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ON COEFFICIENT BOUNDS FOR M-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS DEFINED BY SUBORDINATIONS IN TERM OF MODIFIED SIGMOID FUNCTION

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Abstract. In this work, the authors investigated the unpredictable coefficient bounds for m-fold symmetric bi-univalent functions defined by subordinations in term of modified sigmoid function. The early few coefficient bounds of $|a_{m+1}|$ and $|a_{2m+1}|$ for the new subclasses of analytic functions were obtained.

1. Introduction and Some Basic Definitions

Special functions are composed of large number of highly interconnected processing elements (neurons) working together to solve a specific task. They play a vital role in univalent function theory. These functions have been overshadowed by other fields like algebra, differential equations, topology, functional analysis and real analysis, among others, because they work in the same way the brain does. An example of such functions is the activation function. Activation function increases the size of hypothesis space that a network can represent. The most popular activation function is the sigmoid function. A sigmoid function is a bounded differentiable real function that is defined for all real input values and has a positive derivative at each point. It is a monotone function, and very useful in compressing outputs to lie between 0 and 1.

The sigmoid function is defined mathematically as

$$\Phi(s) = \frac{1}{1+e^{-s}}, \quad s \geq 0. \tag{1}$$

Fadipe-Joseph et al. [1] studied the modified sigmoid function

$$\Phi(z) = \frac{2}{1+e^{-z}} \tag{2}$$

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and obtained another series of modified sigmoid function as

$$\begin{aligned}\Phi(z) &= 1+\sum_{m=1}^{\infty}\frac{(-1)^m}{2^m}\left(\frac{(-1)^n}{n!}z^n\right)^m, \\ &= 1+\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\cdots. \end{aligned} \tag{3}$$

See details in [1], [2], [3] and [4].

Let Γ denotes the class of functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k, \tag{4}$$

which are analytic in the open unit disc $\mathbb{E}=\{z: z\in\mathbb{C}\,\text{and}\,|z|<1\}$. Recall that, S denotes the subclass of functions in Γ which are univalent in $\mathbb{E}$. Also, let $S^*(\beta)$ and $K(\beta)$ be two subclasses of S consisting from starlike and convex functions of order $\beta\,(0\leq\beta<1)$ which their geometric conditions satisfy $Re\left\{\frac{zf'(z)}{f(z)}\right\}>\beta$ and $Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\beta$, respectively. The two subclasses of functions lying in a certain domain starlike with respect to 1 in the right half plane.

Two functions, $f(z)$ and $g(z)$ are said to be subordinate to each other, written $f(z)\prec g(z)$, if there exists an analytic function ω with $\omega(0)=0$ and $|\omega(z)|<1$ for $z\in\mathbb{E}$ such that $f(z)=g(\omega(z))$.

Ma and Minda [5] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1+\frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. They considered an analytic function φ with positive real part in the unit disc $\mathbb{E}$, $\varphi(0)=1$, $\varphi'(0)>0$ and φ map $\mathbb{E}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The classes of starlike and convex functions defined by [5] consists of function $f\in\Gamma$ satisfying the subordination $\frac{zf'(z)}{f(z)}\prec\varphi(z)$ and $1+\frac{zf''(z)}{f'(z)}\prec\varphi(z)$. It is usually known that every univalent function has an inverse function defined by

$$f^{-1}(f(z))=z, \quad (z\in\mathbb{E}), \tag{5}$$

and

$$f(f^{-1}(\omega))=\omega, \quad \left(|\omega|<r_0(f);\, r_0(f)\geq\frac{1}{4}\right), \tag{6}$$

then we say that the function $f\in\Gamma$ is bi-univalent in $\mathbb{E}$. The inverse function has an analytic function of the form

$$f^{-1}(\omega)=\omega-a_2\omega^2+(2a_2^2-a_3)\omega^3-(5a_2^3-5a_2a_3+a_4)\omega^4+\cdots. \tag{7}$$

A function f is bi-starlike or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex and they are denoted by $S^*_\sigma(\varphi)$ and $K_\sigma(\varphi)$.

Lewin [6] examined the class of bi-univalent functions σ and obtained a bound $|a_2|\leq 1.5$. Brannan and Taha [7] found non-sharp inequalities on the early few coefficients $|a_2|$ and $|a_3|$ of the functions in the class $S^*_\sigma(\beta)$ and $K_\sigma(\beta)$. In the recent time, researchers have worked extensively on bi-univalent functions in different directions with different perspectives and their interesting results are too numerous to discuss. Just to mention but few [7], [8], [9], [10] and [11].

