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ON THE DYNAMICS OF SOME RECURSIVE SEQUENCES

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Abstract. In this paper we deal with the investigation of some qualitative behavior such as the global convergence, boundedness and periodicity of the solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_{n-1} x_n + \gamma x_{n-2}^2}{A x_n^2 + B x_{n-1} x_n + C x_{n-2}^2}, \; n = 0, 1, ...,$$

where the parameters α, β, γ, A, B and C are nonnegative numbers with $A + B + C > 0$, $\alpha + \beta + \gamma > 0$ and the initial conditions x_{-2}, x_{-1} and x_0 are arbitrary positive real numbers.

1. Introduction

In this paper we deal with the behavier of the solutions of the following difference equation

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_{n-1} x_n + \gamma x_{n-2}^2}{A x_n^2 + B x_{n-1} x_n + C x_{n-2}^2}, \quad n = 0, 1, ..., \tag{1}$$

where the parameters α, β, γ, A, B, and C are nonnegative numbers with $A + B + C > 0$, $\alpha + \beta + \gamma > 0$ and the initial conditions x_{-2}, x_{-1} and x_0 are arbitrary positive real numbers.

In [11], Kulenović et al. investigated the solutions of the difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n + C x_{n-1}}.$$

In [2], Elabbasy et al. studied the periodicity and the global stability of the difference equation

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1} + \gamma x_{n-2}}{A x_n + B x_{n-1} + C x_{n-2}}.$$

In [3], El-Metwally. studied qualitative the properties of some higher order the difference equations of the form

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$$x_{n+1}=\frac{ax_{n-k_0}^{l_0}x_{n-k_1}^{l_1}...x_{n-k_i}^{l_i}+bx_{n-r_0}^{s_0}x_{n-r_1}^{s_1}...x_{n-r_j}^{s_j}}{cx_{n-k_0}^{l_0}x_{n-k_1}^{l_1}...x_{n-k_i}^{l_i}+dx_{n-r_0}^{s_0}x_{n-r_1}^{s_1}...x_{n-r_j}^{s_j}},$$

and

$$y_{n+1}=\frac{\alpha_0 y_n+\alpha_1 y_{n-1}+...+\alpha_t y_{n-t}}{\beta_0 y_n+\beta_1 y_{n-1}+...+\beta_t y_{n-t}}.$$

Other related results on rational difference equations can be found in [[3],[8],[9]]

Let I be some interval of real numbers and let

$$f: I^{\kappa+1}\to I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-\kappa}, x_{-\kappa+1}, ..., x_0 \in I$, the difference equation

$$x_{n+1}=f(x_n, x_{n-1}, ..., x_{n-\kappa}),\quad n=0,1,..., \tag{2}$$

has a unique solution $\{x_n\}_{n=-\kappa}^{\infty}$. A solution of Eq.(2) that is constant for all $n\geq -k$ is called **an equilibrium** solution of Eq.(2). If

$$x_n=\bar{x},\quad \text{for all}\quad n\geq -k,$$

is an equilibrium solution of Eq.(2) then $\bar{x}$ is called **an equilibrium point** or simply an equilibrium of Eq.(2).

The following definitions and previous results will be useful for the proof of our results in this paper.

Definition 1 [1] (**stability**)

(i) An equilibrium point $\bar{x}$ of Eq.(2) is called **locally stable** if, for every $\varepsilon>0$, there exists $\delta>0$ such that if $\{x_n\}_{n=-k}^{\infty}$ is a solution of Eq.(2) with

$$|x_{-\kappa}-\bar{x}|+|x_{-\kappa+1}-\bar{x}|+...+|x_0-\bar{x}|<\delta,$$

then

$$|x_n-\bar{x}|<\varepsilon,\quad \text{for all}\quad n\geq -k.$$

(ii) An equilibrium point $\bar{x}$ of Eq.(2) is called **locally asymptotically stable** if $\bar{x}$ is locally stable and if in addition there exists $\gamma>0$ such that if $\{x_n\}_{n=-\kappa}^{\infty}$ is a solution of Eq.(2) with

$$|x_{-\kappa}-\bar{x}|+|x_{-\kappa+1}-\bar{x}|+...+|x_0-\bar{x}|<\gamma,$$

then

$$\lim_{n\to\infty}x_n=\bar{x}.$$

(iii) An equilibrium point $\bar{x}$ of Eq.(2) is called **a global attractor** if, for every solution $\{x_n\}_{n=-1}^{\infty}$ of Eq.(2) we have

$$\lim_{n\to\infty}x_n=\bar{x}.$$

(iv) An equilibrium point $\bar{x}$ of Eq.(2) is called **globally asymptotically stable** if $\bar{x}$ is **locally stable** and $\bar{x}$ is also **a global attractor** of Eq.(2).

(v) An equilibrium point $\bar{x}$ of Eq.(2) is called **unstable** if $\bar{x}$ is not locally stable.

Definition 2 [1]

(1) A solution $\{x_n\}$ of Eq.(2) is said to be **periodic** with period p if

$$x_{n+p}=x_n\quad \text{for all}\quad n\geq -1. \tag{3}$$

(2) A solution $\{x_n\}$ of Eq.(2) is said to be **periodic with prime period** p, or **a p-cycle** if it is periodic with period p and p is the least positive integer for which (3) holds.

Definition 3 [7] **(permanence):** Eq.(2) is said to be permanent and bounded if there exists number $\boldsymbol{m}$ and $\boldsymbol{M}$ with $0 < \boldsymbol{m} < \boldsymbol{M} < \infty$ such that for any initial condition $x_{-\kappa}, x_{-\kappa+1}, ..., x_0 \in (0, \infty)$ there exists a positive integer N which depends on these initial conditions such that $\boldsymbol{m} < x_n < \boldsymbol{M}$ for all $n \geq N$.

