Journal of Fractional Calculus and Applications
Vol. 10(2) July 2019, pp. 216-224.
ISSN: 2090-5858.
http://math-frac.oreg/Journals/JFCA/

ON CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS WITH RIEMANN LIOUVILLE q-DERIVATIVE DISTRIBUTION SERIES

N. RAVIKUMAR, S. LATHA, B.A. FRASIN

Abstract. By making use of the concepts of fractional q- calculus, we define the subclasses $\mathcal{S}_p^q(\alpha,\beta,\delta,b)$ and $\mathcal{TS}_p^q[\alpha,\beta,\delta,b]$ of analytic function. For functions belonging to these classes, we obtain coefficient estimates, distortion bounds and many more properties.

1. Introduction

The fractional q-calculus is the extension of the ordinary fractional calculus in the q-theory. The theory of q-calculus operators in recent past have been applied in the areas of ordinary fractional calculus, optimal control problems and in finding solutions of the q-difference and q-integral equations, and in q-transform analysis and also in the geometric function theory of complex analysis. For more details on the subject, one may refer to [6], [1], [3], [9] and [16].

Let $\mathcal{S}$ denote the family of functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \tag{1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Also denote by $\mathcal{T}$, the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z - \sum_{m=2}^{\infty} |a_m| z^m \tag{2}$$

which are univalent and normalized in $\mathcal{U}$. For $f \in \mathcal{S}$ and of the form (1) and $g(z) \in \mathcal{S}$ given by $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$, we define the convolution (or Hadamard product) $f * g$ of two power series f and g by $(f * g)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m$.

The q-shifted factorial is defined for α, $q \in \mathbb{C}$ as a product of n factors by

$$(\alpha, q)_n = \begin{cases} 1, & n = 0 \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}), & n \in \mathbb{N}, \end{cases} \tag{3}$$

1991 *Mathematics Subject Classification.* 30C45, 30C50.

Key words and phrases. Univalent functions, Bernardi operator, fractional derivative, q-derivative.

Submitted Feb. 25, 2018. Revised Nov. 22, 2018.

and in terms of the basic analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)},\ (n>0), \tag{4}$$

where the q-gamma functions [6, 7] is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty}\ \ (0<q<1). \tag{5}$$

Note that, if $|q|<1$, the q-shifted factorial (3), remains meaningful for $n=\infty$ as a convergent infinite product

$$(\alpha; q)_\infty = \prod_{m=0}^{\infty}(1-\alpha q^m).$$

Now recall the following q-analogue definitions given by Gasper and Rahman [6]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x), \text{ where, } [x]_q = \frac{(1-q^x)}{(1-q)}, \tag{6}$$

and called q-analogue of x.

Jackson's q-derivative and q-integral of a function f defined on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [6])

$$D_q f(z) = \frac{f(z)-f(zq)}{z(1-q)},\ \ (z\neq 0,\, q\neq 0). \tag{7}$$

$$\int_0^z f(t) d_q(t) = z(1-q)\sum_{m=0}^{\infty} q^m f(zq^m). \tag{8}$$

In view of the relation

$$\lim_{q\to 1^-}\frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n, \tag{9}$$

we observe that the q-shifted fractional (3), reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n+1)$.

Now recall the definition of the fractional q-calculus operators of a complex-valued function $f(z)$, which were recently studied by Purohit and Raina [13].

Definition 1 The fractional q-integral operator $I^\delta_{q,z}$ of a function $f(z)$ of order $\delta\,(\delta>0)$ is defined by

$$I^\delta_{q,z} = D^{-\delta}_{q,z} f(z) = \frac{1}{\Gamma_q(\delta)}\int_0^z (z-tq)_{1-\delta} f(t) d_q t, \tag{10}$$

where $f(z)$ is a analytic in a simply connected region in the z-plane containing the origin. Here, the term $(z-tq)_{\delta-1}$ is a q-binomial function defined by

$$(z-tq)_{\delta-1} = z^{\delta-1}\prod_{m=0}^{\infty}\left[\frac{1-(\frac{tq}{z})q^m}{1-(\frac{tq}{z})q^\delta+m-1}\right] \tag{11}$$

$$= z^{\delta}\,{}_1\phi_0\left[q^{-\delta+1};-;q,\frac{tq^{\delta}}{z}\right].$$