Furthermore, Strivastava et al. [12] considered examples for the class of m-fold symmetric bi-univalent functions and as an application found the the coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ for a new subclass of functions. For each function $f \in S$, the function $h(z) = \sqrt[m]{f(z^m)} \quad (z \in \mathbb{E}, m \in \mathbb{N})$ is univalent and maps the unit disc $\mathbb{E}$ into a region with m-fold symmetry.

A function is said to be m-fold symmetric (see [13]) if it has the normallized form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad z \in \mathbb{E}, \tag{8}$$

and its inverse is given as follows:

$$\begin{aligned} g(\omega) = \omega - a_{m+1}\omega^{m+1} + \big[(m+1)a_{m+1}^2 - a_{2m+1}\big]\omega^{2m+1} \\ - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]\omega^{3m+1} + \cdots . \end{aligned} \tag{9}$$

Setting $m = 1$ in (9), it coincides with (7) of the class σ. Also, we denote by P, the class of analytic functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ such that $Re(p(z)) > 0$ in $\mathbb{E}$. In view of Pommenrenke [13], the m-fold symmetric functions in the class P is of the form

$$p(z) = 1 + c_m z^m + c_{2m} z^{2m} + c_{3m} z^{3m} + \cdots .$$

Motivated by the earlier works by [2], [12] and [14], this paper aims at introducing and studying several new subclasses of bi-univariate Ma-Minda functions in which both f and f^{-1} are m-fold symmetric analytic functions with derivative P and modified sigmoid function. The early few coefficient bounds of $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these new subclasses were obtained.

For the purpose of our results, the below definitions are necessary.

Definition 1. A function $f(z)$, given by (8) is said to be in the class $H_{\sigma,m}(\Phi)$ if the following conditions are satisfied:

$$f \in \sigma_m,\ f'(z) \prec \Phi(z) \,\text{and}\, g'(\omega) \prec \Phi(\omega),\ g(\omega) = f^{-1}(\omega),$$

where the function g is defined by (9).

Definition 2. A function $f(z)$, given by (8), is said to be in the class $M_{\sigma,m}(\lambda, \Phi)$ if the following conditions are satisfied:

$$f \in \sigma_m,\ (1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \Phi(z) \tag{10}$$

and

$$(1-\lambda)\frac{\omega g'(\omega)}{g(\omega)} + \lambda\left(1 + \frac{\omega g''(\omega)}{g'(\omega)}\right) \prec \Phi(\omega)\ g(\omega) = f^{-1}(\omega), \tag{11}$$

where the function g is defined by (9).

Definition 3. A function $f(z)$, given by (8), is said to be in the class $N_{\sigma,m}(\lambda, \Phi)$, if the following conditions are satisfied:

$$f \in \sigma_m,\ \frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)} \prec \Phi(z),\ \lambda \geq 0 \tag{12}$$

and

$$\frac{\omega g'(\omega)}{g(\omega)} + \frac{\lambda \omega^2 g''(\omega)}{g(\omega)} \prec \Phi(\omega),\ g(\omega) = f^{-1}(\omega), \tag{13}$$

where the function g is defined by (9).

2. Main Results

Theorem 1. *Let $f(z)$, given by* (8), *be the class $H_{\sigma,m}(\Phi)$. Then*

$$|a_{m+1}| \leq \frac{1}{\sqrt{(m+1)(6m+5)}} \tag{14}$$

and

$$|a_{2m+1}| \leq \left(\frac{1}{2(2m+1)} + \frac{1}{8(m+1)}\right). \tag{15}$$

Proof. Let $f \in H_{\sigma,m}(\Phi)$ and $g = f^{-1}$. Then, there are analytic functions $u, v : \mathbb{U} \to \mathbb{V}$, with $u(0) = v(0) = 0$ satisfying

$$f'(z) = \Phi(u(z)) \, \text{and} \, g'(\omega) = \Phi(v(\omega)). \tag{16}$$

By the definition of the functions p_1 and p_2:

$$p_1(z) = \frac{1+u(z)}{1-u(z)} = 1 + c_m z^m + c_{2m} z^{2m} + c_{3m} z^{3m} + \cdots \tag{17}$$

$$p_2(z) = \frac{1+v(z)}{1-v(z)} = 1 + b_m z^m + b_{2m} z^{2m} + b_{3m} z^{3m} + \cdots. \tag{18}$$