The linearized equation of Eq.(2) about the equilibrium point $\bar{x}$ is

$$z_{n+1} = a_1 z_n + a_2 z_{n-1} + ... + a_{k+1} z_{n-k}, \tag{4}$$

where $a_i = \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \bar{x}, ..., \bar{x})$, $i = 0, 1, ..., k$. The characteristic equation of Eq.(4) is

$$\lambda^{k+1} - \sum_{i=1}^{k+1} a_i \lambda^{k-i+1} = 0.$$

Theorem A.[10] Assume that $p_i \in \mathbb{R}$, $i = 2, ...,$ and $k \in \{0, 1, 2, ...\}$. Then

$$\sum_{i=0}^{k} |p_i| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+1} + p_1 y_{n+k-1} + ... + p_k y_n = 0 \quad , n = 0, 1, 2, ...,$$

consider the following equation

$$x_{n+1} = q(x_n, x_{n-1}, x_{n-2}).$$

Theorem B.[6] Let $\boldsymbol{J}$ be some interval of real numbers, $\mathbf{f} \in C[\boldsymbol{J}^{v+1}, \boldsymbol{J}]$ and let $\{x_n\}_{n=-v}^{\infty}$ be a bounded solution of the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-v}), \quad n = 0, 1, ..., \tag{5}$$

with

$$\boldsymbol{I} = \lim_{n\to\infty} \inf x_n, \quad \boldsymbol{S} = \lim_{n\to\infty} \sup x_n \quad \text{and with } \boldsymbol{I}, \boldsymbol{S} \in \boldsymbol{J}.$$

Then there exist two solutions $\{\boldsymbol{I}_n\}_{n=-\infty}^{\infty}$ and $\{\boldsymbol{S}_n\}_{n=-\infty}^{\infty}$ of Eq.(5) with

$$\boldsymbol{I}_0 = \boldsymbol{I}, \quad \boldsymbol{S}_0 = \boldsymbol{S}, \quad \boldsymbol{I}_n, \boldsymbol{S}_n \in [\boldsymbol{I}, \boldsymbol{S}] \quad \text{for all } n \in \mathbb{Z},$$

and such that for every $N \in \mathbb{Z}$, $\boldsymbol{I}_N$ and $\boldsymbol{S}_N$ are limit points of $\{x_n\}_{n=-v}^{\infty}$.

Furthermore for every $m \leq -v$, there exist two subsequences $\{x_{r_n}\}$ and $\{x_{l_n}\}$ of the solution $\{x_n\}_{n=-v}^{\infty}$ such that the following are true:

$$\lim_{n\to\infty} x_{r_n+N} = \boldsymbol{I}_N \quad \text{and} \quad \lim_{n\to\infty} x_{l_n+N} = \boldsymbol{S}_N \quad \text{for every} \quad N \geq m.$$

The solutions $\{\boldsymbol{I}_n\}_{n=-\infty}^{\infty}$ and $\{\boldsymbol{S}_n\}_{n=-\infty}^{\infty}$ are called **full limiting sequences** of Eq.(5).

Consider the scalar $k^{\ th}$ order linear difference equation

$$x(n+k) + p_1(n)x(n+k-1) + ... + p_k(n)x(n) = 0. \tag{6}$$

where k is a positive integer and $p_i : \mathbb{Z}^+ \to \mathbb{C}$ for $i = 1, ..., k$. Eq.(6) is said to be of Poincarè type if the limits

$$q_i = \lim_{k\to\infty} p_i(n), \quad i = 1, ..., k, \tag{7}$$

exist in $\mathbb{C}$. Under this hypothesis, Eq.(6) can be regarded as a perturbation of the eqaution with constant Coefficients

$$x(n+k)+q_1x(n+k-1)+...+q_kx(n)=0. \tag{8}$$

Theorem C.[12] (Poincarè's Theorem). Suppose condition(7) holds.

Let $\lambda_1, ..., \lambda_k$ be the roots of the characteristic equation

$$\lambda^k+q_1\lambda^{k-1}+...+q_k=0 \tag{9}$$

of Eq.(8) and suppose that

$$|\lambda_i|\neq|\lambda_j| \quad \text{for} \quad i\neq j. \tag{10}$$

If $x(n)$ is a solution of (6) then either $x(n)=0$ for all large n or there exists an index $j\in\{1,...,k\}$ such that

$$\lim_{n\to\infty}\frac{x(n+1)}{x(n)}=\lambda_j. \tag{11}$$

2. Local Stability of the equilibrium points of Eq.(1)

In this section, we investigate the local stability character of the solutions of Eq.(1).

The equilibrium points of Eq.(1) are given by the relation

$$\overline{x}=\frac{\alpha\overline{x}^2+\beta\overline{x}^2+\gamma\overline{x}^2}{A\overline{x}^2+B\overline{x}^2+C\overline{x}^2}.$$

Clearly, the unique positive equilibrium point of Eq.(1) is

$$\overline{x}=(\alpha+\beta+\gamma)/(A+B+C).$$

Let $f:(0,\infty)^3\to(0,\infty)$ be a function defined by

$$f(u,v,w)=\frac{\alpha u^2+\beta vu+\gamma w^2}{Au^2+Buv+Cw^2}. \tag{12}$$

Therefore it follows that

$$\frac{\partial f(u,v,w)}{\partial u}=\frac{(2\alpha u+\beta v)\left(Au^2+Buv+Cw^2\right)-(2Au+Bv)\left(\alpha u^2+\beta vu+\gamma w^2\right)}{\left(Au^2+Buv+Cw^2\right)^2},$$

$$\frac{\partial f(u,v,w)}{\partial v}=\frac{\beta u\left(Au^2+Buv+Cw^2\right)-Bu\left(\alpha u^2+\beta vu+\gamma w^2\right)}{\left(Au^2+Buv+Cw^2\right)^2},$$

and

$$\frac{\partial f(u,v,w)}{\partial w}=\frac{2\gamma w\left(Au^2+Buv+Cw^2\right)-2Cw\left(\alpha u^2+\beta vu+\gamma w^2\right)}{\left(Au^2+Buv+Cw^2\right)^2}.$$