According to Gasper and Rahman [6], the series ${}_1\phi_0[\delta;-;q,z]$ is single-valued when $|arg(z)| < \pi$. Therefore, the function $(z-tq)_{\delta-1}$ in (11), is single-valued when $|arg(\frac{-tq^{\delta}}{z}| < \pi$, $|tq^{\frac{\delta}{z}}| < 1$, and $|arg(z)| < \pi$.

Definition 2 The fractional q-derivative operator $D^{\delta}_{q,z}$ of a $f(z)$ of order $\delta(0 \le \delta < 1)$ is defined by

$$D^{\delta}_{q,z}f(z) = D_{q,z}I^{1-\delta}_{q,z}f(z) = \frac{1}{\Gamma_q(1-\delta)}D_q\int_0^z (z-tq)_{-\delta}f(t)d_qt, \tag{12}$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\delta}$ is removed as in Definition 1 above.

Definition 3 Under the hypotheses of Definition 2, the fractional q-derivative for the function $f(z)$ of order δ is defined by

$$D^{\delta}_{q,z}f(z) = D^{n}_{q,z}I^{n-\delta}_{q,z}f(z), \tag{13}$$

where, $n-1 \le \delta < n$, $n \in \mathbb{N}_0$.

Now we define a fractional q-differintegral operator $\Omega^{\delta}_{q,z}f(z)$ for the function $f(z)$ of the form (1), by

$$\Omega^{\delta}_{q}f(z) = \Gamma_q(2-\delta)z^{\delta}D^{\delta}_{q,z}f(z) \tag{14}$$

$$= z + \sum_{m=2}^{\infty}\frac{\Gamma_q(m+1)\Gamma_q(2-\delta)}{\Gamma_q(m+1-\delta)}a_m z^m,$$

where in $D^{\delta}_{q,z}$ (14), represents, respectively, a fractional q-integral of $f(z)$ of order δ when $-\infty < \delta < 0$ and a fractional q-derivative of $f(z)$ of order δ when $0 < \delta < 2$. We note that $q \to 1^{-}$, the operator Ω^{δ}_{q} reduces the operator Ω^{δ} defined by Owa and Srivastava [10].

Recently, several authors investigated applications of fractional q-calculus operators by introducing certain new classes of functions which are analytic in the open disc, (see for example, [12, 14, 15, 17, 18]).

By making use of the concepts of fractional q- calculus, we now define the following subclasses $\mathcal{S}^q_p(\alpha,\beta,\delta,b)$ and $\mathcal{TS}^q_p[\alpha,\beta,\delta,b]$ of analytic function.

Definition 4 For $-1 \le \alpha < 1$, $\beta \ge 0$, $0 < \delta < 2$, $b \in \mathbb{C}-\{0\}$ and $0 < q < 1$, let $\mathcal{S}^q_p(\alpha,\beta,\delta,b)$ be the subclass of $\mathcal{S}$ consisting of functions of the form (1) and satisfying the analytic criterion

$$\Re\left\{1-\frac{2}{b}+\frac{2}{b}.\frac{zD_q(\Omega^{\delta}_q f(z))}{\Omega^{\delta}_q f(z)}\right\} > \beta\left|\frac{2}{b}.\frac{zD_q(\Omega^{\delta}_q f(z))}{\Omega^{\delta}_q f(z)}-\frac{2}{b}\right|+\alpha,\ z \in \mathcal{U}. \tag{15}$$

Let $\mathcal{TS}^q_p[\alpha,\beta,\delta,b] = \mathcal{S}^q_p(\alpha,\beta,\delta,b) \cap \mathcal{T}$.

