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CONSERVATION LAWS FOR THE SPACE-TIME FRACTIONAL OF CLASSICAL DRINFELD'S SOKOLOV-WILSON (FDSW) SYSTEM

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Abstract. In this paper, we studied the space-time fractional classical Drinfeld's Sokolov-Wilson (FDSW) system. This system does not have a Lagrangian. Noether's approach was used to derive the conservation laws in U and V variables. Finally, by reverting back to our original variables u and v, we obtained the conservation laws for the space-time fractional of classical Drinfeld's Sokolov-Wilson (FDSW) system. This conservation laws consisted of some local and nonlocal conserved vectors.

1. Introduction

Conservation laws are one of the most important gateways to understanding physical properties of various systems, such as conservation of energy, mass, momentum and so on. They are important for investigating integrability and for establishing existence and uniqueness of solutions. They have been used for the development of appropriate numerical methods and construction of exact solutions of partial differential equations. They play an essential role in the development of numerical methods and provide an essential starting point for finding non-locally related systems and potential variables.

Conservation laws play a central role in the solution and reduction of partial differential equations. It is well-known that if differential equation is an Euler-Lagrange equation, then conservation laws can be found using Noether's theorem by variational Lie point symmetries of this equation.

A generalization of Noether's theorem was proved to give not only a necessary but also a sufficient condition for the existence of conservation laws. Based on the fundamental operator identity relating the infinitesimal operator of the transformation group, the Euler–Lagrange differential operator, and the Noether operators, a simple and efficient constructive algorithm for constructing conservation laws was developed using symmetries of differential equations that admit a Lagrangian [14].

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The objective of this paper is to construct the conservation laws of the space-time fractional of classical Drinfeld's Sokolov-Wilson (FDSW) system [2], [9]

$$u_t^\alpha + pvv_x = 0, \tag{1}$$
$$v_t^\alpha + qv_{xxx} + ruv_x + svu_x = 0, \qquad t > 0, \quad 0 < \alpha \leq 1, \tag{2}$$

where p, q, r and s are some nonzero parameter by using the fractional generalization of the Noether operators.

To motivate the generalizing conservation laws, we assume that $r = s = p$. Eqs.(1) and (2) becomes

$$u_t^\alpha + pvv_x = 0, \tag{3}$$
$$v_t^\alpha + qv_{xxx} + p\left(uv_x + vu_x\right) = 0. \tag{4}$$

$u_t^\alpha = \frac{\partial^\alpha u}{\partial t^\alpha} = D_t^\alpha$ denotes the Riemann–Liouville derivative defined by [8], [10], which is defined by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \frac{\partial^m u}{\partial t^m}, & \alpha = m \in \mathbb{N}; \\ \frac{1}{\Gamma(m-\alpha)} \frac{\partial^m}{\partial t^m} \int_0^t \left(t-\tau\right)^{m-\alpha-1} u\left(\tau, x\right) d\tau, & m-1 < \alpha < m, \ m \in \mathbb{N}. \end{cases} \tag{5}$$

The $\alpha-$th extended infinitesimal related to Rieman-Liouville fractional time derivative with Eq.(5) reads [11], [12]

$$\zeta_\alpha^0 = D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u). \tag{6}$$

Here the operator D_t^α denotes the total fractional derivative operator. We would like to recall the generalized Leibnitz rule [3], [1] given by

$$D_t^\alpha(f(t)g(t)) = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n} f(t) D_t^n g(t), \qquad \alpha > 0, \tag{7}$$

where

$$\binom{\alpha}{n} = \frac{(-1)^{n-1}\alpha\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}. \tag{8}$$

Using Leibnitz rule (7), Eq.(6) can be written as

$$\begin{aligned} \zeta_\alpha^0 \quad = \quad & D_t^\alpha(\eta) - \alpha D_t(\tau)\frac{\partial^\alpha u}{\partial t^\alpha} - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x) \\ & - \sum_{n=1}^{\infty} \binom{\alpha}{n+1} D_t^{n+1}(\tau) D_t^{\alpha-n}(u). \end{aligned} \tag{9}$$

