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A NOTE ON FRACTIONAL q-INTEGRALS AND APPLICATIONS TO GENERATING FUNCTIONS AND q-MITTAG–LEFFLER FUNCTION

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Dedicated to George E. Andrews on his 80th birthday

Abstract. Motivated by the fact that fractional q-integrals play important roles in numerous sciences, it's natural to generalize the fractional q-integrals. The main object of this paper is to build the relations between fractional q-integrals and certain generating functions for q-polynomials and to generalize two fractional q-identities of [Fract. Calc. Appl. Anal. **10**(2007), 359–373.] by the method of q-difference equation. As applications, we deduce the mixed and $U(n+1)$ type generating functions for Predrag–Sladjana–Miomir polynomial, gain the transformational fractional q-identities and bulid the relations of fractional q-integrals and q-Mittag–Leffler function.

1. Introduction

The fractional calculus is a very suitable tool in describing and solving a lot of problems in numerous sciences [30], such as physics, electromagnetics, acoustics, electrochemistry and material science. Their treatment from the point view of the q-calculus can open new perspectives as it did, for example, in optimal control problems [5]. For further information about q-integral and fractional q-integrals, see [2, 3, 4, 6, 16, 18, 19, 20, 21, 22, 26, 27, 38, 39, 40, 41, 42].

In this paper, we follow the notations and terminology in [17] and suppose that $0 < q < 1$. We first show a list of various definitions and notations in q-calculus which are useful to understand the subject of this paper. The basic hypergeometric series ${}_r\phi_s$

$$ {}_r\phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \tag{1} $$

converges absolutely for all z if $r \le s$ and for $|z| < 1$ if $r = s+1$ unless it terminates. The q-real number $[a]_q$ and compact q-shifted factorials are defined by

$$ [a]_q := \frac{1-q^a}{1-q}, \quad (a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \tag{2} $$

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and $(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$, where $m \in \mathbb{N} := \{1, 2, 3, \cdots\}$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The q-gamma function is defined by [17]

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad (x \in \mathbb{R}\backslash\{0,-1,-2,\ldots\}). \tag{3}$$

The Thomae–Jackson q-integral is defined by [17, 23, 37]

$$\int_a^b f(x)\,\mathrm{d}_q x = (1-q)\sum_{n=0}^\infty \Big[bf(bq^n) - af(aq^n)\Big]q^n. \tag{4}$$

The Riemann–Liouville fractional q-integral operator was introduced in [1]

$$\Big(I_q^\alpha f\Big)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} f(t)\,\mathrm{d}_q t. \tag{5}$$

The generalized Riemann–Liouville fractional q-integral operator for $\alpha \in \mathbb{R}^+$ is given by [32]

$$\Big(I_{q,a}^\alpha f\Big)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_a^x (qt/x;q)_{\alpha-1} f(t)\,\mathrm{d}_q t, \tag{6}$$

which can be rewritten equivalently as follows by (4)

$$\Big(I_{q,a}^\alpha f\Big)(x) = \frac{x^{\alpha-1}(1-q)}{\Gamma_q(\alpha)}\sum_{n=0}^\infty \Big[x(q^{n+1};q)_{\alpha-1} f(xq^n) - a(aq^{n+1}/x;q)_{\alpha-1} f(aq^n)\Big]q^n. \tag{7}$$

Predrag–Sladjana–Miomir [31] obtained the following fractional q-identities.

Proposition 1 ([31, Corollary 4.1])**.** *For $\alpha \in \mathbb{R}^+$ and $0 < a < x < 1$, the following fractional q-integrals are valid*

$$I_{q,a}^\alpha\left\{\frac{1}{(x;q)_\infty}\right\} = \frac{(1-q)^\alpha}{(a;q)_\infty}\sum_{n=0}^\infty \frac{x^{\alpha+n}(a/x;q)_{\alpha+n}}{(q;q)_{\alpha+n}}, \tag{8}$$

$$I_{q,a}^\alpha\Big\{(-x;q)_\infty\Big\} = (1-q)^\alpha(-a;q)_\infty\sum_{n=0}^\infty \frac{q^{\binom{n}{2}}x^{\alpha+n}(a/x;q)_{\alpha+n}}{(-a;q)_n(q;q)_{\alpha+n}}. \tag{9}$$

In this paper, we consider the Predrag–Sladjana–Miomir polynomial.