Or equivalently,

$$u(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_m z^m + \left(c_{2m} - \frac{c_m^2}{2}\right) z^{2m} + \cdots\right) \tag{19}$$

and

$$v(z) = \frac{p_2(z)-1}{p_2(z)+1} = \frac{1}{2}\left(b_m z^m + \left(b_{2m} - \frac{b_m^2}{2}\right) z^{2m} + \cdots\right). \tag{20}$$

Then, p_1 and p_2 are analytic in $\mathbb{U}$ with $p_1(0) = p_2(0) = 1$. Since $u, v : \mathbb{U} \to \mathbb{V}$, the functions p_1 and p_2 have positive real part in $\mathbb{U}$ and $|c_k| \leq 2$, $|b_k| \leq 2$, for all k.

From (16), (19) and (20),

$$f'(z) = \Phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) \tag{21}$$

and

$$g'(\omega) = \Phi\left(\frac{p_2(\omega)-1}{p_2(\omega)+1}\right). \tag{22}$$

Using (19) and (20) together with Equation (3), we get

$$\Phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{4} c_m z^m + \frac{1}{4}\left(c_{2m} - \frac{c_m^2}{2}\right) z^{2m} + \cdots \tag{23}$$

and

$$\Phi\left(\frac{p_2(z)-1}{p_2(z)+1}\right) = 1 + \frac{1}{4} b_m z^m + \frac{1}{4}\left(b_{2m} - \frac{b_m^2}{2}\right) z^{2m} + \cdots. \tag{24}$$

Since

$$f'(z) = 1 + (m+1)a_{m+1} z^m + (2m+1)a_{2m+1} z^{2m} + \cdots,$$

and

$$g'(\omega) = 1 - (m+1)a_{m+1}\omega^m + (2m+1)\big[(m+1)a_{m+1}^2 - a_{2m+1}\big]\omega^{2m} + \cdots.$$

It follows from (21), (22), (23) and (24) that

$$\begin{aligned}(m+1)a_{m+1} &= \frac{c_m}{4}, &(25)\\ (2m+1)a_{2m+1} &= \frac{1}{4}\left(c_{2m}-\frac{c_m^2}{2}\right), &(26)\end{aligned}$$

$$-(m+1)a_{m+1}=\frac{b_m}{4}, \tag{27}$$

and

$$(2m+1)[(m+1)a_{m+1}^2-a_{2m+1}]=\frac{1}{4}\left(b_{2m}-\frac{b_m^2}{2}\right). \tag{28}$$

From (25) and (27), we have

$$c_m=-b_m. \tag{29}$$

Also, from (26), (27), (28) and (29), we obtain

$$a_{m+1}^2=\frac{c_{2m}+b_{2m}}{4(m+1)(6m+5)}. \tag{30}$$

Thus, in view of the inequalities $|c_{2m}|\leq 2$ and $|b_{2m}|\leq 2$ for $|a_{m+1}|$, we get the desired result as in (14). By subtracting (28) from (26) and further computations using (25) and (29) yields

$$a_{2m+1}=\frac{c_{2m}-b_{2m}}{8(2m+1)}+\frac{c_m^2}{32(m+1)}, \tag{31}$$

which in view of the inequalities $|c_{2m}|\leq 2$, $|c_m|\leq 2$ and $|b_{2m}|\leq 2$ for $|a_{2m+1}|$, we get the desired result as in (15). This completes the proof of Theorem 1. $\square$

For the case when $m = 1$, we have

Corollary 1. *Let $f(z)$, given by (8), be the class $H_{\sigma,1}(\Phi)$. Then*

$$|a_2| \leq \frac{1}{\sqrt{22}} \tag{32}$$

and

$$|a_3| \leq \frac{11}{48}. \tag{33}$$

Theorem 2. *Let $f(z)$ given by (8), be the class $M_{\sigma,m}(\lambda, \Phi)$, $\lambda \geq 0$. Then*

$$|a_{m+1}| \leq \frac{1}{m\sqrt{2(1+\lambda m)(3+2\lambda m)}} \tag{34}$$

and

$$|a_{2m+1}| \leq \frac{1}{4m(1+2\lambda m)} \left(1 + \frac{1+m+2\lambda m+2\lambda m^2}{2m(1+\lambda m)^2}\right). \tag{35}$$

