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ON APPROXIMATE SOLUTIONS OF FOKKER PLANCK EQUATION BY THE MODIFIED RESIDUAL POWER SERIES METHOD

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Abstract. In this article, a reliable method for solving the Fokker Planck equation based on the modified residual power series method is presented. Some of our numerical examples are presented. The results show that the proposed method is accurate.

The Fokker-Planck equation (FPE) is used to model the diffusion problem of the form

$$\frac{\partial \Psi}{\partial x}(r,x) - \frac{\partial \Psi}{\partial r}(r,x) + \frac{\partial^2 \Psi}{\partial r^2}(r,x) = 0, r, x > 0, \tag{1}$$

$$\Psi(r,0) = g(r), \tag{2}$$

where $\Psi(r,x)$ is the external potential and $\frac{\partial \Psi}{\partial r}(r,x)$ is the negative external forces. In [1], several methods were developed to solve such problem. In [2]-[5], authors presented variety of applications of the fractional FPE. Several numerical methods are used to solve the fractional FPE such as the operator method [6], the predictor–corrector [7], the variational iteration method and the Adomian decomposition method [8], and He's variational iteration method [9]. For more references, see [10]-[19]. The fractional derivative which we use in this paper is the Caputo derivative which is given by the following definition.

Definition 1 [20] Let k be the smallest integer that exceed θ, then the Caputo derivative is given by

$$D_r^{\theta}\Psi(r,x) = \frac{1}{\Gamma(k-\theta)} \int_0^r (r-s)^{k-\theta-1} \frac{\partial^k \Psi(s,x)}{\partial s^k} ds \tag{3}$$

for $k-1 < \theta < k$ and $D_r^k \Psi(r,x) = \frac{\partial^k \Psi(r,x)}{\partial r^k}$.

The power rule of the Caputo derivative is given by

$$D^{\theta} r^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\theta+1)} r^{\mu-\theta}, \mu > k-1, k-1 < \theta < k \tag{4}$$

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and

$$D^{\theta}r^{\mu}=0,\mu\leq k-1,k-1<\theta<k. \tag{5}$$

Our goal is to generate convergence series that converges to exact solution. The next theorem presents the convergence theorem.

Theorem 1 [21]-[30] If $\Psi(r,x)$ has a multiple fractional fractional power series (FRPS) of the form

$$\Psi(r,x)=\sum_{k=0}^{\infty}\mu_k(r)(x-x_0)^{k\theta},r\in I,x_0\leq x\leq x_0+R \tag{6}$$

and $D_x^{k\theta}\Psi(r,x)$, for $k=0,1,...$ are continuous on $I\times(x_0,x_0+R)$, then $\mu_k(r)=\frac{D_x^{k\theta}\Psi(r,x_0)}{\Gamma(1+k\theta)}$.

In this paper, we consider the fractional FPE of the form

$$D_x^{\theta}\Psi(r,x)+D_r^{\theta}\Psi(r,x)-D_r^{2\theta}\Psi(r,x) = 0, \tag{7}$$

$$\Psi(r,0) = g(r) \tag{8}$$

where $r,x>0$ and $\theta\in(0,1]$. In Section 2, we present the method of solution while in Section 3, we present some of our examples.

1. Method of solution

In this section, we present the method of solution to the fractional FPE of the form

$$D_x^{\theta}\Psi(r,x)+D_r^{\theta}\Psi(r,x)-D_r^{2\theta}\Psi(r,x) = 0, \tag{9}$$

$$\Psi(r,0) = g(r) \tag{10}$$

where $r,x>0$ and $\theta\in(0,1]$. Approximate $\Psi(r,x)$ by

$$\Psi(r,x)=\sum_{k=0}^{\infty}\mu_k(r)\frac{x^{k\theta}}{\Gamma(1+k\theta)}. \tag{11}$$

Its n^{th} truncated series of the form

$$\Psi_n(r,x)=\sum_{k=0}^{n}\mu_k(r)\frac{x^{k\theta}}{\Gamma(1+k\theta)}. \tag{12}$$

Since $\Psi(r,0)=g(r)$,

$$\Psi_n(r,x)=g(r)+\sum_{k=1}^{n}\mu_k(r)\frac{x^{k\theta}}{\Gamma(1+k\theta)}. \tag{13}$$

