

ON THE NATURE OF THE CONFORMABLE DERIVATIVE AND ITS APPLICATIONS TO PHYSICS

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ABSTRACT. The purpose of this work is to show that the Khalil and Katagampoula conformable derivatives are equivalent to the simple change of variables $x \rightarrow x^\alpha/\alpha$, where α is the order of the derivative operator, when applied to differential functions. Although this means no “new mathematics” is obtained by working with these derivatives, it is a second purpose of this work to argue that there is still significant value in exploring the mathematics and physical applications of these derivatives. This work considers linear differential equations, self-adjointness, Sturm-Liouville systems, and integral transforms. A third purpose of this work is to contribute to the physical interpretation when these derivatives are applied to physics and engineering. Quantum mechanics serves as the primary backdrop for this development.

1. INTRODUCTION

The concept of a fractional derivative has been receiving a lot of attention in the literature in recent years,[51, 38, 37, 40, 22] with entire journals devoted to fractional analysis [56]. Many of the authors of these papers mention the famous correspondence between Leibniz and L’Hôpital in 1695. Over the intervening years many definitions of a fractional derivative have appeared; well known examples being the Riemann-Liouville and the Caputo definitions. The current activity clearly suggests the extension of derivatives of non-integer power is not straightforward to say the least.

In fact, the defining properties of such derivatives are not agreed upon. It is typical that a particular definition captures only some of the properties of the conventional derivative. Ortigueira and Machado have recently compared and contrasted definitions of fractional derivatives and have set forth criteria for such derivatives [41]. This has led to some definitions of fractional derivatives to be reclassified as conformable derivatives. Zhao and Luo [54] provide a good account heredity/nonhereditary and locality/nonlocality [48]. In a very recent work, Tarasov clearly discusses nonlocality in the context of a number of familiar fractional derivatives including the Khalil and Katugampoula definitions, which are the focus of

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this current paper.[46] Tarasov points out that equations involving these two conformable derivatives can be reduced to ordinary differential equations. The current paper elaborates on that assertion.

In 2014 Khalil suggested the definition [31],

$$\begin{aligned} D^\alpha[f(x)] &= \lim_{\epsilon \rightarrow 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon}, \quad x > 0 \\ D^\alpha[f(0)] &= \lim_{t \rightarrow 0^+} D^\alpha[f(x)]. \end{aligned} \quad (1)$$

Katugampola shortly thereafter worked out a few additional technical details [27, 28]. For brevity we shall refer to the above derivative as simply the conformable derivative in this work. The conformable derivative was subsequently generalized in several ways [1]. Many papers have appeared based on exploring properties [1, 5, 3, 6, 7] and physical applications [4, 11, 23, 49, 26, 55, 53] of the conformable derivative.

A case of particular interest, particularly with an eye toward applications in physics and engineering, is applying the conformable derivative to differentiable functions. In this case the conformable derivative becomes

$$D^\alpha[f(x)] = x^{1-\alpha} \frac{df(x)}{dx}. \quad (2)$$

In operator language,

$$D^\alpha \equiv x^{1-\alpha} \frac{d}{dx}. \quad (3)$$

This leads to the main point of the current work: *The conformable derivative for differentiable functions is equivalent to a simple change of variable.* Precisely, $u = x^\alpha/\alpha$. It should be noted that a criticism of the conformable derivative is that, although conformable at the limit $\alpha \rightarrow 1$ ($\lim_{\alpha \rightarrow 1} D^\alpha f = f'$), it is not conformable at the other limit, $\alpha \rightarrow 0$, ($\lim_{\alpha \rightarrow 0} D^\alpha f \neq f$). From the point of view of the assertion about the equality of the conformable derivative to a change of variables, one can say that the conformable derivative is not conformable as $\alpha \rightarrow 0$ because t^α/α is undefined at $\alpha = 0$.

As such, the conformable derivative does not contribute “new mathematics.” That said, exploration of the conformable derivative and its generalizations can still be interesting and valuable. The focus of this paper is to elucidate the nature of this change of variables in a variety of settings. Further, we hope to provide some physical insight to assist with use in the applied setting. We focus on application in quantum mechanics but some of the discussion about how to interpret physical units and spaces related via Fourier transformation are relevant in general.

