_3) *$F : J \times R \longrightarrow P_{cp}(R)$ has the property that $F(\cdot, u) : J \to P_{cp}(R)$ is measurable for each $u \in R$,*

(H_4) *there exist a function* $p \in C([0,T],\mathbb{R}_+)$ *such that*

$$H_d(F(t,z_1),F(t,z_2)) \leq p(t)\|z_1 - z_2\| \text{ for all } z_1,\ z_2 \in \mathbb{R},$$

and

$$d(0,F(t,0)) \leq p(t),\ t \in J.$$

If

$$\frac{T^\alpha \|p\|_\infty \Gamma(\alpha)}{\Gamma(2\alpha)} < 1, \tag{4}$$

then the problem (1)-(2) has at least one solution.

Remark 4.1. *For each* $y \in \mathcal{C}_\alpha$ *the set* $S_{F,y}$ *is nonempty since by* (H_3), F *has a measurable selection (see* [8], *Theorem III.6).*

Proof. We shall show that N satisfies the assumptions of Lemma (2.1). The proof will be given in two steps.

Step 1: $N(y) \in P_{cl}\mathcal{C}_\alpha(J,\mathbb{R})$ *for each* $y \in \mathcal{C}_\alpha(J,\mathbb{R})$.

Indeed, let $(y_n)_{n\geq 0} \in N(y)$ such that $y_n \longrightarrow \tilde{y}$ in $\mathcal{C}_\alpha(J,\mathbb{R})$. Then, $\tilde{y} \in \mathcal{C}_\alpha(J,\mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in J$,

$$y_n(t) = t^{\alpha-1}c + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_n(s)ds.$$

Using the fact that N has compact values and from (H_4), we may pass to a subsequence if necessary to get that v_n converges weakly to v in $L^1_w(J,R)$ (the space endowed with the weak topology). An application of lemma (2.2) implies that v_n converges strongly to v and hence $v \in S_{F,y}$. Then, for each $t \in J$,

$$y_n(t) \longrightarrow \tilde{y}(t) = t^{\alpha-1}c + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds.$$

So, $\tilde{y} \in N(y)$.

Step 2: *There exists* $\gamma < 1$ *such that*

$$H_d(N(y),N(\overline{y})) \leq \gamma\|y - \overline{y}\|_\infty \text{ for each } y,\overline{y} \in \mathcal{C}_\alpha.$$

Let $y,\overline{y} \in \mathcal{C}_\alpha$ and $h_1 \in N(y)$. Then there exists $v_1(t) \in F(t,y(t))$ such that for each $t \in J$

$$h_1(t) = t^{\alpha-1}c + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds.$$

From (H_4) it follows that

$$H_d(F(t,y(t)),F(t,\overline{y}(t))) \leq p(t)|y(t) - \overline{y}(t)|.$$

Hence, there exists $w \in N(t,\overline{y}(t))$ such that

$$|v_1(t) - w| \leq p(t)|y(t) - \overline{y}(t)|,\ t \in J.$$

Consider $U : J \to \mathcal{P}(R)$ given by

$$U(t) = \{w \in R : |v_1(t) - w| \leq p(t)|y(t) - \overline{y}(t)|\}.$$

Since the multivalued operator $V(t) = U(t) \cap F(t, \overline{y}(t))$ is measurable (see Proposition III.4 in [8]), there exists a function $v_2(t)$ which is a measurable selection for V. So, $v_2(t) \in F(t, \overline{y}(t))$, and for each $t \in J$,

$$|v_1(t) - v_2(t)| \le p(t)|y(t) - \overline{y}(t)|.$$

Let us define for each $t \in J$

$$h_2(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v_2(s)ds.$$

Then for $t \in J$

$$\begin{array}{rcl} |t^{1-\alpha}h_1(t) - t^{1-\alpha}h_2(t)| & \le & \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v_2(s) - v_1(s)|ds \\ & \le & \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}p(s)s^{1-\alpha}|y(s) - \overline{y}(s)|ds \\ & \le & \frac{t^{1-\alpha}\|p\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\|y - \overline{y}\|_\alpha ds. \end{array}$$

Thus

$$\|h_1 - h_2\|_\alpha \le \frac{T^\alpha \|p\|_\infty \Gamma(\alpha)}{\Gamma(2\alpha)}\|y - \overline{y}\|_\alpha.$$

By an analogous relation, obtained by interchanging the roles of y and $\overline{y}$, it follows that

$$H_d(N(y), N(\overline{y})) \le \left[\frac{T^\alpha \|p\|_\infty \Gamma(\alpha)}{\Gamma(2\alpha)}\right]\|y - \overline{y}\|_\alpha.$$

So by (4), N is a contraction and thus, by Lemma (2.1), N has a fixed point y which is solution to (1)-(2).