Then we see that

$$\frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial u}=\frac{(\alpha B+2\alpha C+\beta C)-(A\beta+2A\gamma+\gamma B)}{(A+B+C)(\alpha+\beta+\gamma)}=-p_2,$$

$$\frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial v}=\frac{(A\beta+\beta C)-(\alpha B+\gamma B)}{(A+B+C)(\alpha+\beta+\gamma)}=-p_1,$$

and

$$\frac{\partial f(\overline{x},\overline{x},\overline{x})}{\partial w}=\frac{2\gamma(A+B)-2C(\alpha+\beta)}{(A+B+C)(\alpha+\beta+\gamma)}=-p_0.$$

Then the linearized equation of Eq.(1) about the positive equilibrium point $\overline{x}$ is

$$y_{n+1} + p_2 y_n + p_1 y_{n-1} + p_0 y_{n-2} = 0. \tag{13}$$

The characteristic equation of the linearized equation is given by

$$\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 = 0. \tag{14}$$

Theorem 1 Assume that

$$(A+B+C)(\alpha+\beta+\gamma) > \max\left\{\begin{array}{c} 4\left|(\alpha C - A\gamma) + (\beta C - B\gamma)\right|, \\ 2\left|(\alpha B - A\beta) + (B\gamma - \beta C)\right|, \\ 2\left|(\alpha B - A\beta) + 2(\alpha C - A\gamma) + (\beta C - B\gamma)\right|. \end{array}\right\} \tag{15}$$

Then the positive equilibrium point of Eq.(1) is locally asymptotically stable.

proof. It is follows by Theorem A that Eq.(13) is locally asymptotically stable if all roots of the Eq.(14) lie in the open disc $|\lambda| < 1$ that is if

$$|p_2| + |p_1| + |p_0| < 1. \tag{16}$$

We consider the following different possibilities:

(1) If $p_2 > 0$, $p_1 > 0$ and $p_0 > 0$. In this case we see from (16) that

$$\begin{aligned} &(\alpha B + 2\alpha C + \beta C) - (A\beta + 2A\gamma + \gamma B) + (A\beta + \beta C) - (\alpha B + \gamma B) \\ &+2\gamma(A+B) - 2C(\alpha+\beta) < (A+B+C)(\alpha+\beta+\gamma) \end{aligned}$$

if and only if

$$(A+B+C)(\alpha+\beta+\gamma) > 0,$$

which is always true.

(2) If $p_2 > 0$, $p_1 > 0$ and $p_0 < 0$. It follows from (16) that

$$\begin{aligned} &(\alpha B + 2\alpha C + \beta C) - (A\beta + 2A\gamma + \gamma B) + (A\beta + \beta C) - (\alpha B + \gamma B) \\ &-2\gamma(A+B) + 2C(\alpha+\beta) < (A+B+C)(\alpha+\beta+\gamma) \end{aligned}$$

if and only if

$$4(\alpha C - A\gamma) + 4(\beta C - \gamma B) < (A+B+C)(\alpha+\beta+\gamma),$$

which is satisfied by (15).

The proofs of the following cases are similar to the proof of cases 1 and 2 and will be left to the reader.

(3) If $p_2 > 0$, $p_1 < 0$ and $p_0 > 0$.
(4) If $p_2 > 0$, $p_1 < 0$ and $p_0 < 0$.
(5) If $p_2 < 0$, $p_1 > 0$ and $p_0 > 0$.
(6) If $p_2 < 0$, $p_1 > 0$ and $p_0 < 0$.
(7) If $p_2 < 0$, $p_1 < 0$ and $p_0 > 0$.
(8) If $p_2 < 0$, $p_1 < 0$ and $p_0 < 0$.

This completes the proof.

3. Boundedness of the Solutions of Eq.(1)

In the section we study the boundedness of the solutions of Eq.(1).

Lemma 1 Every positive solution of Eq.(1) is bounded and persists.

proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a positive solution of Eq.(1). Then it follows that

$$\begin{aligned} x_{n+1} &= \frac{\alpha x_n^2+\beta x_{n-1}x_n+\gamma x_{n-2}^2}{Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2} \\ &= \frac{\alpha x_n^2}{Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2}+\frac{\beta x_{n-1}x_n}{Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2} \\ &\quad +\frac{\gamma x_{n-2}^2}{Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2} \\ &\leq \frac{\alpha}{A}+\frac{\beta}{B}+\frac{\gamma}{C}:=M. \end{aligned}$$

Then

$$x_n \leq M \quad \text{for all } n \geq 1.$$

By the change of variables $x_n=\frac{1}{z_n}$ for all $n\geq 1$, Eq.(1) can be rewritten in the form

$$\begin{aligned} z_{n+1} &= \frac{Az_{n-1}z_{n-2}^2+Bz_nz_{n-2}^2+Cz_n^2z_{n-1}}{\alpha z_{n-1}z_{n-2}^2+\beta z_nz_{n-2}^2+\gamma z_n^2z_{n-1}} \\ &= \frac{Az_{n-1}z_{n-2}^2}{\alpha z_{n-1}z_{n-2}^2+\beta z_nz_{n-2}^2+\gamma z_n^2z_{n-1}}+\frac{Bz_nz_{n-2}^2}{\alpha z_{n-1}z_{n-2}^2+\beta z_nz_{n-2}^2+\gamma z_n^2z_{n-1}} \\ &\quad +\frac{Cz_n^2z_{n-1}}{\alpha z_{n-1}z_{n-2}^2+\beta z_nz_{n-2}^2+\gamma z_n^2z_{n-1}} \\ &\leq \frac{A}{\alpha}+\frac{B}{\beta}+\frac{C}{\gamma}:=\widetilde{m}. \end{aligned}$$

That is

$$x_n \geq \frac{1}{\widetilde{m}}:=m \quad \text{for all } n\geq 1,$$

and this completes the proof.