Definition 2. *For $\alpha \in \mathbb{R}^+$ and $0 < a < x < 1$, we denote*

$$\mathcal{P}_n(\alpha, a, x|q) \triangleq I_{q,a}^\alpha\{x^n\} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{[k]_q!\,a^{n-k}}{\Gamma_q(\alpha+k+1)} x^{\alpha+k}(a/x;q)_{\alpha+k} \tag{10}$$

and generalize the fractional q-identities (8) and (9) as follows.

Theorem 3. *For $\alpha \in \mathbb{R}^+$ and $0 < a < x < 1$. If $\max\{|at|, |az|\} < 1$, we have*

$$I_{q,a}^\alpha\left\{\frac{(bxz, tx;q)_\infty}{(xs, xz;q)_\infty}\right\} = \frac{(1-q)^\alpha(abz, at;q)_\infty}{(as, az;q)_\infty}\sum_{k=0}^\infty \frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\,{}_3\phi_2\left[\begin{matrix} q^{-k}, as, az \\ at, abz \end{matrix}; q, q\right]. \tag{11}$$

Remark 4. *For $t = z = 0$ and $s = 1$ in Theorem 3, equations* (11) *reduces to* (8)*. For $s = z = 0$ and $t = -1$ in Theorem 3, equations* (11) *reduces to* (9)*.*

The rest of the paper is organized as follows. In section 2, we give the proof of the main results by the method of q-difference equation. In section 3, we gain mixed generating functions for Predrag–Sladjana–Miomir polynomial. In section 4, we give transformational fractional q-identities. In section 5, we obtain $U(n+1)$ type generating functions for Predrag–Sladjana–Miomir polynomial. In section 6, we derive the relations between fractional q-integrals and q-Mittag–Leffler function.

2. Proof of Theorem 3

The Verma–Jain polynomials [9, Eq. (2.6)]

$$\mathcal{I}_n(a,x,y,z|q)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}(a;q)_rP_{n-r}(x,y)z^r,\quad \mathcal{J}_n(a,x,y,z|q)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}(a;q)_rP_{n-r}(y,x)z^r, \tag{12}$$

where $P_n(a,b)$ are Cauchy polynomials

$$P_n(a,b)=(b/a;q)_na^n=(a-b)\cdots(a-q^{n-1}b), \tag{13}$$

are important in q-series, which have close relationship with the other q-polynomials such as Roger–Szegö, continuous q-ultraspherical, Al-Salam-Chihara, q-random walk and so on. For more information, please refer to [36].

Chen, Fu and Zhang [14] defined the homogeneous q-difference operator D_{xy} as follows

$$D_{xy}\{f(x,y)\}=\frac{f(x,q^{-1}y)-f(qx,y)}{x-q^{-1}y}, \tag{14}$$

and the homogeneous q-operator $\mathbb{T}(a,zD_{xy})$ defined by

$$\mathbb{T}(a,zD_{xy})=\sum_{n=0}^{\infty}\frac{(a;q)_n(zD_{xy})^n}{(q;q)_n}. \tag{15}$$

We check that [9, Eq. (2.15) and (2.16)]

$$\mathcal{I}_n(a,x,y,z|q)=\mathbb{T}(a,zD_{xy})\{P_n(x,y)\},\quad \mathcal{I}_n(a,x,y,z|q)=z^n\Omega_n(x/z;y/x,a|q), \tag{16}$$

$$\mathcal{J}_n(a,x,y,z|q)=\mathbb{E}(a,z\theta_{xy})\{P_n(y,x)\},\quad \mathcal{J}_n(a,x,y,z|q)=(-1)^nq^{\binom{n}{2}}\mathcal{I}_n(1/a,y,x,az|q^{-1}). \tag{17}$$

The method of q-difference equations shows itself to be an effective way to deduce many important results involving q-series. For more information, please refer to [8, 9, 10, 11, 12, 15, 24, 25].

Before prove the main results, the following lemmas are necessary.