4.2. **Structure of the solution set.**

The following definitions and lemmas can be found in [6, 17]

Definition 4.1. *A single-valued map $f : [0, a] \times X \to Y$ is said to be measurable locally Lipschitz (mLL) if $f(., x)$ is measurable for every $x \in X$ and for every $x \in X$, there exists a neighborhood V_x of $x \in X$ and an integrable function*

$$L_x : [0, a] \to [0, \infty)$$

such that

$$d'(f(t, x_1), f(t, x_2)) \le L_x(t)d(x_1, x_2) \text{ for a.e. } t \in [0, a] \text{ and } x_1,\ x_2 \in V_x.$$

Definition 4.2. *A mapping $F : [0, a] \times X \to \mathcal{P}(Y)$ is mLL-selectionable provided there exists a measurable, locally-Lipchitzian map*

$$f : [0, a] \times X \to Y \text{ and } f(t) \in F(t, y(t)) \text{ for a.e. } t \in [0, a].$$

Let us present an additional assumption :

(H_5) there exist constants $\overline{a}$ and $\overline{b} \in \mathbb{R}_+$ such that

$$\|F(t, x)\|_{\mathcal{P}} \le \overline{a}|x| + \overline{b}, \text{ for a.e. } t \in J \text{ and each } x \in \mathbb{R}.$$

Theorem 4.2. *Let $F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ be a Carathéodory and mLL-selectionable multi-valued with compact convex values which satisfies condition (H_5). Then for every $c \in R$ the solution set $S(F, c)$ is compact and contractible.*

Let $f \in F$ be measurable and locally Lipschitz selection. Consider the single-valued problem

$$D^{\alpha}y(t) = f(t, y(t)), \text{ a.e. } t \in J = (0, T], \ 0 < \alpha \leq 1, \tag{5}$$

$$\lim_{t \longrightarrow 0^+} t^{1-\alpha}y(t) = c. \tag{6}$$

Arguing as in Theorem (4.1) we can prove that the (5)-(6) has exactly one local solution $\overline{x}$ for every $c \in R$.

Bearing in the mind assumptions (H_5), this solution is wholly defined over $[0, T]$. Furthermore, Theorem (3.1) implies that $S(F, c)$ is nonempty and compact.

We define the homotopy $H : S(F, c) \times [0, 1] \to S(F, c)$ by

$$H(y, \lambda)(t) = \begin{cases} y(t), & 0 < t \leq \lambda T \\ \overline{x}(t) & \lambda T < t \leq T, \end{cases}$$

where $\overline{x} = S(f, c)$ is the unique solution of problem $(5) - (6)$. In particular

$$H(y, \lambda) = \begin{cases} y, & \text{for } \lambda = 1, \\ \overline{x}, & \text{for } \lambda = 0. \end{cases}$$

We prove that H is a continuous homotopy . Let $(y_n, \lambda_n) \in S(F, c) \times [0, T]$ be such that $(y_n, \lambda_n) \longrightarrow (y, \lambda)$, as $n \longrightarrow +\infty$.
We shall prove that $H(y_n, \lambda_n) \longrightarrow H(y, \lambda)$, we have

$$H(y_n, \lambda_n)(t) = \begin{cases} y_n(t), & \text{for } t \in (0, \lambda_n T], \\ \overline{x}(t), & \text{for } t \in (\lambda_n T, t]. \end{cases}$$

We consider several cases,

(a) if $\lim_{n \longrightarrow +\infty} \lambda_n = 0$,

$$H(y, 0)(t) = \overline{x}(t), \ t \in [0, T].$$

Hence

$$|H(y_n, \lambda_n)(t) - H(y, \lambda)(t)|_{\alpha} \leq \|y_n - y\|_{\alpha} + \|y_n - \overline{x}\|_{[0, \lambda_n T]},$$

which tends to 0 as $n \longrightarrow +\infty$.

(b) If $\lim_{n \longrightarrow \infty} \lambda_n = 1$,

it's treated similarly.

If $\lambda_n \neq 0$ and $0 < \lim_{n \longrightarrow \infty} \lambda_n < 1$,

two cases must be treated,

- $t \in (0, \lambda_n]$,

then $H(y_n, \lambda_n)(t) - H(y, \lambda)(t) = y_n(t) - y(t)$,
since $y_n \in S(F, c)$, there exist $v_n \in S_{F, y_n}$ such that

$$y_n(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} v_n(s) ds, \ t \in (0, \lambda_n T].$$

We must show that there exists $v_* \in S_{F, y_*}$ such that, for each $t \in J$,

$$y_*(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} v_*(s) ds, \ t \in (0, \lambda_n T].$$

Since $F(t,\cdot)$ is upper semicontinuous, then for every $\varepsilon > 0$, there exist $n_0(\epsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y_*(t)) + \varepsilon B(0,1), \text{ a.e. } t \in J.$$