Theorem 2 Every solution of Eq.(1) is bounded and persists.

proof. Let $\{x_n\}_{n=-2}^{\infty}$be solution of Eq.(1). Then

$$\begin{aligned} x_{n+1} &= \frac{\alpha x_n^2+\beta x_{n-1}x_n+\gamma x_{n-2}^2}{Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2} \\ &\leq \frac{\max\{\alpha,\beta,\gamma\}\left(x_n^2+x_{n-1}x_n+x_{n-2}^2\right)}{\min\{A,B,C\}\left(x_n^2+x_{n-1}x_n+x_{n-2}^2\right)}\leq\frac{\max\{\alpha,\beta,\gamma\}}{\min\{A,B,C\}}. \end{aligned}$$

Similarly it is easy to see that

$$x_n \geq \frac{\min\{\alpha,\beta,\gamma\}}{\max\{A,B,C\}}.$$

Thus we get

$$0<m:=\frac{\min\{\alpha,\beta,\gamma\}}{\max\{A,B,C\}}\leq x_n\leq\frac{\max\{\alpha,\beta,\gamma\}}{\min\{A,B,C\}}:=M<\infty.$$

Therefore every solution of Eq.(1) is bounded and persists. Hence the result holds.

4. Global attractor of the equilibrium points of Eq.(1)

In this section we study the global attractor of the equilibrium points of Eq.(1).

Lemma 2 For any partial order of the quotients $\frac{\alpha}{A}$, $\frac{\beta}{B}$ and $\frac{\gamma}{C}$, the function $f(u,v,w)$ defined by the equation (12) has the monotonicity behavior in at least one or two of its arguments.

proof. The proof follows by some direct substitutions and will be omitted.

Remark 1 It follows from Eq.(1) when

$$\frac{\alpha}{A}=\frac{\beta}{B}=\frac{\gamma}{C},$$

that

$$x_n=\sigma, \quad n\geq -2 \quad \text{for some constant } \sigma.$$

Whenever the quotients $\frac{\alpha}{A}$, $\frac{\beta}{B}$ and $\frac{\gamma}{C}$ are not equal we get the following result.

Theorem 3 The equilibrium point $\overline{x}$ is global attractor of Eq.(1) if one of the following statements holds

1. $\alpha B\geq A\beta$, $\beta C\geq \gamma B$ and

$$\begin{aligned}&(AB)^2\left(\gamma^3+\gamma C^2\overline{x}^2+A\gamma^2\overline{x}+AC\gamma\overline{x}^2+C\gamma^2\overline{x}\right)\\ \geq\ &\left[C^2\overline{x}\left(\alpha B+A\beta\right)\right]\left[AB\left(\alpha+\beta+C\overline{x}\right)+C\left(\alpha B+A\beta\right)\right]. \qquad (17)\end{aligned}$$

2. $\alpha B\geq A\beta$, $\alpha C\leq A\gamma$ and

$$\begin{aligned}&\frac{C\gamma^2+C\overline{x}\left(A+C\right)\left(\gamma+\overline{x}\left(A+C\right)\right)}{\left(A+C\right)^2}\\ \geq\ &\frac{\left(B\alpha+A\beta\right)\left[\left(A+B\right)\overline{x}+\left(\alpha+\gamma\right)\right]+AB\left(\alpha+\gamma\right)\overline{x}}{AB}. \qquad (18)\end{aligned}$$

3. $\alpha B\geq A\beta$, $\beta C\leq \gamma B$ and

$$\frac{\gamma^2+C^2\overline{x}^2+C\gamma\overline{x}}{C}\geq\frac{\left(B\alpha+A\beta\right)\left[\left(A+B\right)\overline{x}+\left(\alpha+\gamma\right)\right]+AB\left(\alpha+\gamma\right)\overline{x}}{AB}. \qquad (19)$$

4. $\alpha B\leq A\beta$, $\alpha C\geq A\gamma$ and

$$\begin{aligned}&\frac{\gamma\beta^2+\gamma\overline{x}\left(A+B\right)\left[\overline{x}\left(A+B\right)+\beta\right]}{\left(A+B\right)^2}\\ \geq\ &\frac{\overline{x}\left(C\alpha+A\gamma\right)}{\left(AC\right)^2}\left[\left(AC\right)\left(\alpha+\beta+\left(A+C\right)\overline{x}\right)+\left(A+C\right)\left(C\alpha+A\gamma\right)\right]. \qquad (20)\end{aligned}$$

5. $\alpha B\leq A\beta$, $\beta C\geq \gamma B$ and

$$\frac{\gamma\beta^2+\gamma\overline{x}\left(A+B\right)\left(\beta+\overline{x}\left(A+B\right)\right)}{\left(A+B\right)^2}\geq\frac{\gamma\overline{x}}{C^2}\left[C\left(\alpha+\beta\right)+\left(A+C\right)\left(C\overline{x}+\gamma\right)\right]. \qquad (21)$$

6. $\alpha B\leq A\beta$, $\beta C\leq \gamma B$ and

$$BA\left(\gamma^2+C\gamma\overline{x}+C^2\overline{x}^2+\left(\alpha-\gamma\right)\left(\gamma+C\overline{x}\right)\right)\geq C\overline{x}\left(A+B\right)\left(\alpha B+A\beta\right). \qquad (22)$$

proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of Eq.(1) and let f be a function defined by Eq.(12). We will prove the theorem in the first case and the proofs of the other cases are similar and will be omittied.