2. Conservation laws for the space-time fractional of classical Drinfeld's Sokolov-Wilson (FDSW) system

In this section, we construct conservation laws for the space-time fractional of classical Drinfeld's Sokolov-Wilson (FDSW) system (3) and (4). We note that this system does not have a Lagrangian. However, we can put the system into a variational form by letting $u = U_x$, $v = V_x$ in order to get conserved vectors. Then the space-time fractional of classical Drinfeld's Sokolov-Wilson (FDSW) system (3) and (4) transforms into

$$\frac{\partial^\alpha U_x}{\partial t^\alpha} + pV_xV_{xx} = 0, \tag{10}$$

$$\frac{\partial^\alpha V_x}{\partial t^\alpha} + qV_{xxxx} + p\left(U_xV_{xx} + sV_xU_{xx}\right) = 0, \qquad t>0, \quad 0<\alpha\le 1. \tag{11}$$

It can be readily verified that L given by

$$L = \frac{1}{2}\left(-\frac{\partial^\alpha U}{\partial t^\alpha}U_x - \frac{\partial^\alpha V}{\partial t^\alpha}V_x + qV_{xx}^2 - pU_xV_x^2\right), \tag{12}$$

is a Lagrangian for the system (10) and (11). This is because L satisfies

$$\frac{\delta L}{\delta U} = 0 \text{ and } \frac{\delta L}{\delta V} = 0, \tag{13}$$

where $\frac{\delta}{\delta U}$ and $\frac{\delta}{\delta V}$ are the Euler-Lagrange operators defined by

$$\begin{aligned} \frac{\delta}{\delta U} &= \frac{\partial}{\partial U} + \left(D_t^\alpha\right)^* \frac{\partial}{\partial D_t^\alpha U} - D_x\frac{\partial}{\partial U_x} + D_x^2\frac{\partial}{\partial U_{xx}}\cdots, \\ \frac{\delta}{\delta V} &= \frac{\partial}{\partial V} + \left(D_t^\alpha\right)^* \frac{\partial}{\partial D_t^\alpha V} - D_x\frac{\partial}{\partial V_x} + D_x^2\frac{\partial}{\partial V_{xx}}\cdots. \end{aligned} \tag{14}$$

Here $\left(D_t^\alpha\right)^*$ is the adjoint operator of D_t^α. For the Riemann-Liouville and Caputo fractional differential operators, the corresponding adjoint operators have the form [5]

$$\begin{aligned} \left({}_0D_t^\alpha\right)^* &= (-1)^n \ {}_tI_T^{n-\alpha}\left(D_t^n\right) \equiv_t^C D_T^\alpha, \\ \left({}_0^CD_t^\alpha\right)^* &= (-1)^n \ D_t^n\left({}_tI_T^{n-\alpha}\right) \equiv_t D_T^\alpha; \end{aligned}$$

where ${}_tD_T^\alpha$ and ${}_t^CD_T^\alpha$ are the right-sided Riemann-Liouville and Caputo operators of fractional differentiation of order α, respectively.

Here ${}_tI_T^{n-\alpha}$ is the right-sided operator of fractional integration of order $n-\alpha$ defined by

$$\left({}_tI_T^{n-\alpha}f\right)(t,x) = \frac{1}{\Gamma(n-\alpha)}\int_t^T \frac{f(\iota,x)}{(\iota-t)^{\alpha+1-n}}d\iota. \tag{15}$$

Consider the vector field

$$\begin{aligned} \chi^\alpha &= \zeta_{1\alpha}^0\frac{\partial}{\partial(D^\alpha U)} + \zeta_{2\alpha}^0\frac{\partial}{\partial(D^\alpha V)} + \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta_1\frac{\partial}{\partial u} + \eta_2\frac{\partial}{\partial v} \\ &\quad + \zeta_x^1\frac{\partial}{\partial U_x} + \zeta_x^2\frac{\partial}{\partial V_x} + \zeta_{xx}^1\frac{\partial}{\partial U_{xx}} + \zeta_{xx}^2\frac{\partial}{\partial V_{xx}} + \cdots. \end{aligned} \tag{16}$$

where D_x denotes the total derivative with respect to x and D_t with respect to t. The αth extended infinitesimal given in Eq.(16) becomes [15]

$$\begin{aligned} \zeta_{\alpha_1}^0 &= \frac{\partial^\alpha \eta_1}{\partial t^\alpha} + \left(\eta_{1u} - \alpha D_t(\tau)\right)\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_{1u}}{\partial t^\alpha} + \mu_1 \\ &\quad + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n \eta_{1u}}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u) \\ &\quad - \sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x). \end{aligned} \tag{17}$$

and

$$\mu_1 = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\eta_1}{\partial t^{n-m}\partial u^k}. \tag{18}$$