Proof. Let $f \in M_{\sigma,m}(\lambda, \Phi)$. Then there are analytic functions $u, v : \mathbb{U} \to \mathbb{V}$, with $u(0) = v(0) = 0$ satisfying

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) = \Phi(u(z)) \tag{36}$$

and

$$(1-\lambda)\frac{\omega g'(\omega)}{g(\omega)} + \lambda\left(1 + \frac{\omega g''(\omega)}{g'(\omega)}\right) = \Phi(v(\omega)). \tag{37}$$

Since

$$\begin{aligned}(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) &= 1 + m(1+\lambda m)a_{m+1}z^m + \big[2m(1+2\lambda m)a_{2m+1} \\ &\quad - m(1+2\lambda m+\lambda m^2)a_{m+1}^2\big]z^{2m} + \cdots \end{aligned} \tag{38}$$

and

$$\begin{aligned}(1-\lambda)\frac{\omega g'(\omega)}{g(\omega)} + \lambda\left(1 + \frac{\omega g''(\omega)}{g'(\omega)}\right) &= 1 - m(1+\lambda m)a_{m+1}\omega^m + \big[m(1+2\lambda m+2m \\ &\quad + 3\lambda m^2)a_{2m+1}^2 - 2m(1+2\lambda m)a_{2m+1}\big]\omega^{2m} + \cdots . \end{aligned} \tag{39}$$

Then, from (23), (24), (36) and (37), we get

$$m(1+\lambda m)a_{m+1} = \frac{c_m}{4}, \tag{40}$$

$$2m(1+2\lambda m)a_{2m+1} - m\big[1+2\lambda m+\lambda m^2\big]a_{m+1}^2 = \frac{1}{4}\left(c_{2m} - \frac{c_m^2}{2}\right), \tag{41}$$

$$-m(1+\lambda m)a_{m+1} = \frac{b_m}{4}, \tag{42}$$

$$m(1+2\lambda m+2m+3\lambda m^2)a_{m+1}^2 - 2m(1+2\lambda m)a_{2m+1} = \frac{1}{4}\left(b_{2m} - \frac{b_m^2}{2}\right). \tag{43}$$

From (40) and (42), we obtain

$$c_m = -b_m \tag{44}$$

and also from (41), (43) and (44), we have

$$a_{m+1}^2 = \frac{c_{2m} + b_{2m}}{8m^2((1+\lambda m)(3+2\lambda m)]}. \tag{45}$$

In view of the inequalities $|c_{2m}| \leq 2$ and $|b_{2m}| \leq 2$ for $|a_{m+1}|$, we get as asserted in (34).

Next, to find the bound on $|a_{2m+1}|$, by a simple calculations using (41), (43) and (44), we get

$$a_{2m+1} = \frac{c_{2m} - b_{2m}}{16m(1+2\lambda m)} + \frac{(1+2\lambda m + m + 2\lambda m^2)c_m^2}{32m^2(1+\lambda m)^2(1+2\lambda m)} \tag{46}$$

which, in view of the inequalities $|c_m| \leq 2$ and $|b_{2m}| \leq 2$ for $|a_{2m+1}|$, we get our desired inequality as asserted in (35). □

For the case when $m = 1$, we have

Corollary 2. *Let $f(z)$ given by (8), be the class $M_{\sigma,1}(\lambda, \Phi)$, $\lambda \geq 0$. Then*

$$|a_2| \leq \frac{1}{\sqrt{2(1+\lambda)(3+2\lambda)}} \tag{47}$$

and

$$|a_3| \leq \frac{1+2\lambda+\frac{1}{2}\lambda^2}{2(1+2\lambda)(1+\lambda)^2}. \tag{48}$$

Setting $\lambda = 0$ in Corollary 2, it yields

Corollary 3. *Let $f(z)$, given by (8), be the class $M_{\sigma,1}(0, \Phi)$. Then*

$$|a_2| \leq \frac{\sqrt{6}}{6} \tag{49}$$

and

$$|a_3| \leq \frac{1}{2}. \tag{50}$$

Setting $\lambda = 1$ in Corollary 2, it gives

Corollary 4. *Let $f(z)$, given by (8), be the class $M_{\sigma,1}(1, \Phi)$. Then*

$$|a_2| \leq \frac{\sqrt{5}}{10} \tag{51}$$

and

$$|a_3| \leq \frac{7}{48}. \tag{52}$$

Theorem 3. *Let $f(z)$, given by (8), be the class $N_{\sigma,m}(\lambda, \Phi)$, $\lambda \geq 0$. Then*

$$|a_{m+1}| \leq \frac{1}{m\sqrt{2(1+2\lambda(m+1)+2(1+\lambda(m+1))^2)}} \tag{53}$$

and

$$|a_{2m+1}| \leq \frac{1}{4m(1+\lambda(1+2m))} + \frac{m+1}{8m^2(1+\lambda(m+1))^2}. \tag{54}$$