Assume that (17) is ture then it is easy to see that the function $f(u,v,w)$ is non-decreasing in its first argument and non-increasing in its third argument. Thus from Eq.(1) we see that

$$\begin{aligned} x_{n+1} &= \frac{\alpha x_n^2+\beta x_{n-1}x_n+\gamma x_{n-2}^2}{Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2} \le \frac{\alpha x_n^2+\beta x_{n-1}x_n+\gamma(0)}{Ax_n^2+Bx_{n-1}x_n+C(0)} \\ &= \frac{\alpha x_n^2}{Ax_n^2+Bx_{n-1}x_n}+\frac{\beta x_{n-1}x_n}{Ax_n^2+Bx_{n-1}x_n} \le \frac{\alpha}{A}+\frac{\beta}{B}. \end{aligned}$$

Then

$$x_n \le \frac{\alpha}{A}+\frac{\beta}{B}=H \quad \text{for all} \quad n\ge 1. \tag{23}$$

$$\begin{aligned} x_{n+1} &= \frac{\alpha x_n^2+\beta x_{n-1}x_n+\gamma x_{n-2}^2}{Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2} \ge \frac{\alpha(0)^2+\beta(0)(H)+\gamma(H)^2}{A(0)^2+B(0)(H)+C(H)^2} \\ &= \frac{\gamma(H)^2}{C(H)^2}=\frac{\gamma}{C}. \end{aligned}$$

Then

$$x_n \ge \frac{\gamma}{C}=h \ \text{ for all } \ n\ge 1. \tag{24}$$

From Eq.(23) and (24) we see that

$$h=\frac{\gamma}{C}\le x_n\le \frac{\alpha}{A}+\frac{\beta}{B}=H \ \text{ for all } \ n\ge 1.$$

It follows by the Method of Full Limiting Sequences that there exist solutions $\{I_n\}_{n=-\infty}^{\infty}$ and $\{S_n\}_{n=-\infty}^{\infty}$ of Eq.(1) with

$$\frac{\gamma}{C}\le I=I_0=\lim_{n\longrightarrow\infty}\inf\ x_n\le \lim_{n\longrightarrow\infty}\sup\ x_n=S_0=S\le\frac{\alpha}{A}+\frac{\beta}{B},$$

where

$$I_n,S_n\in[I,S], \quad n=0,-1,...$$

It suffices to show that $I=S$. Now it follows from Eq.(1) that

$$\begin{aligned} I &= \frac{\alpha I_{-1}^2+\beta I_{-1}I_{-2}+\gamma I_{-3}^2}{AI_{-1}^2+BI_{-1}I_{-2}+CI_{-3}^2}\ge f(I,I_{-2},S) \\ &= \frac{\alpha I^2+\beta I_{-2}I+\gamma S^2}{AI^2+BI_{-2}I+CS^2}\ge\frac{(\alpha+\beta)I^2+\gamma S^2}{AI^2+BSI+CS^2}, \end{aligned}$$

and so

$$AI^3+BSI^2+CS^2I\ge(\alpha+\beta)I^2+\gamma S^2.$$

Then we obtain

$$AI^3S+BS^2I^2+CS^3I\ge(\alpha+\beta)I^2S+\gamma S^3,$$

or

$$(\alpha+\beta)I^2S+\gamma S^3-AI^3S-CS^3I\le BS^2I^2. \tag{25}$$

Similarly it is easy to see from Eq.(1) that

$$(\alpha+\beta)IS^2+\gamma I^3-AIS^3-CSI^3\ge BS^2I^2. \tag{26}$$

Therefore it follows from Eqs.(25) and (26) that

$$\left((\alpha+\beta)IS^2-(\alpha+\beta)SI^2\right)+\left(\gamma I^3-\gamma S^3\right)+\left(AI^3S-AIS^3\right)+\left(CS^3I-CSI^3\right)\ge 0$$

$$\begin{aligned}
&\Longrightarrow\quad (\alpha+\beta)\, IS\,(S-I)+(A-C)\, IS\left(I^2-S^2\right)+\left(\gamma I^3-\gamma S^3\right)\geq 0\\
&\Longrightarrow\quad (\alpha+\beta)\, IS\,(S-I)+(A-C)\, IS\left(I^2-S^2\right)+\left(\gamma I^3-\gamma I^2 S\right)+\left(\gamma I^2 S-\gamma S^3\right)\geq 0,
\end{aligned}$$

or

$$(I-S)\left[\gamma I^2-(\alpha+\beta)\, IS\right]+(\gamma S+(A-C)\, SI)\left(I^2-S^2\right)\geq 0,$$

or equivalently

$$(I-S)\left[\gamma I^2-(\alpha+\beta)\, IS+(\gamma S+(A-C)\, SI)\,(I+S)\right]\geq 0,$$

and so $I\geq S$ if

$$\gamma I^2+(\gamma S+ASI)\,(I+S)\geq(\alpha+\beta)\, IS+CIS\,(I+S)\,. \tag{27}$$

Now it follows from (17) that

$$\begin{aligned}
&\frac{\gamma^3+\gamma C^2\overline{x}^2+A\gamma^2\overline{x}+AC\gamma\overline{x}^2+C\gamma^2\overline{x}}{C^2}\\
\geq\ &\left(\frac{\overline{x}\,(\alpha B+A\beta)}{(AB)^2}\right)\left[AB\,(\alpha+\beta+C\overline{x})+C\,(\alpha B+A\beta)\right]\\
\Leftrightarrow\ &\frac{\gamma^3}{C^2}+\gamma\overline{x}^2+\frac{\gamma^2\overline{x}}{C}+\frac{A\gamma\overline{x}^2}{C}+\frac{A\gamma^2\overline{x}}{C^2}\\
\geq\ &\left(\frac{\overline{x}\,(\alpha B+A\beta)}{AB}\right)\left(\frac{AB\,(\alpha+\beta)+CAB\overline{x}+C\,(\alpha B+A\beta)}{AB}\right)\\
\Leftrightarrow\ &\gamma\left(\frac{\gamma}{C}\right)^2+\left(\frac{A\gamma\overline{x}}{C}+\gamma\overline{x}\right)\left(\frac{\gamma}{C}+\overline{x}\right)\\
\geq\ &\left(\frac{(\alpha B+A\beta)\,\overline{x}}{AB}\right)\left(\alpha+\beta+C\left(\overline{x}+\frac{(\alpha B+A\beta)}{AB}\right)\right).
\end{aligned}$$