The vector field χ^α, of the form Eq.(16), is called a Noether point symmetry generator corresponding to the Lagrangian L if there exists gauge functions $\beta^1(x,t,U,V)$ and $\beta^2(x,t,U,V)$ such that [4]

$$\chi^\alpha(L) + \{D_t(\tau) + D_x(\xi)\}L = D_t(\beta^1) + D_x(\beta^2). \tag{19}$$

The classical symmetry properties can be expanded if one studies Eqs.(10) and (11) together with Eq.(16) as a over-determined system of fractional partial differential equations. The determining equations for a Noether point symmetry generator χ^α are now obtained by equating to zero the coefficients of the independent coordinates [7]. By disbanding this system of linear fractional partial differential equations for $\xi(x,t,U,V)$, $\tau(x,t,U,V)$, $\eta_1(x,t,U,V)$, $\eta_2(x,t,U,V)$, $\beta^1(x,t,U,V)$ and $\beta^2(x,t,U,V)$, we procure:

$$\begin{aligned}
\eta_1(x,t,U,V) &= \frac{3C_1\, U}{2} + A_4(t),\\
\eta_2(x,t,U,V) &= \frac{3C_1\, V}{2} + A_2(t),\\
\xi(x,t,U,V) &= C_1 x + C_2,\\
\tau(x,t,U,V) &= \frac{3C_1}{\alpha}\, t + C_3,\\
\beta^1(x,t,U,V) &= 0,\\
\beta^2(x,t,U,V) &= \frac{1}{2}\frac{\partial^\alpha A_4}{\partial t^\alpha}U + \frac{1}{2}\frac{\partial^\alpha A_2}{\partial t^\alpha}V;
\end{aligned} \tag{20}$$

where C_1, C_2, C_3 are arbitrary constants and $A_2[t]$, $A_4[t]$ are arbitrary functions. The above results will now be used to find the components of the conserved vectors for the α-order Lagrangian L.

3. Fractional Noether Operators

The x-component conserved vector can be given by the formula for differential equations of integer order. This formula has the form:

$$\begin{aligned}
N^x &= -\beta^1 + \xi_i L + W^1\left[\frac{\partial L}{\partial U_x} - D_x\frac{\partial L}{\partial U_{xx}} + D_x^2\frac{\partial L}{\partial U_{xxx}} + \dots\right]\\
&+ D_x\left(W^1\right)\left[\frac{\partial L}{\partial U_{xx}} - D_x\frac{\partial L}{\partial U_{xxx}}\right] + D_x^2\left(W^1\right)\left[\frac{\partial L}{\partial U_{xxx}}\right]\\
&+ W^2\left[\frac{\partial L}{\partial V_x} - D_x\frac{\partial L}{\partial V_{xx}} + D_x^2\frac{\partial L}{\partial V_{xxx}} + \dots\right] + D_x\left(W^2\right)\left[\frac{\partial L}{\partial V_{xx}} - D_x\frac{\partial L}{\partial V_{xxx}}\right]\\
&+ D_x^2\left(W^2\right)\left[\frac{\partial L}{\partial V_{xxx}}\right],
\end{aligned} \tag{21}$$

where $W_i^1 = \eta_1 - \tau_i U_t - \xi_i U_x$ and $W_i^2 = \eta_2 - \tau_i V_t - \xi_i V_x$ are the characteristic functions corresponding to the Lie symmetries χ_1, χ_2 and χ_3 [6].