First, the basic calculus of the conformable derivative is laid out. Second, self-adjointness is discussed and the Sturm-Liouville system under this change of variable is presented with examples. Finally, integral transforms, specifically the Fourier and Laplace transforms, are discussed. The interpretation of the meaning of the physical units is presented in context of each of these settings. Application to quantum mechanics follows the mathematical development and proceeds concluding remarks. For ease of discussion, the word “conformable” will be used as an adjective to describe the conformable derivative type of change of variable. For example, “conformable Bessel function,” “conformable Laplace transform” etc. This

does not imply that there is something fundamentally new about a conformable object compared to its standard counter-part. *They are always related via a simple change of variable.*

2. THE BASIC CALCULUS OF THE CONFORMABLE DERIVATIVE

In this section the equivalence of calculus of differentiable functions using the conformable derivative and the change of variable $u = x^\alpha/\alpha$ is demonstrated.

The conformable derivative has the following important properties. This definition yields the following results (from Theorem 2.3 of Katugampola [27])

- $D^\alpha[af + bg] = aD^\alpha[f] + bD^\alpha[g]$ (linearity).
- $D^\alpha[fg] = fD^\alpha[g] + gD^\alpha[f]$ (product rule).
- $D^\alpha[f(g)] = \frac{df}{dg}D^\alpha[g]$ (chain rule).
- $D^\alpha[f] = x^{1-\alpha}f'$, where $f' = \frac{df}{dx}$.

To see the equivalence of the conformable derivative and the change of variables $u = x^\alpha/\alpha$, consider direct substitution and the chain rule in

$$D^\alpha f(x) \equiv x^{1-\alpha} \frac{df(x)}{dx}. \quad (4)$$

Then,

$$x^{1-\alpha} \frac{df(x)}{dx} = x^{1-\alpha} \frac{df(u)}{du} \frac{du}{dx} = x^{1-\alpha} \frac{df(u)}{du} x^{\alpha-1} = \frac{df(u)}{du}. \quad (5)$$

2.1. Second order linear differential equation. Consider the general second order linear differential equation (SOLDE)

$$p(u) \frac{d^2y(u)}{du^2} + q(u) \frac{dy(u)}{du} + r(u)y(u) = s(u). \quad (6)$$

Now let

$$u = \frac{x^\alpha}{\alpha}, \quad du = x^{\alpha-1} dx. \quad (7)$$

Thus,

$$\frac{dy(u)}{du} = \frac{dy\left(\frac{x^\alpha}{\alpha}\right)}{dx} \frac{dx}{du} = x^{1-\alpha} \frac{dy\left(\frac{x^\alpha}{\alpha}\right)}{dx} = D^\alpha y\left(\frac{x^\alpha}{\alpha}\right) \quad (8)$$

and

$$\frac{d^2y(u)}{du^2} = x^{2-2\alpha} \frac{d^2y}{dx^2} + (1-\alpha)x^{1-2\alpha} \frac{dy}{dx} \equiv \hat{C}_{2\alpha} y\left(\frac{x^\alpha}{\alpha}\right). \quad (9)$$

Therefore Eq. (6) becomes

$$\begin{aligned} p\left(\frac{x^\alpha}{\alpha}\right) \hat{C}_{2\alpha} y\left(\frac{x^\alpha}{\alpha}\right) + q\left(\frac{x^\alpha}{\alpha}\right) D_x^\alpha y\left(\frac{x^\alpha}{\alpha}\right) \\ + r\left(\frac{x^\alpha}{\alpha}\right) y\left(\frac{x^\alpha}{\alpha}\right) = s\left(\frac{x^\alpha}{\alpha}\right). \end{aligned} \quad (10)$$

This provides a recipe for translating any normal SOLDE into a conformable SOLDE. The notion of a “natural” variable, $\frac{x^\alpha}{\alpha}$, for the conformable derivative arises. The simple change of variable $x \longleftrightarrow \frac{x^\alpha}{\alpha}$ pulls all SOLDEs into conformable SOLDEs and vice versa.