Now

$$\gamma\left(\frac{\gamma}{C}\right)^2+\left(\frac{A\gamma\overline{x}}{C}+\gamma\overline{x}\right)\left(\frac{\gamma}{C}+\overline{x}\right)\leq\gamma I^2+(AIS+\gamma S)\,(I+S)\,, \tag{28}$$

and

$$\left(\frac{(\alpha B+A\beta)\,\overline{x}}{AB}\right)\left(\alpha+\beta+C\left(\overline{x}+\frac{(\alpha B+A\beta)}{AB}\right)\right)\geq(\alpha+\beta)\, IS+CSI\,(I+S)\,. \tag{29}$$

Then we see from (28) and (29) that

$$\gamma I^2+(\gamma S+ASI)\,(I+S)\geq(\alpha+\beta)\, IS+CIS\,(I+S)\,,$$

which yields that (27) is satisfied and then the proof is complete.

5. Rate of Convergence of Eq.(1)

In this section we will recognize the rate of convergence of the solutions that converge to the unique positive equilibrium point of Eq.(1).

Theorem 4 Assume that (15) holds. Then all solutions of Eq.(1) which are eventually approach to the equilibrium satisfy the following.

$$\lim_{n\to\infty}\frac{x_{n+1}-\bar{x}}{x_n-\bar{x}}=\lambda_i,$$

where λ_i are the roots of Eq.(14).

proof. We have

$$\begin{aligned}
x_{n+1}-\bar{x} &= \frac{A\beta x_n x_{n-1}-A\beta x_n^2+A\gamma x_{n-2}^2-A\gamma x_n^2+B\alpha x_n^2-B\alpha x_n x_{n-1}}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)} \\
&+\frac{B\gamma x_{n-1}^2-B\gamma x_n x_{n-1}+C\alpha x_n^2-C\alpha x_{n-2}^2+C\beta x_n x_{n-1}-C\beta x_{n-2}^2}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)} \\
&= \frac{(B\alpha-A\beta)\,x_n^2-(B\alpha-A\beta)\,x_n\bar{x}+(B\alpha-A\beta)\,x_n\bar{x}}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)} \\
&+\frac{(C\alpha-A\gamma)\,x_n^2-(C\alpha-A\gamma)\,x_n\bar{x}+(C\alpha-A\gamma)\,x_n\bar{x}}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)} \\
&+\frac{(A\gamma-C\alpha)\,x_{n-2}^2-(C\alpha-A\gamma)\,x_n x_{n-2}-(A\gamma-C\alpha)\,x_{n-2}\bar{x}}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)} \\
&+\frac{(A\gamma-C\alpha)\,x_{n-2}\bar{x}+(C\alpha-A\gamma)\,x_n x_{n-2}+(B\gamma-C\beta)\,x_{n-2}^2}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)} \\
&+\frac{(B\gamma-C\beta)\,x_{n-2}\bar{x}-(B\gamma-C\beta)\,x_{n-2}\bar{x}+(C\beta-B\gamma)\,x_n x_{n-2}}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)} \\
&+\frac{(C\beta-B\gamma)\,x_n x_{n-1}-(C\beta-B\gamma)\,x_n\bar{x}+(C\beta-B\gamma)\,x_n\bar{x}}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)} \\
&+\frac{(A\beta-B\alpha)\,x_n x_{n-1}-(C\beta-B\gamma)\,x_n x_{n-2}}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)} \\
&+\frac{(A\beta-B\alpha)\,x_n\bar{x}-(A\beta-B\alpha)\,x_n\bar{x}}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)} \\
&= \frac{((B\alpha-A\beta)+(C\alpha-A\gamma))\,x_n}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)}\,(x_n-\bar{x}) \\
&+\frac{((C\alpha-A\gamma)+(C\beta-B\gamma))\,x_{n-2}}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)}\,(x_n-\bar{x}) \\
&+\frac{((C\beta-B\gamma)+(A\beta-B\alpha))\,x_n}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)}\,(x_{n-1}-\bar{x}) \\
&+\frac{((A\gamma-C\alpha)+(B\gamma-C\beta))\,x_{n-2}}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)}\,(x_{n-2}-\bar{x}) \\
&+\frac{((B\gamma-C\beta)+(A\gamma-C\alpha))\,x_n}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)}\,(x_{n-2}-\bar{x})
\end{aligned}$$

Put

$$e_n=x_n-\bar{x},e_{n-1}=x_{n-1}-\bar{x},e_{n-2}=x_{n-2}-\bar{x}.$$

Then we obtain

$$e_{n+1}-\gamma_n e_n-\delta_n e_{n-1}-\zeta_n e_{n-2}=0,$$

where

$$\gamma_n=\frac{((B\alpha-A\beta)+(C\alpha-A\gamma))\,x_n+((C\alpha-A\gamma)+(C\beta-B\gamma))\,x_{n-2}}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)},$$

$$\delta_n=\frac{((C\beta-B\gamma)+(A\beta-B\alpha))\,x_n}{\left(Ax_n^2+Bx_{n-1}x_n+Cx_{n-2}^2\right)(A+B+C)},$$

and

$$\zeta_n = \frac{((A\gamma - C\alpha) + (B\gamma - C\beta))\, x_{n-2} + ((B\gamma - C\beta) + (A\gamma - C\alpha))\, x_n}{\left(Ax_n^2 + Bx_{n-1}x_n + Cx_{n-2}^2\right)(A+B+C)}.$$