To find $\mu_j(r)$, for $1\leq j\leq n$, we solve the fractional equation

$$D_x^{(j-1)\theta}R\Psi_n(r,0)=0 \tag{14}$$

where

$$\begin{aligned} R\Psi_n(r,x) &= D_x^{\theta}\Psi_n(r,x) + D_r^{\theta}\Psi_n(r,x) - D_r^{2\theta}\Psi_n(r,x) \\ &= \sum_{k=1}^{n}\mu_k(r)\frac{x^{(k-1)\theta}}{\Gamma(1+(k-1)\theta)} + D_r^{\theta}g(r) \\ &\quad + \sum_{k=1}^{n}\frac{x^{k\theta}}{\Gamma(1+k\theta)}D_r^{\theta}\mu_k(r) - D_r^{2\theta}g(r) \\ &\quad - \sum_{k=1}^{n}\frac{x^{k\theta}}{\Gamma(1+k\theta)}D_r^{2\theta}\mu_k(r). \end{aligned} \tag{15}$$

Therefore,

$$D_x^{(j-1)\theta}R\Psi_n(r,0) = \mu_j(r) + D_r^{\theta}\mu_{j-1}(r) - D_r^{2\theta}\mu_{j-1}(r) = 0 \tag{16}$$

or

$$\begin{aligned} \mu_j(r) &= D_r^{2\theta}\mu_{j-1}(r) - D_r^{\theta}\mu_{j-1}(r), \\ \mu_0 &= g(r), \end{aligned} \tag{17}$$

for $j = 1, 2, ..., n$. Thus,

$$\begin{aligned} \mu_0 &= g(r), \\ \mu_1(r) &= D_r^{2\theta}g(r) - D_r^{\theta}g(r), \\ \mu_2(r) &= D_r^{2\theta}\left(D_r^{2\theta}g(r) - D_r^{\theta}g(r)\right) - D_r^{\theta}\left(D_r^{2\theta}g(r) - D_r^{\theta}g(r)\right), \\ &\vdots \end{aligned} \tag{18}$$

2. Numerical Results

In this section, we present three examples to show the efficiency of the proposed method.

Example 1: Consider the following problem

$$D_x^{\theta}\Psi(r,x) + D_r^{\theta}\Psi(r,x) - D_r^{2\theta}\Psi(r,x) = 0, \tag{19}$$

$$\Psi(r,0) = \frac{r^{2\theta}}{\Gamma(2\theta+1)}, \tag{20}$$

where $r, x > 0$ and $\theta \in (0,1]$. Approximate $\Psi(r,x)$ by

$$\Psi(r,x) = \sum_{k=0}^{\infty}\mu_k(r)\frac{x^{k\theta}}{\Gamma(1+k\theta)}. \tag{21}$$

Its n^{th} truncated series of the form

$$\Psi_n(r,x) = \sum_{k=0}^{n}\mu_k(r)\frac{x^{k\theta}}{\Gamma(1+k\theta)}. \tag{22}$$

Since $\Psi(r,0) = \frac{r^{2\theta}}{\Gamma(2\theta+1)}$,

$$\Psi_n(r,x) = \frac{r^{2\theta}}{\Gamma(2\theta+1)} + \sum_{k=1}^{n}\mu_k(r)\frac{x^{k\theta}}{\Gamma(1+k\theta)}. \tag{23}$$

To find $\mu_j(r)$, for $1 \leq j \leq n$, we solve the fractional equation

$$D_x^{(j-1)\theta} R\Psi_n(r,0) = 0 \tag{24}$$

where

$$\begin{aligned} R\Psi_n(r,x) &= D_x^{\theta}\Psi_n(r,x) + D_r^{\theta}\Psi_n(r,x) - D_r^{2\theta}\Psi_n(r,x) \\ &= \sum_{k=1}^{n} \mu_k(r) \frac{x^{(k-1)\theta}}{\Gamma(1+(k-1)\theta)} + \frac{r^{\theta}}{\Gamma(\theta+1)} \\ &\quad + \sum_{k=1}^{n} \frac{x^{k\theta}}{\Gamma(1+k\theta)} D_r^{\theta}\mu_k(r) - 1 \\ &\quad - \sum_{k=1}^{n} \frac{x^{k\theta}}{\Gamma(1+k\theta)} D_r^{2\theta}\mu_k(r). \end{aligned} \tag{25}$$