It is interesting to expand Eq. (10).

$$\begin{aligned}
 p\left(\frac{x^\alpha}{\alpha}\right)\hat{C}_{2\alpha}y + q\left(\frac{x^\alpha}{\alpha}\right)D_x^\alpha y + r\left(\frac{x^\alpha}{\alpha}\right)y &= s\left(\frac{x^\alpha}{\alpha}\right) \\
 p\left(\frac{x^\alpha}{\alpha}\right)(x^{2-2\alpha}y'' + (1-\alpha)x^{1-2\alpha}y') + q\left(\frac{x^\alpha}{\alpha}\right)x^{1-\alpha}y' \\
 + r\left(\frac{x^\alpha}{\alpha}\right)y &= s\left(\frac{x^\alpha}{\alpha}\right) \\
 x^{2-2\alpha}p\left(\frac{x^\alpha}{\alpha}\right)y'' + \left((1-\alpha)x^{1-2\alpha}p\left(\frac{x^\alpha}{\alpha}\right) + x^{1-\alpha}q\left(\frac{x^\alpha}{\alpha}\right)\right)y' \\
 + r\left(\frac{x^\alpha}{\alpha}\right)y &= s\left(\frac{x^\alpha}{\alpha}\right) \\
 P(x)y'' + Q(x)y' + R(x)y &= S(x), \quad (11)
 \end{aligned}$$

where

$$\begin{aligned}
 P(x) &= p\left(\frac{x^\alpha}{\alpha}\right)x^{2-2\alpha} \\
 Q(x) &= (1-\alpha)x^{1-2\alpha}p\left(\frac{x^\alpha}{\alpha}\right) + x^{1-\alpha}q\left(\frac{x^\alpha}{\alpha}\right) \\
 R(x) &= r\left(\frac{x^\alpha}{\alpha}\right) \\
 S(x) &= s\left(\frac{x^\alpha}{\alpha}\right). \quad (12)
 \end{aligned}$$

Example 1

Let's consider some examples. First Bessel's equation

$$u^2 \frac{d^2 y(u)}{du^2} + u \frac{dy(u)}{du} + (u^2 + v^2)y(u) = 0, \quad (13)$$

which has solution

$$y(u) = C_1 J_v(u) + C_2 Y_v(u). \quad (14)$$

The corresponding conformable Bessel's function according to the recipe is

$$\left(\frac{x^\alpha}{\alpha}\right)^2 \hat{C}_{2\alpha}y\left(\frac{x^\alpha}{\alpha}\right) + \left(\frac{x^\alpha}{\alpha}\right) D_x^\alpha y\left(\frac{x^\alpha}{\alpha}\right) \quad (15)$$

$$+ \left(\left(\frac{x^\alpha}{\alpha}\right)^2 - v^2\right)y\left(\frac{x^\alpha}{\alpha}\right) = 0, \quad (16)$$

which has solution

$$y\left(\frac{x^\alpha}{\alpha}\right) = C_1 J_v\left(\frac{x^\alpha}{\alpha}\right) + C_2 Y_v\left(\frac{x^\alpha}{\alpha}\right). \quad (17)$$

The expanded SOLDE becomes via

$$\begin{aligned} P(x) &= \left(\frac{x^\alpha}{\alpha}\right)^2 x^{2-2\alpha} = \frac{x^2}{\alpha^2} \\ Q(x) &= (1-\alpha)x^{1-2\alpha} \left(\frac{x^\alpha}{\alpha}\right)^2 + x^{1-\alpha} \left(\frac{x^\alpha}{\alpha}\right) = \frac{x}{\alpha^2} \\ R(x) &= \left(\left(\frac{x^\alpha}{\alpha}\right)^2 - v^2\right) \\ S(x) &= 0. \end{aligned} \tag{18}$$

So one obtains

$$\begin{aligned} \frac{x^2}{\alpha^2}y'' + \frac{x}{\alpha^2}y' + \left(\left(\frac{x^\alpha}{\alpha}\right)^2 - v^2\right)y &= 0 \\ x^2y'' + xy' + (x^{2\alpha} - \alpha^2v^2)y &= 0. \end{aligned} \tag{19}$$