It can be seen that, the special cases of the class $\mathcal{TS}^q_p[\alpha,\beta,\delta,b]$ as $q \to 1^{-}$ and for different choices of the parameters we get the results obtained by Altintas and Owa [2], Bharthi, Parvtham and Swaminathan [5], Padamanabhan and Jayamala [11], Owa and Srivastava [10], Kim and Ronning [8].

In this paper, we obtain coefficient estimates, distortion bounds and many more properties for functions in the classes $\mathcal{S}_p^q(\alpha,\beta,\delta,b)$ and $\mathcal{TS}_p^q[\alpha,\beta,\delta,b]$.

2. The classes $\mathcal{S}_p^q(\alpha,\beta,\delta,b)$ and $\mathcal{TS}_p^q[\alpha,\beta,\delta,b]$.

In this section we obtain a necessary, sufficient condition and extreme points for functions $f(z)$ in the class $\mathcal{TS}_p^q[\alpha,\beta,\delta,b]$

Theorem 1 A sufficient condition for a function $f(z)$ of the form (1) to be in $\mathcal{S}_p^q(\alpha,\beta,\delta,b)$ is that

$$\sum_{m=2}^{\infty}[2(1+\beta)([m]_q-1)+b(1-\alpha)]K_q(m,\delta)(\delta)a_m \leq b(1-\alpha), \tag{16}$$

where, $K_q(m,\delta)=\frac{\Gamma_q(m+1)\Gamma_q(2-\delta)}{\Gamma_q(m+1-\delta)}$, $-1\leq\alpha<1,\ \beta\geq 0,\ 0<\delta<2,\ b\in\mathbb{C}-\{0\}$ and $0<q<1$.

Proof. Suppose $f\in\mathcal{S}_p^q(\alpha,\beta,\delta,b)$ then,

$$\begin{aligned}
&\beta\left|\frac{2}{b}\left(\frac{zD_q\Omega_q^\delta f(z)}{\Omega_q^\delta f(z)}-1\right)\right|-\Re\left\{\frac{2}{b}\left(\frac{zD_q\Omega_q^\delta f(z)}{\Omega_q^\delta f(z)}-1\right)\right\}\\
\leq\ & (1+\beta)\left|\frac{2}{b}\left(\frac{zD_q\Omega_q^\delta f(z)}{\Omega_q^\delta f(z)}-1\right)\right|\\
\leq\ & \frac{2(1+\beta)}{b}\frac{\sum_{m=2}^{\infty}([m]_q-1)K_q(m,\delta)|a_m||z|^{m-1}}{1-\sum_{m=2}^{\infty}K_q(m,\delta)|a_m||z|^{m-1}}\\
\leq\ & \frac{2(1+\beta)}{b}\frac{\sum_{m=2}^{\infty}([m]_q-1)K_q(m,\delta)|a_m|}{1-\sum_{m=2}^{\infty}K_q(m,\delta)|a_m|}.
\end{aligned}$$

This is bounded above by $1-\alpha$ if

$$\sum_{m=2}^{\infty}[\,2\,(1+\beta)\,([m]_q-1)+b(1-\alpha)]K_q(m,\delta)|a_m|\leq b(1-\alpha).$$

This completes the proof.

Theorem 2 A necessary and sufficient condition for f of the form (2) namely $f(z)=z-\sum_{m=2}^{\infty}|a_m|z^m,\quad z\in\mathcal{U}$ to be in $\mathcal{TS}_p^q[\alpha,\beta,\delta,b]$, $-1\leq\alpha<1,\ \beta\geq 0$, $0<\delta<2,\ b\in\mathbb{C}-\{0\}$ and $0<q<1$ is that

$$\sum_{m=2}^{\infty}\frac{[\,2\,(\beta-1)\,(1-[m]_q)+(1-\alpha)b]}{b(1-\alpha)}K_q(m,\delta)|a_m|\leq 1. \tag{17}$$