Lemma 5 ([9, Theorem 3]). *Let $f(x,y,z)$ be a three variables analytic function in a neighbourhood of $(x,y,z)=(0,0,0)\in\mathbb{C}^3$. If $f(x,y,z)$ satisfies the equation*

$$\begin{aligned}&\Big(x-q^{-1}y\Big)[f(a,x,y,z)-f(a,x,y,qz)]\\&\quad=z\Big[f\Big(a,x,q^{-1}y,z\Big)-f(a,qx,y,z)\Big]-az\Big[f\Big(a,x,q^{-1}y,qz\Big)-f(a,qx,y,qz)\Big],\end{aligned} \tag{18}$$

then we have

$$f(a,x,y,z)=\mathbb{T}(a,zD_{xy})\{f(a,x,y,0)\}. \tag{19}$$

Lemma 6 ([9, Eq. (2.12)]). *Suppose that* $\max\{|xt|,|zt|\}<1$*, we have*

$$\mathbb{T}(a,zD_{xy})\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\}=\frac{(azt,yt;q)_\infty}{(xt,zt;q)_\infty}. \tag{20}$$

Lemma 7 (q-Chu-Vandermonde formula)**.** *For $n \in \mathbb{N}$, we have*

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, a \\ c \end{matrix}; q, q\right] = \frac{(c/a;q)_n a^n}{(c;q)_n}. \tag{21}$$

Lemma 8. *For $\alpha \in \mathbb{R}^+$, $0 < a < x < 1$ and $|as| < 1$, we have*

$$I_{q,a}^{\alpha}\left\{\frac{(xt;q)_\infty}{(xs;q)_\infty}\right\} = \frac{(1-q)^\alpha (at;q)_\infty}{(as;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}(t/s;q)_k s^k}{(q;q)_{\alpha+k}(at;q)_k}. \tag{22}$$

Proof of Lemma 8. The left-hand side (LHS) of equation (22) is equal to

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(t/s;q)_n s^n}{(q;q)_n} I_{q,a}^{\alpha}\{x^n\}\\
&=\sum_{n=0}^{\infty}\frac{(t/s;q)_n t^n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q;q)_n a^{n-k}}{(q;q)_k (q;q)_{n-k}}\frac{(q;q)_k (1-q)^{-k}}{(q;q)_{\alpha+k}(1-q)^{-\alpha-k}} x^{\alpha+k}(a/x;q)_{\alpha+k}\\
&=(1-q)^\alpha\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}(t/s;q)_k s^k}{(q;q)_{\alpha+k}}\sum_{n=0}^{\infty}\frac{(tq^k/s;q)_n (as)^n}{(q;q)_n},
\end{aligned}$$

which equals the right-hand side (RHS) of equation (22) after simplification. The proof is complete. □

Lemma 9. *For* $\max\{|as|, |az|\} < 1$, *we have*

$$\begin{aligned}
&\left(s - q^{-1}t\right)\left[\frac{(abz,at;q)_\infty}{(as,az;q)_\infty} - \frac{(abzq,at;q)_\infty}{(as,azq;q)_\infty}\right]\\
&= z\left[\frac{(abz,atq^{-1};q)_\infty}{(as,az;q)_\infty} - \frac{(abz,at;q)_\infty}{(asq,az;q)_\infty}\right] - bz\left[\frac{(abzq,atq^{-1};q)_\infty}{(as,azq;q)_\infty} - \frac{(abzq,at;q)_\infty}{(asq,azq;q)_\infty}\right].
\end{aligned} \tag{23}$$

Proof of Lemma 9. The LHS of equation (23) is equal to

$$LHS\,of(23) = \frac{(abz,at;q)_\infty}{(as,az;q)_\infty}\cdot\frac{z(b-1)(atq^{-1}-as)}{1-abz}. \tag{24}$$

The RHS of equation (23) equals

$$\begin{aligned}
RHS\,of(23) &= \frac{(abz,at;q)_\infty}{(as,az;q)_\infty}\cdot z\Big[(1-atq^{-1}) - (1-as)\Big]\\
&\quad - \frac{(abz,at;q)_\infty}{(as,az;q)_\infty}\cdot bz\left[\frac{(1-az)(1-atq^{-1})}{1-abz} - \frac{(1-as)(1-az)}{1-abz}\right]\\
&= \frac{(abz,at;q)_\infty}{(as,az;q)_\infty}\cdot\frac{z(b-1)(atq^{-1}-as)}{1-abz}.
\end{aligned}$$

So the equation (23) is valid. The proof is complete. □

Proof of Theorem 3. We denote the RHS of (11) by $f(b,s,t,z)$, and write $f(b,s,t,z)$ equivalently by

$$f(b,s,t,z) = (1-q)^\alpha\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k (q;q)_{\alpha+k}}\sum_{j=0}^{k}\frac{(q^{-k};q)_j q^j}{(q;q)_j}\cdot\frac{(abzq^j, atq^j;q)_\infty}{(asq^j, azq^j;q)_\infty}. \tag{25}$$