Example 2

Consider the differential equation for confluent hypergeometric limit function

$$u \frac{d^2y(u)}{du^2} + b \frac{dy(u)}{du} - y(u) = 0. \tag{20}$$

This has solution

$$y(u) = C_1 {}_0F_1(; a; u) + C_2 u^{1-b} {}_0F_1(; 2-a; u). \tag{21}$$

So

$$\frac{x^\alpha}{\alpha} \hat{C}_{2\alpha} y + b D_x^\alpha y - y = 0 \tag{22}$$

has solutions

$$y(x) = C_1 {}_0F_1\left(; b; \frac{x^\alpha}{\alpha}\right) + C_2 \left(\frac{x^\alpha}{\alpha}\right)^{1-b} {}_0F_1\left(; 2-b; \frac{x^\alpha}{\alpha}\right). \tag{23}$$

Here

$$\begin{aligned} P(x) &= \left(\frac{x^\alpha}{\alpha}\right) x^{2-2\alpha} = \frac{x^{2-\alpha}}{\alpha} \\ Q(x) &= (1-\alpha)x^{1-2\alpha} \left(\frac{x^\alpha}{\alpha}\right) + bx^{1-\alpha} = \left(\frac{1}{\alpha} - 1 + b\right)x^{1-\alpha} \\ R(x) &= -1 \\ S(x) &= 0, \end{aligned} \tag{24}$$

so,

$$\frac{x^{2-\alpha}}{\alpha}y'' + \left(\frac{1}{\alpha} - 1 + b\right)x^{1-\alpha}y' - y = 0. \tag{25}$$

Example 3

Finally, consider Airy's differential equation

$$y'' - xy = 0, \tag{26}$$

which has solutions

$$y = C_1 \text{Ai}[x] + C_2 \text{Bi}[x]. \tag{27}$$

So,

$$\hat{C}_{2\alpha}y - \frac{x^\alpha}{\alpha}y = 0 \quad (28)$$

has solutions

$$y = C_1 \text{Ai} \left[\frac{x^\alpha}{\alpha} \right] + C_2 \text{Bi} \left[\frac{x^\alpha}{\alpha} \right]. \quad (29)$$

3. SELF ADJOINTNESS AND STURM-LIOUVILLE SYSTEMS

Several properties of D^α have recently been investigated.[4] In that work the conformable analogue of $D^2 = \frac{d^2}{dx^2}$ was developed by first simply considering $D^\alpha D^\alpha$ (which is not equal to $D^{2\alpha}$). This, however, is not self-adjoint but can be made so by standard methods [21]. Doing so results in the self-adjoint operator [4]

$$\hat{A}_{2\alpha} = \frac{d}{dx} \left[x^{1-\alpha} \frac{d}{dx} \right], \quad (30)$$

note that $\lim_{\alpha \rightarrow 1} \hat{A}_{2\alpha} = D^2$. The eigenvalue equation

$$\hat{A}_{2\alpha}y + E_n y = 0 \quad (31)$$

was solved in reference [4]. This is the simplest conformable Sturm-Liouville system and its solutions are thoroughly explored in the remainder of this section. Several other Sturm-Liouville systems are discussed more briefly at the end of this section. For the special case of boundary conditions $y(0) = y(1) = 0$, the (normalized) solutions are

$$y = \mathbb{J}_n^{(\alpha)}(x) \equiv \frac{\sqrt{x^\alpha} J_\eta \left(n_\eta (x^\alpha)^{\frac{1}{2\eta}} \right)}{\sqrt{(\eta-1)J_{\eta-1}(n_\eta)J_{\eta+1}(n_\eta)}}, \quad (32)$$

where J is Bessel's function, $\eta = \frac{\alpha}{1+\alpha}$ and n_η is the n^{th} zero of $J_\eta(x)$. The eigenvalues are

$$E_n = \frac{(1+\alpha)^2 n_\eta^2}{4}. \quad (33)$$

The first three ($n = 1, 2, 3$) $\mathbb{J}_n^{(\alpha)}$ are plotted in Fig 1 for $\alpha = 1/4, 1/2, 3/4$, and 1.