Proof. In view of Theorem 1, we need to prove the necessity. If $f\in\mathcal{TS}_p^q[\alpha,\beta,\delta,b]$ and z is real then

$$\Re\left\{1-\frac{2}{b}+\frac{2}{b}.\frac{zD_q(\Omega_q^\delta f(z))}{\Omega_q^\delta f(z)}\right\}-\alpha\geq\beta\left[\frac{2}{b}.\frac{zD_q(\Omega_q^\delta f(z))}{\Omega_q^\delta f(z)}-\frac{2}{b}\right].$$

That is

$$\frac{2}{b}(\beta-1)\left[\sum_{m=2}^{\infty}(1-[m]_q)K_q(m,\delta)|a_m|z^{m-1}\right]\leq(1-\alpha)\left[1-\sum_{m=2}^{\infty}K_q(m,\delta)|a_m|z^{m-1}\right].$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality,

$$\sum_{m=2}^{\infty} \frac{[\,2\,(\beta-1)\,(1-[m]_q)+(1-\alpha)b]}{b(1-\alpha)} K_q(m,\delta)|a_m| \leq 1.$$

Theorem 3 Let $f \in \mathcal{TS}_p^q[\alpha,\beta,\delta,b]$, $-1 \leq \alpha < 1$, $\beta \geq 0$, $0 < \delta < 2$, $b \in \mathbb{C}-\{0\}$ and $0 < q < 1$. Define $f_1(z) = z$ and

$$f_m(z) = z - \frac{b(1-\alpha)}{[2(\beta-1)(1-[m]_q)+(1-\alpha)b]\,K_q(m,\delta)} z^m, \quad m = 2,3,\cdots,$$

$z \in \mathcal{U}$. Then $f \in \mathcal{TS}_p^q[\alpha,\beta,\delta,b]$ if and only if f can be expressed as

$$f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z), \tag{18}$$

where $\mu_m \geq 0$ and $\sum_{m=1}^{\infty} \mu_m = 1$.

Proof. If $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ with $\sum_{m=1}^{\infty} \mu_m = 1, \quad \mu_m \geq 0$, then

$$\sum_{m=2}^{\infty} [2(\beta-1)(1-[m]_q)+(1-\alpha)b]\,K_q(m,\delta)\mu_m \cdot \frac{b(1-\alpha)}{[2(\beta-1)(1-[m]_q)+(1-\alpha)b]K_q(m,\delta)}$$
$$= \sum_{m=2}^{\infty} \mu_m(b(1-\alpha)) = (1-\mu_1)(b(1-\alpha)) \leq b(1-\alpha).$$

Hence $f \in \mathcal{TS}_p^q[\alpha,\beta,\delta,b]$.
Conversely, let $f(z) = z - \sum_{m=2}^{\infty} |a_m| z^m \in \mathcal{TS}_p^q[\alpha,\beta,\delta,b]$, define

$$\mu_m = \frac{[2(\beta-1)(1-[m]_q)+(1-\alpha)b]\,K_q(m,\delta)|a_m|}{b(1-\alpha)}, \quad m = 2,3,\cdots$$

and define $\mu_1 = 1 - \sum_{m=2}^{\infty} \mu_m$. From Theorem 2, $\sum_{m=2}^{\infty} \mu_m \leq 1$ and so $\mu_1 \geq 0$. Therefore, we can see that $f(z)$ can be expressed in the form (18).

Corollary 1 Let $f \in \mathcal{TS}_p^q[\alpha,\beta,\delta,b]$ then

$$|a_m| < \frac{b(1-\alpha)}{[2(\beta-1)(1-[m]_q)+(1-\alpha)b]\,K_q(m,\delta)}, \qquad m = 2,3,4,...$$

Theorem 4 Let $\delta_1 < \delta_2$ then $\mathcal{TS}_p^q[\alpha,\beta,\delta_2,b] \subset \mathcal{TS}_p^q[\alpha,\beta,\delta_1,b]$.