We check that $f(b,s,t,z)$ satisfies equation (18) by (23), so we have

$$
\begin{aligned}
f(b,s,t,z) &= \mathbb{T}(b,zD_{st})\{f(b,s,t,0)\}\\
&= \mathbb{T}(b,zD_{st})\left\{(1-q)^{\alpha}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\sum_{j=0}^{k}\frac{(q^{-k};q)_j q^j}{(q;q)_j}\cdot\frac{(atq^j;q)_\infty}{(asq^j;q)_\infty}\right\}\\
&= \mathbb{T}(b,zD_{st})\left\{\frac{(1-q)^{\alpha}(at;q)_\infty}{(as;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}{}_2\phi_1\left[\begin{matrix}q^{-k},as\\ at\end{matrix};q,q\right]\right\} by\,(22)\\
&= I_{q,a}^{\alpha}\left\{\mathbb{T}(b,zD_{st})\left\{\frac{(xt;q)_\infty}{(xs;q)_\infty}\right\}\right\},
\end{aligned}
$$

which equals the LHS of equation (11) after using (20). The proof is complete. □

3. Mixed generating functions for Predrag–Sladjana–Miomir polynomial

In this section, we study the mixed generating functions for Predrag–Sladjana–Miomir polynomial, before the main results, the following lemmas are necessary. For more information about the mixed generating functions, please refer to [9, 13].

Lemma 10. *For $\alpha\in\mathbb{R}^+$, $0<a<x<1$, we have*

$$
\sum_{n=0}^{\infty}\mathcal{P}_n(\alpha,a,x|q)\frac{w^n}{(q;q)_n}=\frac{(1-q)^{\alpha}}{(aw;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}w^k}{(q;q)_{\alpha+k}}. \tag{26}
$$

Proof of Lemma 10. The LHS of equation (26) is equal to

$$
I_{q,a}^{\alpha}\left\{\sum_{n=0}^{\infty}\frac{(xw)^n}{(q;q)_n}\right\}=I_{q,a}^{\alpha}\left\{\frac{1}{(xw;q)_\infty}\right\}, \tag{27}
$$

which equals the RHS of equation (26) after using (22). The proof is complete. □

Lemma 11 ([9, Eq. (2.20)])**.** *For* $\max\{|zt|,|xt|\}<1$, *we have*

$$
\sum_{n=0}^{\infty}\mathcal{I}_n(a,x,y,z|q)\frac{t^n}{(q;q)_n}=\frac{(azt,yt;q)_\infty}{(zt,xt;q)_\infty}. \tag{28}
$$

Theorem 12. *For $\alpha\in\mathbb{R}^+$, $0<a<x<1$ and* $\max\{|atw|,|arw|\}<1$, *we have*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{P}_n(\alpha,a,x|q)\mathcal{I}_n(b,r,s,t|q)\frac{w^n}{(q;q)_n}\\
&\quad=\frac{(1-q)^{\alpha}(abtw,asw;q)_\infty}{(atw,arw;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}{}_3\phi_2\left[\begin{matrix}q^{-k},atw,arw\\ asw,abtw\end{matrix};q,q\right].
\end{aligned} \tag{29}
$$

Remark 13. *For $s=t=0$ in Theorem 12, equation* (29) *reduces to* (26) *by using equation* (21).

Proof of Theorem 12. The LHS of equation (29) is equal to

$$
I_{q,a}^{\alpha}\left\{\sum_{n=0}^{\infty}\mathcal{I}_n(b,r,s,t|q)\frac{(xw)^n}{(q;q)_n}\right\}=I_{q,a}^{\alpha}\left\{\frac{(btwx,swx;q)_\infty}{(twx,rwx;q)_\infty}\right\}, \tag{30}
$$

which equals the RHS of equation (29) after using (11). The proof is complete. □

4. Transformational fractional q-identities

In this section, we give the following transformational fractional q-identities by the method of the homogeneous q-difference equation. For more information, please refer to [9, 12, 14].