The $\mathbb{J}_n^{(\alpha)}$ functions form a complete, orthonormal set over the domain $0 \leq x \leq 1$ and are a generalization of the set of orthonormal sine functions over the same domain, $y_n = \sqrt{2} \sin n\pi x$. The $\mathbb{J}_n^{(\alpha)}$ functions serve to introduce a parameterized (by α) extension of the harmonic functions in a manner that has a bit more richness than a Fourier-Bessel series.

Several aspects of the $\mathbb{J}_n^{(\alpha)}$ functions are now investigated. This is offered as an example of how results arising from conformable derivative based equations can still offer interesting subject matter to study despite the fact that the results can be obtained via a simple change of variable. Many relations can be obtained analytically but often one must resort to numerical calculations. The properties of the zeros of $\mathbb{J}_n^{(\alpha)}$ and a scaling factor for the $\mathbb{J}_n^{(\alpha)}$ functions are first explored. Then the expansion of an arbitrary function is investigated with some representative examples.

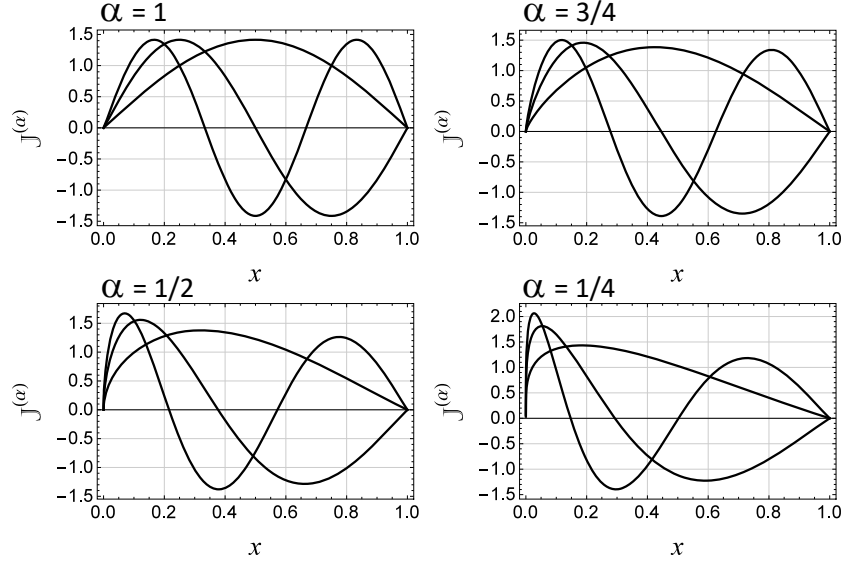


FIGURE 1. Each panel shows plots of $\mathbb{J}_1^{(\alpha)}$, $\mathbb{J}_2^{(\alpha)}$, and $\mathbb{J}_3^{(\alpha)}$ for $\alpha = 1, 3/4, 1/2,$ and $1/4$ (clockwise from the top left panel). These functions are solutions of Eq. (31) with boundary conditions $\mathbb{J}_n^{(\alpha)}(0) = \mathbb{J}_n^{(\alpha)}(1) = 0$ and they form a complete orthonormal set. The curves in each panel can be identified by the fact that $\mathbb{J}_n^{(\alpha)}$ has $n - 1$ zeros between $0 < x < 1$. The most noticeable characteristic of the $\mathbb{J}_n^{(\alpha)}$ functions is the skewing towards lower values of x as α decreases.

3.1. Zeros and scaling of $\mathbb{J}_n^{(\alpha)}$. The position of the zeros of $\mathbb{J}_n^{(\alpha)}$ are determined by a combination of the particular Bessel function involved and by the x^α appearing in its argument. The position of the k^{th} zero of $\mathbb{J}_n^{(\alpha)}$, $\mathbb{N}_n^{(\alpha)}(k)$, is given by the formula

$$\mathbb{N}_n^{(\alpha)}(k) = \left(\frac{k_\eta}{n_\eta} \right)^{\frac{2}{1+\alpha}}. \quad (34)$$