Proof. Let $f \in \mathcal{TS}_p^q[\alpha,\beta,\delta_2,b]$ then we have

$$\sum_{m=2}^{\infty} \frac{[\,2\,(\beta-1)\,(1-[m]_q)+(1-\alpha)b]}{b(1-\alpha)} K_q(m,\delta_2)|a_m| \leq 1,$$

but hence $K_q(m,\delta)$ is an increasing function of δ therefore $K_q(m,\delta_1) < K_q(m,\delta_2)$, so we have

$$\sum_{m=2}^{\infty} \frac{[\,2\,(\beta-1)\,(1-[m]_q)+(1-\alpha)b]}{b(1-\alpha)} K_q(m,\delta_1)|a_m|$$
$$< \sum_{m=2}^{\infty} \frac{[\,2\,(\beta-1)\,(1-[m]_q)+(1-\alpha)b]}{b(1-\alpha)} K_q(m,\delta_2)|a_m| \leq 1,$$

then $f \in \mathcal{TS}_p^q[\alpha,\beta,\delta_1,b]$.

Corollary 2 Let $0 \leq \alpha_2 < \alpha_1 < 1$ and $f \in \mathcal{TS}_p^q[\alpha_1,\beta,\delta,b]$ then $f \in \mathcal{TS}_p^q[\alpha_2,\beta,\delta,b]$.

Theorem 5 The class $\mathcal{TS}_p^q[\alpha,\beta,\delta,b]$ is convex set.

Proof. Let f and g be the arbitrary elements of $\mathcal{TS}_p^q[\alpha,\beta,\delta,b]$ then for every $t(0<t<1)$, we show that $(1-t)f(z)+tg(z)\in \mathcal{TS}_p^q[\alpha,\beta,\delta,b]$, thus we have

$$(1-t)f(z)+tg(z)=z-\sum_{m=2}^{\infty}[(1-t)|a_m|+t|b_m|]z^m$$

and

$$\sum_{m=2}^{\infty}\frac{[\,2\,(\beta-1)\,(1-[m]_q)+(1-\alpha)b]}{b(1-\alpha)}[(1-t)|a_m|+t|b_m|]K_q(m,\delta)$$
$$=(1-t)\sum_{m=2}^{\infty}\frac{[\,2\,(\beta-1)\,(1-[m]_q)+(1-\alpha)b]}{b(1-\alpha)}|a_m|K_q(m,\delta)$$
$$+t\sum_{m=2}^{\infty}\frac{[\,2\,(\beta-1)\,(1-[m]_q)+(1-\alpha)b]}{b(1-\alpha)}|b_m|K_q(m,\delta)<1.$$

Corollary 3 Suppose that $f(z)$ and $g(z)$ belong to $\mathcal{TS}_p^q[\alpha,\beta,\delta,b]$ then the function $h(z)$ defined by $h(z)=\frac{1}{2}[f(z)+g(z)]$ also belongs to $\mathcal{TS}_p^q[\alpha,\beta,\delta,b]$.

Theorem 6 Let the function $f(z)=z-\sum_{m=2}^{\infty}|a_m|z^m$ be the class $\mathcal{TS}_p^q[\alpha,\beta,\delta,b]$ for $-1\le\alpha<1,\ \beta\ge 0,\ 0<\delta<2,\ b\in\mathbb{C}-\{0\}$ and $0<q<1$, then

$$|z|+\frac{b(1-\alpha)(1-q^{2-\delta})}{[2q(1-\beta)+(1-\alpha)b](1-q)\Gamma_q(3)}|z|^2\le|f(z)|\le|z|-\frac{b(1-\alpha)(1-q^{2-\delta})}{[2q(1-\beta)+(1-\alpha)b](1-q)\Gamma_q(3)}|z|^2.$$

$$1+\frac{b(1-\alpha)(1-q^{2-\delta})}{[2q(1-\beta)+(1-\alpha)b](1-q)^2\Gamma_q(3)}|z|\le|D_qf(z)|\le 1+\frac{b(1-\alpha)(1-q^{2-\delta})}{[2q(1-\beta)+(1-\alpha)b](1-q)^2\Gamma_q(3)}|z|.$$

Proof. Since $f\in\mathcal{TS}_p^q[\alpha,\beta,\delta,b]$, then in view of Theorem 2 , we first show that the function

$$\phi(m)=\frac{\Gamma_q(m+1)\Gamma_q(2-\delta)}{\Gamma_q(m+1-\delta)},\ m\ge 2$$

is an increasing function of m for $0\le\delta<2$.