Theorem 14. *Let $A(n,k)$ and $B(n,k)$ be independent of a and b. For $n \in \mathbb{N}$, if $A(n,k)$ and $B(n,k)$ satisfies*
$$\sum_{k=0}^{n} A(n,k)\frac{(sx;q)_k}{(tx;q)_k} = \sum_{k=0}^{n} B(n,k)\frac{P_k(s,t)x^k}{(tx;q)_k}, \tag{31}$$
then for $\alpha \in \mathbb{R}^+$, $0 < a < x < 1$, we have
$$\begin{aligned}&\sum_{k=0}^{n} A(n,k)\sum_{l=0}^{\infty}\frac{x^{\alpha+l}(a/x;q)_{\alpha+l}}{a^l(q;q)_{\alpha+l}}{}_3\phi_2\left[\begin{matrix} q^{-l}, asq^k, azq^k \\ atq^k, abzq^k \end{matrix}; q, q\right] \\ &= \sum_{k=0}^{n} B(n,k)\sum_{j=0}^{k}\frac{(q^{-k};q)_j q^j}{(q;q)_j}\sum_{l=0}^{\infty}\frac{x^{\alpha+l}(a/x;q)_{\alpha+l}}{a^l(q;q)_{\alpha+l}}{}_3\phi_2\left[\begin{matrix} q^{-l}, asq^j, azq^j \\ atq^j, abzq^j \end{matrix}; q, q\right].\end{aligned} \tag{32}$$

Corollary 15. *Let $A(n,k)$ and $B(n,k)$ be independent of a and b. For $n \in \mathbb{N}$, if $A(n,k)$ and $B(n,k)$ satisfies*
$$\sum_{k=0}^{n} A(n,k)\frac{(sx;q)_k}{(tx;q)_k} = \sum_{k=0}^{n} B(n,k)\frac{P_k(s,t)x^k}{(tx;q)_k}, \tag{33}$$
then for $\alpha \in \mathbb{R}^+$, $0 < a < x < 1$, we have
$$\begin{aligned}&\sum_{k=0}^{n} A(n,k)\sum_{l=0}^{\infty}\frac{x^{\alpha+l}(a/x;q)_{\alpha+l}(t/s;q)_l(sq^k)^l}{(q;q)_{\alpha+l}(atq^k;q)_l} \\ &= \sum_{k=0}^{n} B(n,k)\sum_{j=0}^{k}\frac{(q^{-k};q)_j q^j}{(q;q)_j}\sum_{l=0}^{\infty}\frac{x^{\alpha+l}(a/x;q)_{\alpha+l}(t/s;q)_l(sq^j)^l}{(q;q)_{\alpha+l}(atq^j;q)_l}.\end{aligned} \tag{34}$$

Remark 16. *For $z = 0$ in Theorem 14, equation (32) reduces to (34).*

Proof of Theorem 14. By the terminating Jackson's formula [17, Eq. (Ⅲ.4)]
$${}_2\phi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, z\right] = \frac{(az;q)_\infty}{(z;q)_\infty}{}_2\phi_2\left[\begin{matrix} a, c/b \\ c, az \end{matrix}; q, bz\right], \tag{35}$$
formula (31) is valid for cases
$$A(n,k) = \frac{(q^{-n};q)_k z^k}{(q;q)_k} \quad \text{and} \quad B(n,k) = \frac{(zq^{-n};q)_n(q^{-n};q)_k z^k(-1)^k q^{\binom{k}{2}}}{(q, zq^{-n};q)_k}. \tag{36}$$
We can write equation (31) equivalently by
$$\sum_{k=0}^{n} A(n,k)\cdot\frac{(txq^k;q)_\infty}{(sxq^k;q)_\infty} = \sum_{k=0}^{n} B(n,k)\sum_{j=0}^{k}\frac{(q^{-k};q)_j q^j}{(q;q)_j}\cdot\frac{(txq^j;q)_\infty}{(sxq^j;q)_\infty}. \tag{37}$$
Using the method of the homogeneous q-difference equation by Lemma 5, we have
$$\sum_{k=0}^{n} A(n,k)\frac{(txq^k, bzxq^k;q)_\infty}{(sxq^k, zxq^k;q)_\infty} = \sum_{k=0}^{n} B(n,k)\sum_{j=0}^{k}\frac{(q^{-k};q)_j q^j}{(q;q)_j}\cdot\frac{(txq^j, bzxq^j;q)_\infty}{(sxq^j, zxq^j;q)_\infty}. \tag{38}$$
Applying the fractional q-integral operator $I_{q,a}^{\alpha}$ on both sides of equation (38), we deduce the equation (32) by using equation (11). The proof is complete. □

5. $U(n+1)$ type generating functions for Predrag–Sladjana–Miomir polynomial

Various authors have researched that multiple basic hypergeometric series associated with the unitary $U(n+1)$ group. For more information, please refer to [7, 10, 28].