As the positive equilibrium is a global attractor, we get

$$\begin{aligned}\lim_{n\to\infty}\gamma_n &= \frac{(B\alpha - A\beta) + 2\,(C\alpha - A\gamma) + (C\beta - B\gamma)}{\bar{x}\,(A+B+C)^2}\\ &= \frac{(\alpha B + 2\alpha C + \beta C) - (A\beta + 2A\gamma + \gamma B)}{(\alpha+\beta+\gamma)\,(A+B+C)},\end{aligned}$$

$$\lim_{n\to\infty}\delta_n = \frac{(C\beta - B\gamma) + (A\beta - B\alpha)}{\bar{x}\,(A+B+C)^2} = \frac{(A\beta + C\beta) - (\alpha B + \gamma B)}{(A+B+C)\,(\alpha+\beta+\gamma)},$$

and

$$\begin{aligned}\lim_{n\to\infty}\zeta_n &= \frac{2\,(A\gamma - C\alpha) + 2\,(B\gamma - C\beta)}{\bar{x}\,(A+B+C)^2}\\ &= \frac{2\gamma\,(A+B) - 2C\,(\alpha+\beta)}{(A+B+C)\,(\alpha+\beta+\gamma)}.\end{aligned}$$

Thus the limiting equation of Eq.(1) is the linearized equation (13). Then the result follows as an immediate consequence of Poincare's theorem C and the proof is complete.

6. **Periodic solutions**

In this section we present some results for the existence of minimal period-two solutions of Equation (1).

Theorem 5 Eq.(1) has a positive periodic solution of prime period two if and only if

$$(A+C-B)\,(\beta - (\alpha+\gamma)) > 4B\,(\alpha+\gamma)\,. \tag{30}$$

proof. First assume that there exists a periodic solution of prime period two $\{..., p,\ q,\ p,\ q, ...\}$ of Eq.(1). From (30) we get

$$p = \frac{(\alpha+\gamma)\,q^2 + \beta pq}{(A+C)\,q^2 + Bpq},$$

and

$$q = \frac{(\alpha+\gamma)\,p^2 + \beta pq}{(A+C)\,p^2 + Bpq}.$$

Then

$$(A+C)\,pq + Bp^2 = (\alpha+\gamma)\,q + \beta p, \tag{31}$$

and

$$(A+C)\,pq + Bq^2 = (\alpha+\gamma)\,p + \beta q. \tag{32}$$

Subtracting (32) from (31)

$$B(p^2 - q^2) = (\beta - (\alpha+\gamma))\,(p-q).$$

Since $p \neq q$, it follows that

$$p + q = \frac{\beta - (\alpha+\gamma)}{B}. \tag{33}$$

Substituting (33) in (31) we obtain

$$\left(B-\left(C+A\right)\right)p^{2}+\frac{\left(B\left(C+A\right)-B\right)\left(\beta-\left(\alpha+\gamma\right)\right)}{B}p-\frac{\left(\beta-\left(\alpha+\gamma\right)\right)\left(\alpha+\gamma\right)}{B}=0, \tag{34}$$

from which

$$p_{\pm}=\frac{\left(\beta-\left(\alpha+\gamma\right)\right)}{2B}\left(1\pm\sqrt{1-4B\left(\alpha+\gamma\right)/\left(A+C-B\right)\left(\beta-\left(\alpha+\gamma\right)\right)}\right)$$

and so

$$1-4B\left(\alpha+\gamma\right)/\left(A+C-B\right)\left(\beta-\left(\alpha+\gamma\right)\right)>0,$$

or

$$4B\left(\alpha+\gamma\right)<\left(A+C-B\right)\left(\beta-\left(\alpha+\gamma\right)\right).$$

Since $A+C-B$ and $\beta-\alpha-\gamma$, have the same sign then (30) holds.

Second suppose that the condition (30) is true. We will show that Eq.(1) has positive prime period two solutions. Assume that p and q are distinct positive real numbers.

Now choose $x_{-2}=p$, $x_{-1}=q$ and $x_0=p$. It is easy to prove that

$$x_1=x_{-1} \quad \text{and} \quad x_2=x_0.$$

Then it follows by induction that

$$x_{2n}=p \quad \text{and} \quad x_{2n+1}=q \quad \text{for all} \quad n\geq -1.$$

Thus Eq.(1) has the positive prime period two solution

$$\ldots,\ p,\ q,\ p,\ q,\ldots$$

where p and q are the distinct roots of the quadratic equation (34) and the proof is completed.

Theorem 6 Assume that $\{x_n\}_{n=-2}^{\infty}$be solution of Eq.(1) with $x_{N+2}>x_N>\bar{x}>x_{N-1}>x_{N+1}>x_{N+3}$ for some $N\geq 0$. Then $\{x_n\}_{n=-2}^{\infty}$ converges to a two cycle solution of Eq.(1) .

proof. We have

$$x_{N+4}=f\left(x_{N+3},x_{N+2},x_{N+1}\right)>f\left(x_{N+1},x_N,x_{N-1}\right)=x_{N+2},$$

and

$$x_{N+5}=f\left(x_{N+4},x_{N+3},x_{N+2}\right)<f\left(x_{N+2},x_{N+1},x_N\right)=x_{N+3}.$$

By induction on $N\geq 0$ one can easy establishes the monotonic of the two sequences $\{x_{N+2k}\}_{n=-2}^{\infty}$, $\{x_{N+2k}\}_{n=-2}^{\infty}$ and since $\{x_n\}_{n=-2}^{\infty}$ is bounded, the sequence $\{x_{N+2k}\}$ is increasing and bounded above by M and the sequence $\{x_{N+2k}\}$ is decreasing and bounded below by m. i.e

$$x_{N+2k}\leq M \quad \text{and} \quad x_{N+2k+1}\geq m \quad \text{for} \quad k=0,1,\ldots.$$

Therefore, both subsequences $\{x_{N+2k}\}_{n=-2}^{\infty}$, $\{x_{N+2k}\}_{n=-2}^{\infty}$ converges to, say ψ and φ respectively, that is

$$\lim_{\kappa\to\infty}x_{N+2\kappa}=\psi \quad \text{and} \quad \lim x_{\substack{N+2\kappa+1\\ \kappa\to\infty}}=\phi.$$

This completes the proof.