Figure 2 shows $\mathbb{N}_n^{(\alpha)}(k)$ for a variety of different n and k values as a function of α . Figure 2a shows the positions of the nine zeros of $\mathbb{J}_{10}^{(\alpha)}$ as a function of α and Fig. 2b shows the position of the first zero for $\mathbb{J}_n^{(\alpha)}$ where $n = 2, \dots, 10$. The limits of $\mathbb{N}_n^{(\alpha)}(k)$ are

$$\lim_{\alpha \rightarrow 1} \mathbb{N}_n^{(\alpha)}(k) = \frac{k_{\frac{1}{2}}}{n_{\frac{1}{2}}} = \frac{k}{n} \quad (35)$$

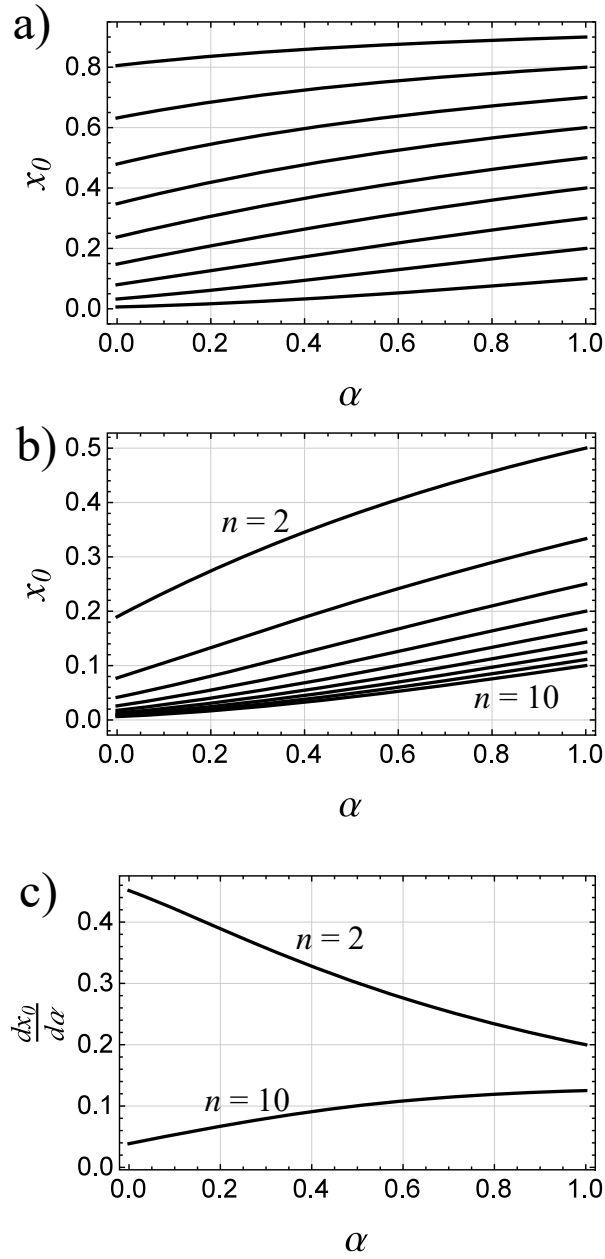


FIGURE 2. The position of the $x = x_0$ zeros of $\mathbb{J}_n^{(\alpha)}$. **(a)** The position of the first 9 zeros of $\mathbb{J}_{10}^{(\alpha)}$. Each of these zeros decreases with decreasing α . **(b)** The position of the first zeros of $\mathbb{J}_2^{(\alpha)}$ through $\mathbb{J}_{10}^{(\alpha)}$. The position of these zeros also decrease with decreasing α but not in the same fashion. **(c)** The derivative of the top ($n = 2$) and bottom ($n = 10$) curves in (b).

and

$$\lim_{\alpha \rightarrow 0} \mathbb{N}_n^{(\alpha)}(k) = \frac{k_0^2}{n_0^2}. \quad (36)$$

Although the graphs shown in Fig.2 are simply manifestations of the properties of the zeros of the Bessel function, it is insightful to point out some features. All zeros move to smaller values as α decreases but do so along different trajectories such that the spacing between zeros is the same for $\alpha = 1$ but is increasing for $\alpha < 1$. Figure 2c shows the derivative of the $\mathbb{J}_2^{(\alpha)}$ and $\mathbb{J}_{10}^{(\alpha)}$ curves of Fig. 2b. These exhibit opposite behavior with α .