We have that

$$\begin{aligned}\frac{\phi(m+1)}{\phi(m)} &= \frac{\Gamma_q(m+2)\Gamma_q(m+1-\delta)}{\Gamma_q(m+1)\Gamma_q(m+2-\delta)}, \qquad (m\ge 2)\\ &= \frac{1-q^{m+1}}{1-q^{m+1-\delta}}, \qquad (0<q<1).\end{aligned}$$

The function $\phi(m)$ is a increasing function of m if $\frac{\phi(m+1)}{\phi(m)}\ge 1$ and this gives

$$\frac{1-q^{m+1}}{1-q^{m+1-\delta}}\ge 1, \qquad (0<q<1).$$

Thus $\phi(m)\,(m\ge 2)$ is increasing of m for $0<\delta<2,\ 0<q<1$.

$$\begin{aligned}&[2q(1-\beta)+(1-\alpha)b]K_q(2,\delta)\sum_{m=2}^{\infty}|a_m|\\ \le&\ \sum_{m=2}^{\infty}[\,2\,(\beta-1)\,(1-[m]_q)+(1-\alpha)b]\,|a_m|\\ \le&\ b(1-\alpha).\end{aligned}$$

That is,

$$\sum_{m=2}^{\infty} |a_m| \leq \frac{b(1-\alpha)(1-q^{2-\delta})}{[2q(1-\beta)+(1-\alpha)b](1-q)\Gamma_q(3)}$$

and this last inequality in conjunction with the following inequality,

$$|f(z)| \leq |z| - \ |z|^2 \sum_{m=2}^{\infty} |a_m| \text{ and } |f(z)| \geq |z| + |z|^2 \sum_{m=2}^{\infty} |a_m|.$$

So,

$$|z| + \frac{b(1-\alpha)(1-q^{2-\delta})}{[2q(1-\beta)+(1-\alpha)b](1-q)\Gamma_q(3)} |z|^2 \leq |f(z)| \leq |z| - \frac{b(1-\alpha)(1-q^{2-\delta})}{[2q(1-\beta)+(1-\alpha)b](1-q)\Gamma_q(3)} |z|^2.$$

Again,

$$|D_q f(z)| = \left| 1 - \sum_{m=2}^{\infty} [m]_q |a_m| z^{m-1} \right| \leq 1 - |z| \sum_{m=2}^{\infty} [m]_q |a_m|$$

and

$$|D_q f(z)| \geq 1 + |z| \sum_{m=2}^{\infty} [m]_q |a_m|.$$

We have,

$$\sum_{m=2}^{\infty} [m]_q |a_m| \leq \frac{b(1-\alpha)(1-q^{2-\delta})}{[2q(1-\beta)+(1-\alpha)b](1-q)}.$$

So,

$$1 + \frac{b(1-\alpha)(1-q^{2-\delta})}{[2q(1-\beta)+(1-\alpha)b](1-q)^2\Gamma_q(3)} |z| \leq |D_q f(z)| \leq 1 + \frac{b(1-\alpha)(1-q^{2-\delta})}{[2q(1-\beta)+(1-\alpha)b](1-q)^2\Gamma_q(3)} |z|.$$

Bernardi Libera's integral operator is defined as

$$L_\gamma f(z) = \frac{\gamma+1}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt,$$

which was studied by Bernardi in [4].

Theorem 7 Let $f \in \mathcal{TS}_p^q[\alpha, \beta, \delta, b]$. The q-analogous Bernardi's integral operator defined by

$$L_{q,\gamma} f(z) = \frac{[\gamma+1]_q}{z^\gamma} \int_0^z t^{\gamma-1} f(t) d_q t,$$

then $L_{q,\gamma} f \in \mathcal{TS}_p^q[\alpha, \beta, \delta, b]$.