7. Numerical Examples

Example 1 We assume $x_{-2} = 4, x_{-1} = 5, x_0 = 0.5, \alpha = 2, \beta = 9, \gamma = 6,$ $A = 12, B = 7, C = 9$.(See Figure 1).
Example 2 (See Figure 2) since $x_{-2} = 0.5, x_{-1} = 0.6, x_0 = 0.6, \alpha = 0.9,$ $\beta = 3.9, \gamma = 3, A = 8, B = 1.5, C = 2.$
Example 3 (See Figure 3) since $x_{-2} = 5, x_{-1} = 1, x_0 = 5, \alpha = 0.02, \beta = 5,$ $\gamma = 0.9, A = 1.2, B = 2, C = 0.8.$
Example 4 We consider $x_{-2} = 9, x_{-1} = 5, x_0 = 9, \alpha = 0.05, \beta = 15,$ $\gamma = 0.3, A = 2, B = 0.5, C = 2$.(See Figure 4).
Example 5 (See Figure 5) since $x_{-2} = 3, x_{-1} = 2, x_0 = 1, \alpha = 5, \beta = 0.8,$ $\gamma = 3, A = 2, B = 0.4, C = 0.02.$
Example 6 (See Figure 6) since $x_{-2} = 15, x_{-1} = 5, x_0 = 1, \alpha = 13, \beta = 15,$ $\gamma = 0.2, A = 2, B = 0.5, C = 2.$

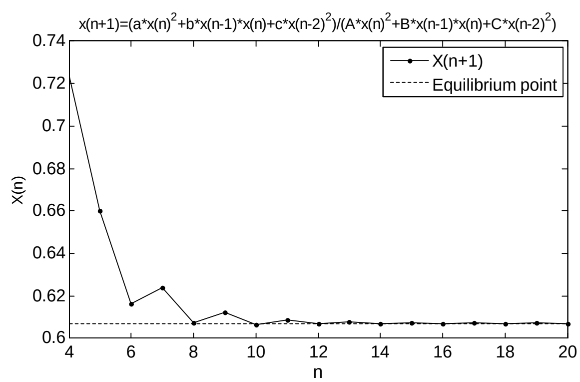

Figure (1)

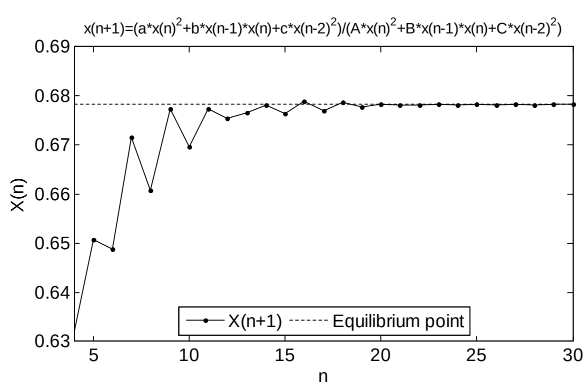

Figure (2)

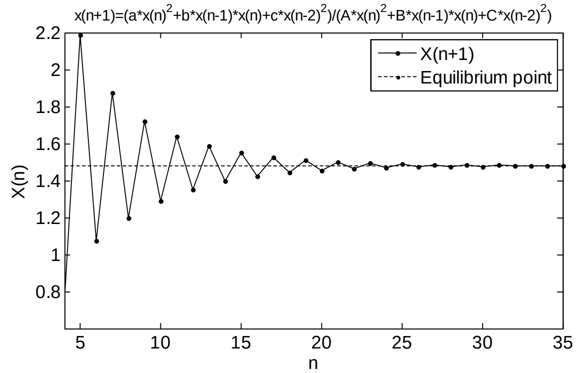

Figure (3)

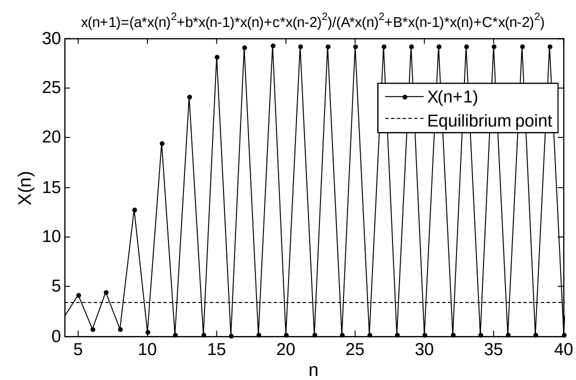

Figure (4)

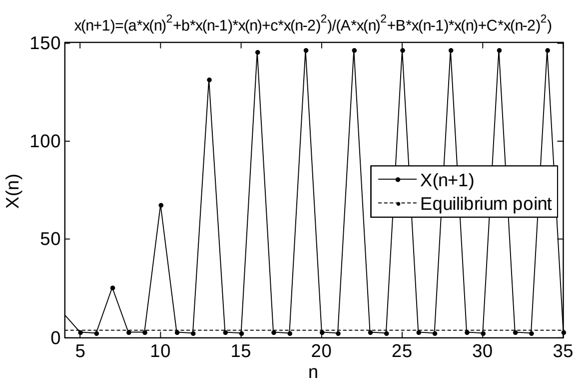

Figure (5)

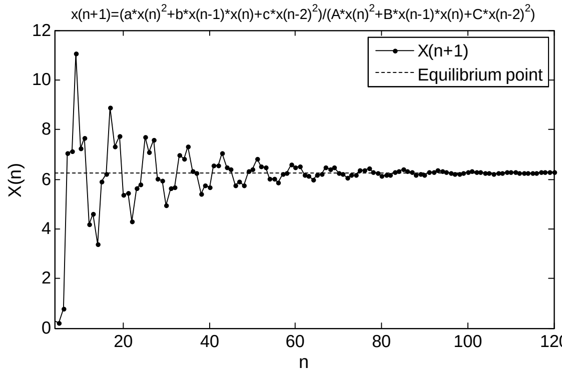

Figure (6)

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