Based on the fractional generalizations of the Noether operators, the t-component conserved vector can be given by [13]:

$$\begin{aligned} N^t &= -\beta^2 + \tau_i L + \sum_{k=0}^{n-1}((-1)^k \ {}_0D_t^{\alpha-1-k}\left(W^1\right) D_t^k \frac{\partial}{\partial\left({}_0D_t^{\alpha}U\right)} \\ &+ (-1)^k \ {}_0D_t^{\alpha-1-k}\left(W^2\right) D_t^k \frac{\partial}{\partial\left({}_0D_t^{\alpha}V\right)}) - (-1)^n J\left(W^1, D_t^n \frac{\partial}{\partial\left({}_0D_t^{\alpha}U\right)}\right) \\ &- (-1)^n J\left(W^2, D_t^n \frac{\partial}{\partial\left({}_0D_t^{\alpha}V\right)}\right), \end{aligned} \tag{22}$$

where J is the integral

$$J\left(f,g\right) = \frac{1}{\Gamma\left(n-\alpha\right)} \int_0^t \int_t^T \frac{f\left(s,x\right) g\left(\mu,x\right)}{\left(\mu-s\right)^{\alpha+1-n}} d\mu ds. \tag{23}$$

This integral has a property

$$D_t J\left(f,g\right) = f \ {}_tI_T^{n-\alpha} g - g \ {}_0I_t^{n-\alpha} f. \tag{24}$$

Thus, using Eqs.(21) and (22) together with Eq.(20) and $u = U_x$, $v = V_x$ yields the following independent conserved vectors for our system (3) and (4).

3.1. **Conservation laws through X_1.** For this symmetry operator, the Lie characteristics are

$$W_1^1 = \frac{3}{2}U - \frac{3}{\alpha}tU_t - xU_x, \quad W_1^2 = \frac{3}{2}V - \frac{3}{\alpha}tV_t - xV_x. \tag{25}$$

Effecting these values into the vector components Eqs.(21) and (22), we get

$$\begin{aligned} C_1^x &= q\left(v - \frac{3t}{\alpha}v_t - \frac{x}{2}v_x\right)v_x + pxuv + qxvv_{xx}, \\ C_1^t &= \frac{3t}{2\alpha}(qv_x^2 - puv^2) - \frac{1}{2}v_0D_t^{\alpha-1}\left(\frac{3}{2}\int v_x dx - xv\right) \\ &-\frac{1}{2}u_0D_t^{\alpha-1}\left(\frac{3}{2}\int u_x dx - xu\right) - \frac{3}{4}J\left(\int u_x dx, u_t\right) \\ &+\frac{3t}{2\alpha}\left({}_0D_t^{\alpha-1}u\int u_t dx - u_t \ {}_tD_T^{\alpha-1}\int u_t dx\right) \\ &+\frac{3}{\alpha}J\left(tu_t - \left(\alpha-1\right)u, \int u_t dt\right) + \frac{x}{2}J\left(u,u_t\right) - \frac{3}{4}J\left(\int v_x dx, v_t\right) \\ &+\frac{3t}{2\alpha}\left({}_0D_t^{\alpha-1}v\int v_t dx - v_t \ {}_tD_T^{\alpha-1}\int v_t dx\right) \\ &+\frac{3}{\alpha}J\left(tv_t - \left(\alpha-1\right)v, \int v_t dt\right) + \frac{x}{2}J\left(v,v_t\right). \end{aligned} \tag{26}$$

3.2. **Conservation laws through X_2.** Substituting the Lie characteristics

$$W_2^1 = -U_x, \quad W_2^2 = -V_x. \tag{27}$$

into Eqs.(21) and (22) gives

$$\begin{aligned} C_2^x &= -\frac{3}{2}qv_x^2+\frac{3}{2}puv^2-\frac{p}{2}uv-qv_{xx}, \\ C_2^t &= \frac{1}{2}u_0D_t^{\alpha-1}u-\frac{1}{2}J\left(u,u_t\right)+\frac{1}{2}v_0D_t^{\alpha-1}v-\frac{1}{2}J\left(v,v_t\right) \end{aligned} \tag{28}$$