In [28], Milne studied the theory and application of the $U(n+1)$ generalization of the classical Bailey transform and Bailey lemma, which involving the following nonterminating $U(n+1)$ generalization of the q-binomial theorem.

Proposition 17 ([28, Theorem 5.42])**.** *Let b, z and $x_1, \ldots, x_n$ be indeterminate, and let $n \geq 1$. Suppose that none of the denominators in the following identity vanishes, and that $0 < |q| < 1$ and $|z| < |x_1, \ldots, x_n||x_m|^{-n}|q|^{(n-1)/2}$, for $m = 1, 2, \ldots, n$. Then*

$$\sum_{\substack{y_k\geq 0\\ k=1,2\ldots,n}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s};q\right)_{y_r}^{-1} \prod_{i=1}^{n} (x_i)^{ny_i-(y_1+\ldots+y_n)}(-1)^{(n-1)(y_1+\ldots+y_n)} \right.$$
$$\left. \times q^{y_2+2y_3+\ldots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\ldots+\binom{y_n}{2}\right]-e_2(y_1,\ldots,y_n)}(b;q)_{y_1+\ldots+y_n} z^{y_1+\ldots+y_n} \right\} = \frac{(bz;q)_\infty}{(z;q)_\infty}, \quad (39)$$

where $e_2(y_1, \ldots, y_n)$ is the second elementary symmetric function of $\{y_1, \ldots, y_n\}$.

Lemma 18 ([9, Eq. (2.20)])**.** *For $k \in \mathbb{N}$ and $\max\{|tx|, |rx|\} < 1$, we have*

$$\sum_{n=0}^{\infty} \mathcal{I}_{n+j}(b,r,s,t|q)\frac{x^{n+j}}{(q;q)_n} = \frac{(btx,sx;q)_\infty}{(tx,rx;q)_\infty}\, {}_3\phi_2\left[\begin{matrix} q^{-j}, tx, rx \\ btx, sx \end{matrix}; q, q\right]. \quad (40)$$

Theorem 19. *Let b, z and $x_1, \ldots, x_n$ be indeterminate, and let $n \geq 1$. Suppose that none of the denominators in the following identity vanishes, and that $0 < |q| < 1$ and $|z| < |x_1, \ldots, x_n||x_m|^{-n}|q|^{(n-1)/2}$, for $m = 1, 2, \ldots, n$. Then*

$$\sum_{\substack{y_k\geq 0\\ k=1,2\ldots,n}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s};q\right)_{y_r}^{-1} \prod_{i=1}^{n} (x_i)^{ny_i-(y_1+\ldots+y_n)}(-1)^{(n-1)(y_1+\ldots+y_n)} \right.$$
$$\left. \times q^{y_2+2y_3+\ldots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\ldots+\binom{y_n}{2}\right]-e_2(y_1,\ldots,y_n)} \mathcal{I}_{y_1+\ldots+y_n+j}(b,r,s,t|q)\mathcal{P}_{y_1+\ldots+y_n+j}(\alpha,a,x|q) \right\}$$
$$= \frac{(1-q)^\alpha(abt,as;q)_\infty}{(ar,at;q)_\infty} \sum_{i=0}^{j} \frac{(q^{-j},ar,at;q)_i q^i}{(q,abt,as;q)_i} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\, {}_3\phi_2\left[\begin{matrix} q^{-k}, arq^i, atq^i \\ asq^i, abtq^i \end{matrix}; q, q\right], \quad (41)$$

where $e_2(y_1, \ldots, y_n)$ is the second elementary symmetric function of $\{y_1, \ldots, y_n\}$.

Remark 20. *For $j = s = t = 0$ in Theorem 19, equation* (41) *reduces to* (26).