Proof. Let $f \in N_{\sigma,m}(\lambda, \Phi)$. Then there are analytic functions $u, v : \mathbb{U} \to \mathbb{V}$ with $u(0) = v(0) = 0$ satisfying

$$\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)} = \Phi(u(z)) \tag{55}$$

and

$$\frac{\omega g'(\omega)}{g(\omega)} + \frac{\lambda \omega^2 g''(\omega)}{g(\omega)} = \Phi(v(\omega)),\ g(\omega) = f^{-1}(\omega). \tag{56}$$

Since

$$\frac{zf'(z)}{f(z)} + \frac{\lambda z^2 f''(z)}{f(z)} = 1 + m(1+\lambda(m+1))a_{m+1}z^m + \big[2m(1+\lambda(2m+1))a_{2m+1} - m(1+\lambda(m+1))a_{m+1}^2\big]z^{2m} + \cdots,$$

and

$$\frac{\omega g'(\omega)}{g(\omega)} + \frac{\lambda \omega^2 g''(\omega)}{g(\omega)} = 1 - m(1+\lambda(m+1))a_{m+1}\omega^m + \big[(m(2m+1) + \lambda m(m+1)(4m+1))a_{m+1}^2 - 2m(1+\lambda(2m+1))a_{2m+1}\big]\omega^{2m} + \cdots,$$

then from (23), (24), (55) and (56), we have

$$m(1+\lambda(m+1))a_{m+1} = \frac{c_m}{4}, \tag{57}$$

$$2m(1+\lambda(2m+1))a_{2m+1} - m(1+\lambda(m+1))a_{m+1}^2 = \frac{1}{4}\left(c_{2m} - \frac{c_m^2}{2}\right), \tag{58}$$

$$-m(1+\lambda(m+1))a_{m+1} = \frac{b_m}{4} \tag{59}$$

and

$$\big[m(2m+1)+\lambda m(m+1)(4m+1)\big]a_{m+1}^2 - 2m(1+\lambda(2m+1))a_{2m+1} = \frac{1}{4}\left(b_{2m} - \frac{b_m^2}{2}\right). \tag{60}$$

From (57) and (59), we get

$$c_m = -b_m. \tag{61}$$

Also, from (58), (60) and (61), we obtain

$$a_{m+1}^2 = \frac{c_{2m}+b_{2m}}{8m^2(1+2\lambda(m+1)+2(1+\lambda(m+1))^2)} \tag{62}$$

which, in view of the inequalities $|c_{2m}| \le 2$ and $|b_{2m}| \le 2$ for $|a_{m+1}|$, we get the desired result as asserted in (53).

Now, by simple calculations from (57), (58), (60) and (61), we get

$$a_{2m+1} = \frac{c_{2m}-b_{2m}}{16m(1+\lambda(1+2m))} + \frac{c_m^2(m+1)}{32m^2(1+\lambda(1+m))^2}, \tag{63}$$

which, in view of the inequalities $|c_m| \le 2$, $|c_{2m}| \le 2$ and $|b_{2m}| \le 2$ for $|a_{2m+1}|$, we get the desired result as asserted in (54). □

For the case when $m = 1$, we have

Corollary 5. *Let $f(z)$, given by (8), be the class $N_{\sigma,1}(\lambda,\Phi)$, $\lambda \ge 0$. Then*

$$|a_2| \le \frac{1}{\sqrt{2[1+4\lambda+2(1+2\lambda)^2]}} \tag{64}$$

and

$$|a_3| \le \frac{2+7\lambda+4\lambda^2}{4(1+3\lambda)(1+2\lambda)^2}. \tag{65}$$

Setting $\lambda = 1$ in Corollary 5, it gives

Corollary 6. *Let $f(z)$, given by (8), be the class $N_{\sigma,1}(1, \Phi)$. Then*

$$|a_2| \leq \frac{1}{\sqrt{46}} \tag{66}$$

and

$$|a_3| \leq \frac{13}{144}. \tag{67}$$

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