In much the same way that $y_1 = \sin \pi x$ can be scaled to $y_2 = \sin 2\pi x$ by letting $x \rightarrow 2x$, one can determine a scaling factor, s , such that $\mathbb{J}_1^{(\alpha)}(sx) \propto \mathbb{J}_2^{(\alpha)}(x)$. A second amplitude scaling factor, N_s , is needed to create the equality, $N_s \mathbb{J}_1^{(\alpha)}(sx) = \mathbb{J}_2^{(\alpha)}(x)$. More generally, the scaling factors such that $N_s \mathbb{J}_n^{(\alpha)}(sx) = \mathbb{J}_{n+1}^{(\alpha)}(x)$ are,

$$s = 1/\mathbb{N}_{n+1}^{(\alpha)}(n) \quad (37)$$

and

$$N_s = \sqrt{\frac{1}{\sqrt{s}} \cdot \frac{J_{\eta-1}(n_\eta) J_{\eta+1}(n_\eta)}{J_{\eta-1}((n+1)_\eta) J_{\eta+1}((n+1)_\eta)}}. \quad (38)$$

3.2. Integrals of $\mathbb{J}_n^{(\alpha)}$. Most of the integrals involving $\mathbb{J}_n^{(\alpha)}$ need to be evaluated numerically, including showing orthonormality in the general case. One important class of integrals is the moments of $\left(\mathbb{J}_n^{(\alpha)}\right)^2$,

$$M(m) = \int_0^1 x^m \left(\mathbb{J}_n^{(\alpha)}\right)^2. \quad (39)$$

Figure 3a shows the first four moments of $\left(\mathbb{J}_1^{(\alpha)}\right)^2$ as a function of α and Fig. 3b shows the standard deviation, skewness, and kurtosis of $\left(\mathbb{J}_1^{(\alpha)}\right)^2$ also as a function of α . The standard deviation and the kurtosis are weak functions of α . The kurtosis increases slightly with decreasing α but the standard deviation slightly increases with decreasing α until about $\alpha = 2/5$ then it slightly decreases as α tends to zero. The skewness on the other hand is a stronger function of α as it rises from zero at $\alpha = 1$ to approximately 0.18 as $\alpha \rightarrow 0$.

One interesting integral to consider is

$$\int_{\mathbb{N}_n^{(\alpha)}(k)}^{\mathbb{N}_n^{(\alpha)}(k+1)} \left(\mathbb{J}_n^{(\alpha)}\right)^2 dx. \quad (40)$$

This is the area between adjacent zeros and is shown in Fig. 4 for $\mathbb{J}_2^{(\alpha)}$ and $\mathbb{J}_3^{(\alpha)}$. Note that the area between $x = 0$ and $x = \mathbb{N}_n^{(\alpha)}(1)$ decreases with decreasing α whereas the areas between higher zeros increase. The difference in area for the higher zeros gets smaller (not shown for $n > 3$).