Since $F(\cdot,\cdot)$ has compact values, then there exists a subsequence $v_{n_m}(\cdot)$ such that

$$v_{n_m}(\cdot) \to v_*(\cdot) \text{ as } m \to \infty$$

and

$$v_*(t) \in F(t, y_*(t)), \text{ a.e. } t \in J.$$

Since y_n converges to y, there exists $M > 0$ such that $\|y_n\|_\alpha \leq M$.
Hence, from (H_4), we have

$$|v_n(t)| \leq \overline{a}t^{\alpha-1}M + \overline{b}, \text{ a.e. } t \in J,$$

which implies

$$v_n(t) \in \overline{a}t^{\alpha-1}M + \overline{b}B(0,1).$$

From the Lebesgue dominated convergence theorem, yields

$$y_*(t) = t^{\alpha-1}c + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_*(s)ds, \ t \in (0, \lambda T].$$

If $t \in (\lambda_n T, T]$, then

$$H(y_n, \lambda_n)(t) = H(y, \lambda)(t).$$

Thus

$$\|H(y_n, \lambda_n) - H(y, \lambda)\|_\alpha \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

In the case where $\lambda = 1$, we have

$$H(y_n, \lambda_n) = H(y, \lambda) = y.$$

Therefore H is a continuous function, proving that $S(F, c)$ is contractible to the point $\overline{x}$.

5. Examples

Consider the problem

$$D^{\frac{1}{2}}y(t) \in F_1(t, y(t)), \text{ a.e. } t \in J = (0,1], \ \alpha = \frac{1}{2}, \tag{7}$$

$$\lim_{t \longrightarrow 0^+} t^{\frac{1}{2}}y(t) = 4. \tag{8}$$

- Let $F_1 : (0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be a multivalued map given by

$$F_1(t,x) = \left[|x| + \frac{3x^2}{3x^2+1} + t^3 + 2t, \ \frac{|x|}{|x|+2} + t^2 + 2\right]. \tag{9}$$

For $f \in F_1$, we have

$$|f(t)| \leq \max(|x| + \frac{3x^2}{3x^2+1} + t^3 + 2t, \ \frac{|x|}{|x|+2} + t^2 + 2) \leq 4 + |x|, \ x \in \mathbb{R}.$$

Thus

$$\|F_1(t,x)\|_{\mathcal{P}} = \sup\{|y| : \ y \in F(t,x)\} \leq a|x| + b, \ x \in \mathbb{R},$$

with $a = 1$ and $b = 4$. Hence by Theorem (3.1), the problem (7)-(8) with F given by (9) has at least one solution on $[0,1]$ and the solution set $S_{F_1}(4)$ is compact.

Consider the multivalued map $F_2 : (0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ given by

$$F_2(t,x) = \left[0, \ \frac{1}{3}\sin x + \frac{|x|}{t+9} + \frac{1}{9}\right], \tag{10}$$

and the fractional differential inclusion defined by

$$D^{\frac{1}{2}} y(t) \in F_2(t, y(t)), \text{ a.e. } t \in J = (0,1], \ \alpha = \frac{1}{2}, \tag{11}$$

$$\lim_{t \longrightarrow 0^+} t^{\frac{1}{2}} y(t) = 4. \tag{12}$$

Clearly

$$\begin{aligned} \|F_2(t,x)\| &= \sup\{|v| : v \in F_2(t,x)\} \\ &\leq \frac{1}{3}|\sin x| + \frac{|x|}{t+9} + \frac{1}{9} \\ &\leq \frac{1}{9}|x| + \frac{4}{9}, \end{aligned}$$

and

$$H_d(F_2(t,x), F_2(t,y)) \leq \left(\frac{1}{3} + \frac{1}{t+9}\right)|x-y|.$$

Let $p(t) = \frac{1}{3} + \frac{1}{t+9}$. Then $\|p\||_\infty = \frac{4}{9}$ and $\frac{T^\alpha \|p\|_\infty \Gamma(\alpha)}{\Gamma(2\alpha)} \approx 0.7877572 < 1$. Hence by Theorem (4.2), the problem (11)-(12) with F_2 given by (10) has at least one solution. It is clear that F_2 is a Carathéodory multivalued map with compact convex values and satisfies the growth condition (H_5). Then the solution set $S_{F_2}(4)$ is a contractible compact set.

6. Conclusion

We deal in this paper with the existence and topological structure of solutions set for fractional differential inclusion. We give the first result when the nonlinearity is upper semi-continuous and takes convex values, we prove then that solution sets is nonempty and compact, our main tool is the Leray-Schauder alternative. The second one is treated when the nonlinearity takes non-convex values, by usefulness of Covitz Nadler contraction we prove that (1)-(2) has one solution, the compactness, contractibility of solution sets is also proved. An example is given to illustrate our results.

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