3.3. **Conservation laws through X_3.** For this symmetry operator, the Lie characteristics are

$$W_3^1=-U_t,\quad W_3^2=-V_t. \tag{29}$$

These reduce the vector components Eqs.(21) and (22) to

$$\begin{aligned} C_3^x &= \frac{1}{2}\int u_t dx\left(\int\frac{\partial^\alpha u_x}{\partial t^\alpha}dx+pv^2\right)+\int u_t dx\left(\frac{1}{2}\int\frac{\partial^\alpha v_x}{\partial t^\alpha}dx+puv-qv_{xx}\right)-qv_tv_x, \\ C_3^t &= \frac{q}{2}v_x^2-\frac{p}{2}uv^2-\frac{1}{2}J\left(\int u_t dx,u_t\right)-\frac{1}{2}J\left(\int v_t dx,v_t\right). \end{aligned} \tag{30}$$

3.4. **Conservation laws through X_{A_4} and X_{A_2}.** For this generator, the Lie characteristics are

$$W_{A_4}^1=A_4\left[t\right],\quad W_{A_2}^2=A_2\left[t\right]. \tag{31}$$

Substituting these values in Eqs.(21) and (22), we obtain

$$\begin{aligned} C_{(A_4,A_2)}^x &= -\frac{1}{2}A_2(puv+qv_{xx}), \\ C_{(A_4,A_2)}^t &= -\frac{1}{2}\frac{\partial^\alpha A_4}{\partial t^\alpha}\int u_x dx-\frac{1}{2}\frac{\partial^\alpha A_2}{\partial t^\alpha}\int v_x dx-\frac{1}{2}u_0D_t^{\alpha-1}A_4 \\ &\quad -\frac{1}{2}J\left(A_4,u_t\right)-\frac{1}{2}v_0D_t^{\alpha-1}A_2-\frac{1}{2}J\left(A_2,v_t\right). \end{aligned} \tag{32}$$

Since $A_2\left[t\right]$ and $A_4\left[t\right]$ are arbitrary functions, Eq.(32) gives infinitely many conserved vectors. We now extract one special case from the conserved vector Eq.(32) by letting $A_2\left[t\right]=1$ and $A_4\left[t\right]=0$, which gives a local conserved vector

$$\begin{aligned} C_{(A_4,A_2)}^x &= -\frac{1}{2}(puv+qv_{xx}), \\ C_{(A_4,A_2)}^t &= -\frac{1}{2}J\left(1,u_t\right). \end{aligned} \tag{33}$$

It should be noted that the conserved vectors Eqs.(26) and (30) are nonlocal conserved vectors and Eq.(28) is a local conserved vector for the system (3) and (4). We note that since the functions $A_2(t)$ and $A_4(t)$ are arbitrary, one obtains infinitely many local and nonlocal conservation laws for the system (3) and (4) from the conserved vector Eq.(32).

4. Conservation laws for the space-time fractional of classical Drinfeld's Sokolov-Wilson (FDSW) system with the Caputo fractional derivative

Thus, using Eqs.(21) and (22) together with Eq.(20) and $u=U_x$, $v=V_x$ yields the following independent conserved vectors for our system (3) and (4):

4.1. **Conservation laws through** X_1 **:.** For this symmetry operator, the Lie characteristics are

$$W_1^1 = \frac{3}{2}U - \frac{3}{\alpha}tU_t - xU_x, \quad W_1^2 = \frac{3}{2}V - \frac{3}{\alpha}tV_t - xV_x. \tag{34}$$

Effecting these values into the vector components Eqs.(21) and (22), we get

$$\begin{aligned}
C_1^x &= q\left(v - \frac{3t}{\alpha}v_t - \frac{x}{2}v_x\right)v_x + pxuv + qxvv_{xx}, \\
C_1^t &= \frac{3t}{2\alpha}(qv_x^2 - puv^2) - \left(\frac{3}{4}\int v_x dx - \frac{xv}{2}\right)_t D_T^{\alpha-1}(v) \\
&- \left(\frac{3}{4}\int u_x dx - \frac{xu}{2}\right)_t D_T^{\alpha-1}u - \frac{3}{4}J\left(\int u_x dx, u_t\right) \\
&+ \frac{3t}{2\alpha}\left({}_tD_T^{\alpha-1}u\int u_t dx - \left(\int u_t dx\right)_t D_T^{\alpha-1}u\right) \\
&+ \frac{3}{\alpha}J\left(tu_{tt} - (\alpha-1)u_t, \int u dt\right) + \frac{x}{2}J(u_t, u) - \frac{3}{4}J\left(\int v_{tx}dx, v\right) \\
&+ \frac{3t}{2\alpha}\left({}_tD_T^{\alpha-1}v\int v_t dx - \int v_t dx\; {}_tD_T^{\alpha-1}v_t\right) \\
&+ \frac{3}{\alpha}J\left(tv_{tt} - (\alpha-1)v_t, \int v dt\right) + \frac{x}{2}J(v_t, v).
\end{aligned} \tag{35}$$