Proof. We have

$$\begin{aligned}
L_{q,\gamma}f(z) &= \frac{[\gamma+1]_q}{z^\gamma} z(1-q)\sum_{j=0}^{\infty} q^j (zq^j)^{\gamma-1} f(zq^j)\\
&= [\gamma+1]_q(1-q)\sum_{j=0}^{\infty} q^{j\gamma} f(zq^j)\\
&= [\gamma+1]_q(1-q)\sum_{j=0}^{\infty} q^{j\gamma}\sum_{m=1}^{\infty} q^{jm}|a_m|z^m\\
&= [\gamma+1]_q\sum_{j=0}^{\infty}\sum_{m=1}^{\infty}(1-q)q^{j(\gamma+m)}|a_m|z^m\\
&= z+\sum_{m=2}^{\infty}\frac{[\gamma+1]_q}{[\gamma+m]_q}|a_m|z^m.
\end{aligned}$$

Since $f \in \mathcal{TS}_p^q[\alpha,\beta,\delta,b]$ and since $\frac{[\gamma+1]_q}{[\gamma+m]_q} < 1$ for all $m \geq 2$, we have

$$\sum_{m=2}^{\infty}\left[\,2\,(\beta-1)\,(1-[m]_q)+(1-\alpha)b\right] K_q(m,\delta)|a_m|\frac{[\gamma+1]_q}{[\gamma+m]_q} < b(1-\alpha).$$

Theorem 8 Let $f \in \mathcal{TS}_p^q[\alpha,\beta,\delta,b]$ then $L_{q,\gamma}f(z)$ is q-starlike of order $0 \leq \alpha_3 \leq 1$ in $|z| < R_1$ where

$$R_1 = \inf\left\{\left(\frac{[\gamma+m]_q}{[\gamma+1]_q}.\frac{(1-\alpha_3)\left[\,2\,(\beta-1)\,(1-[m]_q)+(1-\alpha)b\right] K_q(m,\delta)}{([m]_q-\alpha_3)b(1-\alpha)}\right)^{\frac{1}{m-1}} : m \in \mathbb{N}/\{1\}\right\}.$$

Proof. It is sufficient to prove

$$\left|\frac{z(D_qL_{q,\gamma}f(z))}{L_{q,\gamma}f(z)}-1\right| < 1-\alpha_3, \qquad z \in \mathcal{U}.$$

Now

$$\begin{aligned}
&\left|\frac{z(D_qL_{q,\gamma}f(z))}{L_{q,\gamma}f(z)}-1\right|\\
=&\left|\frac{\sum_{m=2}^{\infty}([m]_q-1)|a_m|z^{m-1}\frac{[\gamma+1]_q}{[\gamma+m]_q}}{1+\sum_{m=2}^{\infty}|a_m|z^{m-1}\frac{[\gamma+1]_q}{[\gamma+m]_q}}\right|\\
\leq&\ \frac{\sum_{m=2}^{\infty}([m]_q-1)\frac{[\gamma+1]_q}{[\gamma+m]_q}|a_m|\ |z|^{m-1}}{1-\sum_{m=2}^{\infty}|a_m||z|^{m-1}\left(\frac{[\gamma+1]_q}{[\gamma+m]_q}\right)}.
\end{aligned}$$

This last expression is less than $1-\alpha_3$, since

$$|z|^{m-1} \leq \left(\frac{[\gamma+m]_q}{[\gamma+1]_q}\right)\frac{(1-\alpha_3)\left[\,2\,(\beta-1)\,(1-[m]_q)+(1-\alpha)b\right] K_q(m,\delta)}{([m]_q-\alpha_3)b(1-\alpha)}.$$

Using the fact that f is convex if and only if zD_qf is starlike, we obtain the following

Theorem 9 Let $f \in \mathcal{TS}_p^q[\alpha, \beta, \delta, b]$ then $L_{q,\gamma} f(z)$ is q-convex of order $0 \le \alpha_3 \le 1$ in $|z| < R_2$ where

$$R_2 = \inf \left\{ \left(\frac{[\gamma + m]_q}{[\gamma + 1]_q} \frac{(1-\alpha_3)\left[\,2\,(\beta-1)\,(1-[m]_q) + (1-\alpha)b\right] K_q(m,\delta)}{[m]_q([m]_q - \alpha_3)b(1-\alpha)} \right)^{\frac{1}{m-1}} : m \in \mathbb{N}/\{1\} \right\}.$$

References

[1] M.H. Abu-Risha, M.H. Annaby, M.E.H. Isail and Z.S. Mansour, Linear q-difference equations, Z. Anal. Anwend. 26 (2007), 481–494.