Therefore,

$$D_x^{(j-1)\theta} R\Psi_n(r,0) = \mu_j(r) + D_r^{\theta}\mu_{j-1}(r) - D_r^{2\theta}\mu_{j-1}(r) = 0 \tag{26}$$

or

$$\begin{aligned} \mu_j(r) &= D_r^{2\theta}\mu_{j-1}(r) - D_r^{\theta}\mu_{j-1}(r), \\ \mu_0 &= \frac{r^{2\theta}}{\Gamma(2\theta+1)}, \end{aligned} \tag{27}$$

for $j = 1, 2, ..., n$. Thus,

$$\begin{aligned} \mu_0 &= \frac{r^{2\theta}}{\Gamma(2\theta+1)}, \\ \mu_1 &= 1 - \frac{r^{\theta}}{\Gamma(\theta+1)}, \\ \mu_2 &= 1, \\ \mu_n &= 0, n > 2. \end{aligned} \tag{28}$$

Thus,

$$\begin{aligned} \Psi_2(r,x) &= \sum_{k=0}^{2} \mu_k(r) \frac{x^{k\theta}}{\Gamma(1+k\theta)} \\ &= -\frac{r^{\theta}x^{\theta}}{(\Gamma(\theta+1))^2} + \frac{x^{\theta}}{\Gamma(\theta+1)} + \frac{r^{2\theta}+x^{2\theta}}{\Gamma(2\theta+1)} \end{aligned} \tag{29}$$

which is the exact solution.

Example 2: Consider the following problem

$$D_x^{\theta}\Psi(r,x) + D_r^{\theta}\Psi(r,x) - D_r^{2\theta}\Psi(r,x) = 0, \tag{30}$$

$$\Psi(r,0) = -\frac{r^{3\theta}}{\Gamma(3\theta+1)}, \tag{31}$$

where $r, x > 0$ and $\theta \in (0,1]$. Approximate $\Psi(r,x)$ by

$$\Psi(r,x) = \sum_{k=0}^{\infty} \mu_k(r) \frac{x^{k\theta}}{\Gamma(1+k\theta)}. \tag{32}$$

Its n^{th} truncated series of the form

$$\Psi_n(r,x) = \sum_{k=0}^{n} \mu_k(r) \frac{x^{k\theta}}{\Gamma(1+k\theta)}. \tag{33}$$

Since $\Psi(r,0) = -\frac{r^{3\theta}}{\Gamma(3\theta+1)}$,

$$\Psi_n(r,x) = -\frac{r^{3\theta}}{\Gamma(3\theta+1)} + \sum_{k=1}^{n} \mu_k(r) \frac{x^{k\theta}}{\Gamma(1+k\theta)}. \tag{34}$$

To find $\mu_j(r)$, for $1 \leq j \leq n$, we solve the fractional equation

$$D_x^{(j-1)\theta} R\Psi_n(r,0) = 0 \tag{35}$$

where

$$\begin{aligned} R\Psi_n(r,x) &= D_x^{\theta}\Psi_n(r,x) + D_r^{\theta}\Psi_n(r,x) - D_r^{2\theta}\Psi_n(r,x) \\ &= \sum_{k=1}^{n} \mu_k(r) \frac{x^{(k-1)\theta}}{\Gamma(1+(k-1)\theta)} - \frac{r^{2\theta}}{\Gamma(2\theta+1)} \\ &\quad + \sum_{k=1}^{n} \frac{x^{k\theta}}{\Gamma(1+k\theta)} D_r^{\theta}\mu_k(r) + \frac{r^{\theta}}{\Gamma(\theta+1)} \\ &\quad - \sum_{k=1}^{n} \frac{x^{k\theta}}{\Gamma(1+k\theta)} D_r^{2\theta}\mu_k(r). \end{aligned} \tag{36}$$

Therefore,

$$D_x^{(j-1)\theta} R\Psi_n(r,0) = \mu_j(r) + D_r^{\theta}\mu_{j-1}(r) - D_r^{2\theta}\mu_{j-1}(r) = 0 \tag{37}$$

or

$$\begin{aligned} \mu_j(r) &= D_r^{2\theta}\mu_{j-1}(r) - D_r^{\theta}\mu_{j-1}(r), \\ \mu_0 &= -\frac{r^{3\theta}}{\Gamma(3\theta+1)}, \end{aligned} \tag{38}$$

for $j = 1, 2, ..., n$. Thus,

$$\begin{aligned} \mu_0 &= -\frac{r^{3\theta}}{\Gamma(3\theta+1)}, \\ \mu_1 &= -\frac{r^{\theta}}{\Gamma(\theta+1)} + \frac{r^{2\theta}}{\Gamma(2\theta+1)}, \\ \mu_2 &= 2 - \frac{r^{\theta}}{\Gamma(\theta+1)}, \\ \mu_3 &= 1, \\ \mu_n &= 0, n > 3. \end{aligned} \tag{39}$$