Proof of Theorem 19. Letting $(b, z) = (sq^j/r, xr)$ in equation (39) and multiplying $P_j(r, s)$ on both sides of equation (39) yields

$$\sum_{\substack{y_k\geq 0\\ k=1,2\ldots,n}} \left\{ \prod_{1\leq r<s\leq n} \left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right] \prod_{r,s=1}^{n} \left(q\frac{x_r}{x_s};q\right)_{y_r}^{-1} \prod_{i=1}^{n} (x_i)^{ny_i-(y_1+\ldots+y_n)}(-1)^{(n-1)(y_1+\ldots+y_n)} \right.$$
$$\left. \times q^{y_2+2y_3+\ldots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\ldots+\binom{y_n}{2}\right]-e_2(y_1,\ldots,y_n)} P_{y_1+\ldots+y_n+j}(r,s)x^{y_1+\ldots+y_n+j} \right\} = \frac{x^j P_j(r,s)(rxq^j;q)_\infty}{(sx;q)_\infty}. \quad (42)$$

We denote $f(b,r,s,t)$ by

$$f(b,r,s,t)=\frac{(btx,sx;q)_\infty}{(tx,rx;q)_\infty}{}_3\phi_2\left[\begin{matrix}q^{-j},tx,rx\\ btx,sx\end{matrix};q,q\right] \tag{43}$$

and we check that $f(b,r,s,t)$ satisfies equation (18), so we have

$$f(b,r,s,t)=\mathbb{T}(b,tD_{rs})\{f(b,r,s,0)\}=\mathbb{T}(b,tD_{rs})\left\{\frac{x^jP_j(r,s)(rxq^j;q)_\infty}{(sx;q)_\infty}\right\}\quad by\,(42) \tag{44}$$

$$=\sum_{\substack{y_k\ge0\\ k=1,2\dots,n}}\left\{\prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left(q\frac{x_r}{x_s};q\right)_{y_r}^{-1}\prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\ldots+y_n)}(-1)^{(n-1)(y_1+\ldots+y_n)}\right. \tag{45}$$

$$\left.\times\, q^{y_2+2y_3+\ldots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\ldots+\binom{y_n}{2}\right]-e_2(y_1,\ldots,y_n)}\mathbb{T}(b,tD_{rs})\{P_{y_1+\ldots+y_n+j}(r,s)\}\cdot x^{y_1+\ldots+y_n+j}\right\}, \tag{46}$$

which equals

$$f(b,r,s,t)=\sum_{\substack{y_k\ge0\\ k=1,2\dots,n}}\left\{\prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left(q\frac{x_r}{x_s};q\right)_{y_r}^{-1}\prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\ldots+y_n)}(-1)^{(n-1)(y_1+\ldots+y_n)}\right.$$
$$\left.\times\, q^{y_2+2y_3+\ldots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\ldots+\binom{y_n}{2}\right]-e_2(y_1,\ldots,y_n)}\mathcal{I}_{y_1+\ldots+y_n+j}(b,r,s,t|q)x^{y_1+\ldots+y_n+j}\right\}$$
$$=\frac{(btx,sx;q)_\infty}{(tx,rx;q)_\infty}{}_3\phi_2\left[\begin{matrix}q^{-j},tx,rx\\ btx,sx\end{matrix};q,q\right]. \tag{47}$$

Applying the fractional q-integral operator $I^\alpha_{q,a}$ on both sides of equation (47), we have

$$\sum_{\substack{y_k\ge0\\ k=1,2\dots,n}}\left\{\prod_{1\le r<s\le n}\left[\frac{1-\frac{x_r}{x_s}q^{y_r-y_s}}{1-\frac{x_r}{x_s}}\right]\prod_{r,s=1}^{n}\left(q\frac{x_r}{x_s};q\right)_{y_r}^{-1}\prod_{i=1}^{n}(x_i)^{ny_i-(y_1+\ldots+y_n)}(-1)^{(n-1)(y_1+\ldots+y_n)}\right.$$
$$\left.\times\, q^{y_2+2y_3+\ldots+(n-1)y_n+(n-1)\left[\binom{y_1}{2}+\ldots+\binom{y_n}{2}\right]-e_2(y_1,\ldots,y_n)}\mathcal{I}_{y_1+\ldots+y_n+j}(b,r,s,t|q)\mathcal{P}_{y_1+\ldots+y_n+j}(\alpha,a,x|q)\right\}$$
$$=I^\alpha_{q,a}\left\{\frac{(btx,sx;q)_\infty}{(tx,rx;q)_\infty}{}_3\phi_2\left[\begin{matrix}q^{-j},tx,rx\\ btx,sx\end{matrix};q,q\right]\right\}, \tag{48}$$

which is equivalent to the equation (41). The proof is complete. □

6. Relations between fractional q-integrals and q-Mittag–Leffler function

In 1903, the Swedish mathematician Mittag–Leffler [29] introduced the function $E_\alpha(z)$, defined by