[2] O. Altintas and S. Owa, On subclasses of univalent functions with negative coefficients , Pusan Kyongnam Math.J., 4(1988), 41-56.

[3] G. Bangerzako, Variational calculus on q-nonuniform lattices, J. Math. Anal. Appl. 306, 1 (2005), 161–179.

[4] S. D. Bernardi, Convex and starlike univalent functions. Transactions of the American Mathematical Society, 135(1969),429-446.

[5] R. Bharthi, R. Paravatham and A. Swaminathan, On subclasses of uniformaly convex functions and corresponding class of starlike functions , Tamkang J.Math.,26(1)(1997), 17-32.

[6] G. Gasper and M. Rahman, Basic Hypergeometric Series , vol. 35 of Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge, Mass, USA, 1990.

[7] F. H. Jackson, On q-functions and a certain difference operator, Transactions of the Royal Society of Edinburgh, vol. 46, no. 2,(1909) pp.253-281.

[8] Y. C. Kim and F. Ronning, Integral transforms of certain subclasses of analytic function, J. Math. Anal. Appl., 258(2001), 466-657.

[9] Z.S.I. Mansour, Linear sequential q-difference equations of fractional order, Fract. Calc. Appl. Anal. 12, 2 (2009), 159–178.

[10] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, Canadian Journal of Mathematics, vol. 39, no. 5, (1987), pp.1057 - 1077.

[11] K. S. Padmanabhan and M. Jayamala, A class of univalent functions with varying arguments Internat. J. Math. and Math. Sci., Vol.15, No.3 (1992),517-522.

[12] S. D. Purohit, A new class of multivalently analytic functions associated with fractional q–calculus operators, Fractional Differential Calculus, Volume 2, Number 2 (2012), 129–138.

[13] S. D. Purohit and R. K. Raina, Certain subclasses of analytic functions associated with fractional q-calculus operators, Mathematica Scandinavica., Vol. 109, No. 1,(2011) pp.55-70.

[14] C. Ramachandran, T. Soupramanien and B.A. Frasin, New subclasses of analytic function associated with q-difference operator, European journal of pure and applied mathematics, Vol. 10, No. 2, 2017, 348-362.

[15] N. Ravikumar, Certain classes of analytic functions defined by fractional q-calculus operator, Acta Univ. Sapientiae, Mathematica, 10, 1 (2018) 178–188.

[16] P.M. Rajkovic, S.D. Marinkovic and M.S. Stankovic, Fractional integrals and derivatives in q-calculus, Appl. Anal. Discrete Math. 1 (2007), 311–323.

[17] K.Vijay, Certain Class of Analytic Functions Based on q-difference operator, arXiv:1709.04138v1 [math.CV] 13 Sep 2017.

[18] K. Vijaya, G. Murugusundaramoorthy and S.Yalçın, Certain Class of Analytic Functions Involving Salagean Type q-Difference Operator, Konuralp Journal of Mathematics, 6 (2) (2018) 264-271.

N. Ravikumar, PG Department of Mathematics, JSS College of Arts, Commerce and Science, Mysore - 570 005, INDIA.

E-mail address: ravisn.kumar@gmail.com

S. Latha, Department of Mathematics, Yuvaraja's College, University of Mysore, Mysore - 570 009, INDIA.

E-mail address: drlatha@gmail.com

B.A. Frasin, Faculty of Science, Department of Mathematics, Al al-Bayt University, Mafraq, Jordan

E-mail address: bafrasin@yahoo.com