Thus,

$$\begin{aligned} \Psi_2(r,x) &= \sum_{k=0}^{3} \mu_k(r) \frac{x^{k\theta}}{\Gamma(1+k\theta)} \\ &= -\frac{r^{\theta}x^{\theta}}{(\Gamma(\theta+1))^2} + \frac{2x^{2\theta}}{\Gamma(2\theta+1)} + \frac{r^{2\theta}x^{\theta} - r^{\theta}x^{2\theta}}{\Gamma(2\theta+1)\Gamma(\theta+1)} + \frac{x^{3\theta} - r^{3\theta}}{\Gamma(3\theta+1)} \end{aligned} \tag{40}$$

which is the exact solution.

Example 3: Consider the following problem

$$\begin{aligned} D_x^{\theta}\Psi(r,x)+D_r^{\theta}\Psi(r,x)-D_r^{2\theta}\Psi(r,x) &= 0, && (41)\\ \Psi(r,0) &= E_{\theta}(2r^{\theta}), && (42) \end{aligned}$$

where $r,x>0, E_{\theta}$ is the Mittag-Leffler function, and $\theta\in(0,1]$. Approximate $\Psi(r,x)$ by

$$\Psi(r,x)=\sum_{k=0}^{\infty}\mu_k(r)\frac{x^{k\theta}}{\Gamma(1+k\theta)}. \tag{43}$$

Its n^{th} truncated series of the form

$$\Psi_n(r,x)=\sum_{k=0}^{n}\mu_k(r)\frac{x^{k\theta}}{\Gamma(1+k\theta)}. \tag{44}$$

Since $\Psi(r,0)=E_{\theta}(2r^{\theta})$,

$$\Psi_n(r,x)=E_{\theta}(2r^{\theta})+\sum_{k=1}^{n}\mu_k(r)\frac{x^{k\theta}}{\Gamma(1+k\theta)}. \tag{45}$$

To find $\mu_j(r)$, for $1\le j\le n$, we solve the fractional equation

$$D_x^{(j-1)\theta}R\Psi_n(r,0)=0 \tag{46}$$

where

$$\begin{aligned} R\Psi_n(r,x) &= D_x^{\theta}\Psi_n(r,x)+D_r^{\theta}\Psi_n(r,x)-D_r^{2\theta}\Psi_n(r,x) \qquad (47)\\ &= \sum_{k=1}^{n}\mu_k(r)\frac{x^{(k-1)\theta}}{\Gamma(1+(k-1)\theta)}+D_r^{\theta}E_{\theta}(2r^{\theta})\\ &\quad+\sum_{k=1}^{n}\frac{x^{k\theta}}{\Gamma(1+k\theta)}D_r^{\theta}\mu_k(r)-D_r^{2\theta}E_{\theta}(2r^{\theta})\\ &\quad-\sum_{k=1}^{n}\frac{x^{k\theta}}{\Gamma(1+k\theta)}D_r^{2\theta}\mu_k(r). \end{aligned}$$

Therefore,

$$D_x^{(j-1)\theta}R\Psi_n(r,0)=\mu_j(r)+D_r^{\theta}\mu_{j-1}(r)-D_r^{2\theta}\mu_{j-1}(r)=0 \tag{48}$$

or

$$\begin{aligned} \mu_j(r) &= D_r^{2\theta}\mu_{j-1}(r)-D_r^{\theta}\mu_{j-1}(r), && (49)\\ \mu_0 &= E_{\theta}(2r^{\theta}), \end{aligned}$$

for $j=1,2,...,n$. Thus,

$$\begin{aligned} \mu_0 &= E_{\theta}(2r^{\theta}), && (50)\\ \mu_n &= 2^n E_{\theta}(2r^{\theta}), n>0. \end{aligned}$$

Thus,

$$\begin{aligned}\Psi_2(r,x) &= \sum_{k=0}^{\infty}\mu_k(r)\frac{x^{k\theta}}{\Gamma(1+k\theta)} \\ &= \sum_{k=0}^{\infty}2^k E_\theta(2r^\theta)\frac{2^k x^{k\theta}}{\Gamma(1+k\theta)} \\ &= E_\theta(2r^\theta)E_\theta(2x^\theta)\end{aligned} \tag{51}$$

which is the exact solution.

We notice that the proposed method is accurate and gives the exact solution in the three examples. In addition, there is no influence for the parameter θ on the solution. This is clear since we get the exact solution for any choice of θ. In addition, it is advisable to use this approach for other applications in physics and engineering.

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