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\quad \alpha\in\mathbb{C},\mathbb{R}(\alpha)>0, \tag{49}$$

where $\Gamma(z)$ is the Gamma function. A generalization of equation (49) was given by Wiman [43] in 1905

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\quad \alpha,\beta\in\mathbb{C},\mathbb{R}(\alpha)>0,\mathbb{R}(\beta)>0. \tag{50}$$

The Mittag–Leffler function reduces immediately to the exponential function $e^z = E_1(z)$ when $\alpha = 1$. For $0 < \alpha < 1$ it interpolates between the pure exponential e^z and a geometric function $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n (|z| < 1)$. Its importance has been realized during the last two decades due to its involvement in the problems of applied sciences such as physics, chemistry, biology and engineering. Mittag–Leffler function occurs naturally in the solution of fractional order differential or integral equations. For more information, please refer to [34, 35].

Rajković–Marinković–Stanković [31] defined the following two q-Mittag–Leffler functions, called the small q-Mittag–Leffler function and the big q-Mittag–Leffler function respectively.

Definition 21 ([31, Definition 4.1])**.** *For $|c| < |x|$, $q, x, c, \beta \in \mathbb{C}$ and $Re(\beta) > 0$, we have*

$$e_{q;\alpha,\beta}(x;c) = \sum_{n=0}^{\infty} \frac{x^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1}}{(q;q)_{\alpha n+\beta-1}}, \tag{51}$$

$$E_{q;\alpha,\beta}(x;c) = \sum_{n=0}^{\infty} \frac{q^{\binom{\alpha n+\beta-1}{2}} x^{\alpha n+\beta-1}(c/x;q)_{\alpha n+\beta-1}}{(-c;q)_{\alpha n+\beta-1}(q;q)_{\alpha n+\beta-1}}. \tag{52}$$

Proposition 22 ([31, Theorem 4.2])**.** *For $\alpha \in \mathbb{R}^+$ and $0 < c < x < 1$, we have*

$$I_{q,c}^{\alpha}(e_q(x)) = (1-q)^{\alpha} e_q(c) e_{q;1,\alpha+1}(x;c), \tag{53}$$

$$I_{q,c}^{\alpha}(E_q(x)) = (1-q)^{\alpha} q^{\binom{\alpha+1}{2}} E_q(cq^{-\alpha}) E_{q;1,\alpha+1}(xq^{-\alpha}; cq^{-\alpha}). \tag{54}$$

In this section, we continue to study the relations between fractional q-integrals and q-Mittag–Leffler function.

Theorem 23. *For $\alpha \in \mathbb{R}^+$ and $0 < a < x < 1$, we have*

$$I_{q,a}^{\alpha}\left\{\frac{1}{(xt;q)_\infty}\right\} = \frac{(1-q)^{\alpha}}{t^{\alpha}} e_q(at) e_{q;1,\alpha+1}(xt;at), \tag{55}$$

$$I_{q,a}^{\alpha}\left\{(xt;q)_\infty\right\} = (-t)^{-\alpha}(1-q)^{\alpha} q^{\binom{\alpha+1}{2}} E_q(-atq^{-\alpha}) E_{q;1,\alpha+1}(-xtq^{-\alpha}; -atq^{-\alpha}). \tag{56}$$

Remark 24. *For $t = 1$ and $t = -1$ in Theorem 23, equations* (55) *and* (56) *reduce to* (53) *and* (54) *respectively.*

Proof of Theorem 23. The RHS of equation (55) can be written equivalently by

$$\begin{aligned}
I_{q,a}^{\alpha}\left\{\frac{1}{(xt;q)_\infty}\right\} &= \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} I_{q,a}^{\alpha}\{x^n\} \\
&= \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} \sum_{k=0}^{n} \frac{(q;q)_n a^{n-k}}{(q;q)_k (q;q)_{n-k}} \frac{(q;q)_k (1-q)^{-k}}{(q;q)_{\alpha+k}(1-q)^{-\alpha-k}} x^{\alpha+k} (a/x;q)_{\alpha+k} \\
&= (1-q)^{\alpha} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k} t^k}{(q;q)_{\alpha+k}} \sum_{n=0}^{\infty} \frac{a^n t^n}{(q;q)_n} \\
&= \frac{(1-q)^{\alpha}}{(at;q)_\infty} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k} t^k}{(q;q)_{\alpha+k}},
\end{aligned}$$

which is the LHS of equation (55). Similarly, we can obtain equation (56). The proof of Theorem 23 is complete. ◻

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