4.2. **Conservation laws through** X_2 **:.** Substituting the Lie characteristics

$$W_2^1 = -U_x, \quad W_2^2 = -V_x. \tag{36}$$

into Eqs.(21) and (22) gives

$$\begin{aligned}
C_2^x &= -\frac{3}{2}qv_x^2 + \frac{3}{2}puv^2 - \frac{p}{2}uv - qv_{xx}, \\
C_2^t &= \frac{1}{2}u_tD_T^{\alpha-1}u - \frac{1}{2}J(u_t, u) + \frac{1}{2}v_tD_T^{\alpha-1}v - \frac{1}{2}J(v_t, v).
\end{aligned} \tag{37}$$

4.3. **Conservation laws through** X_3 **:.** For this symmetry operator, the Lie characteristics are

$$W_3^1 = -U_t, \quad W_3^2 = -V_t. \tag{38}$$

These reduce the vector components Eqs.(21) and (22) to

$$\begin{aligned}
C_3^x &= \frac{1}{2}\int u_t dx\left(\int \frac{\partial^\alpha u_x}{\partial t^\alpha}dx + pv^2\right) + \int u_t dx\left(\frac{1}{2}\int \frac{\partial^\alpha v_x}{\partial t^\alpha}dx + puv - qv_{xx}\right) - qv_tv_x, \\
C_3^t &= \frac{q}{2}v_x^2 - \frac{p}{2}uv^2 - \frac{1}{2}J\left(\int u_{tt}dx, u\right) - \frac{1}{2}J\left(\int v_{tt}dx, v\right).
\end{aligned} \tag{39}$$

4.4. **Conservation laws through X_{A_4} and X_{A_2} :.** For this generator, the Lie characteristics are

$$W^1_{A_4} = A_4\left[t\right], \quad W^2_{A_2} = A_2\left[t\right]. \tag{40}$$

Substituting these values in Eqs.(21) and (22), we obtain

$$\begin{aligned} C^x_{(A_4,A_2)} &= -\frac{1}{2}A_2(puv + qv_{xx}), \\ C^t_{(A_4,A_2)} &= -\frac{1}{2}\frac{\partial^\alpha A_4}{\partial t^\alpha}\int u_x dx - \frac{1}{2}\frac{\partial^\alpha A_2}{\partial t^\alpha}\int v_x dx - \frac{1}{2}A_{4t}D_T^{\alpha-1}u \\ &\quad -\frac{1}{2}J\left(A_{4_t}, u\right) - \frac{1}{2}A_{2t}D_T^{\alpha-1}v - \frac{1}{2}J\left(A_{2_t}, v\right). \end{aligned} \tag{41}$$

Since $A_2\left[t\right]$ and $A_4\left[t\right]$ are arbitrary functions, Eq. (41) gives infinitely many conserved vectors.

It should be noted that the conserved vectors Eqs.(35) and (39) are nonlocal conserved vectors and Eq.(37) is a local conserved vector for the system (3) and (4). We note that since the functions $A_2(t)$ and $A_4(t)$ are arbitrary, one obtains infinitely many local and nonlocal conservation laws for the system (3) and (4) from the conserved vector Eq.(41).

5. Conclusion

The space-time fractional of classical Drinfeld's Sokolov-Wilson (FDSW) system (3) and (4) do not have a Lagrangian. We make the transformation $u = U_x$ and $v = V_x$ which convert the space-time fractional of classical Drinfeld's Sokolov-Wilson (FDSW) system into a system of fractional-order partial differential equations in U and V variables, which has a Lagrangian. Noether's approach is then used to construct the conservation laws. Finally, the conservation laws are expressed in the original variables u and v. Some local and nonlocal conserved quantities are found for the space-time fractional of classical Drinfeld's Sokolov-Wilson (FDSW) system (3) and (4).

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