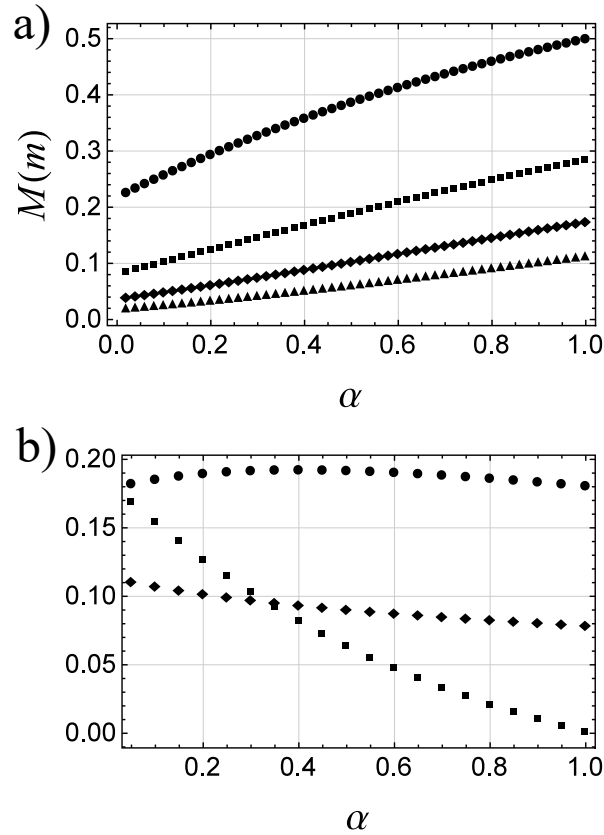


FIGURE 3. The (a) first four moments (\bullet 1, \blacksquare 2, \blacklozenge 3, \blacktriangle 4) and (b) the standard deviation (\bullet), skewness (\blacksquare) and kurtosis (\blacklozenge) for $(\mathbb{J}_1^{(\alpha)})^2$. The standardized moment (rather than the cumulant) definition is being used for skewness and kurtosis. The mean and skewness are relatively strong functions of α whereas the standard deviation and kurtosis are weak functions of α .

3.3. Relation to the Fourier sine series and the Fourier-Bessel series.

Since the $\mathbb{J}_n^{(\alpha)}$ functions form a complete orthonormal set, they can serve as a basis for expansion of arbitrary functions over the domain $0 \leq x \leq 1$

$$f(x) = \sum_{n=1}^{\infty} a_n \mathbb{J}_n^{(\alpha)}(x), \tag{41}$$

where coefficients are obtained in the usual way via

$$a_n = \int_0^1 f(x) \mathbb{J}_n^{(\alpha)}(x) dx. \tag{42}$$

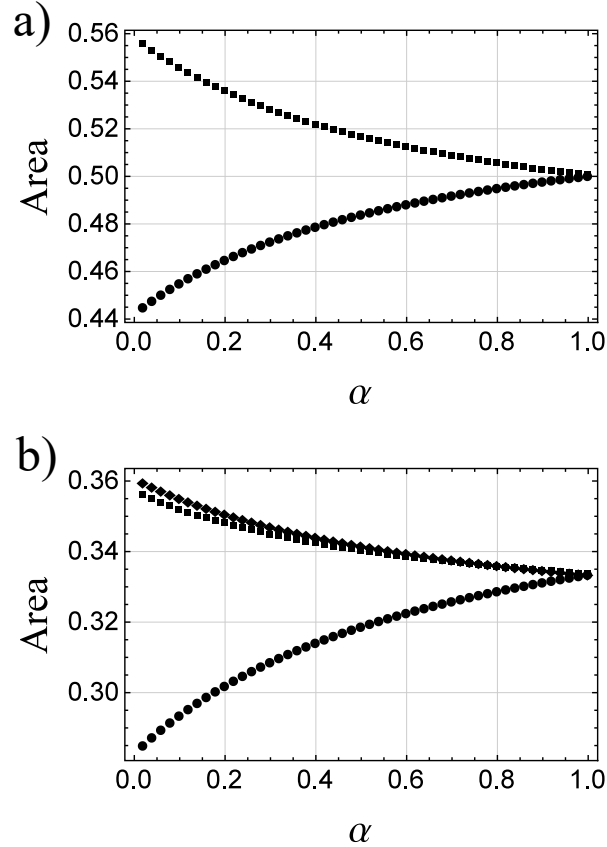


FIGURE 4. A comparison of the integral of expression (40) for (a) $\mathbb{J}_2^{(\alpha)}$ and (b) $\mathbb{J}_3^{(\alpha)}$. The (●) represents $\int_0^{\mathbb{N}_n^{(\alpha)}(1)} \left(\mathbb{J}_n^{(\alpha)}\right)^2 dx$, (■) represents $\int_{\mathbb{N}_n^{(\alpha)}(1)}^{\mathbb{N}_n^{(\alpha)}(2)} \left(\mathbb{J}_n^{(\alpha)}\right)^2 dx$, (◆) represents $\int_{\mathbb{N}_n^{(\alpha)}(1)}^1 \left(\mathbb{J}_n^{(\alpha)}\right)^2 dx$. The relative area on $\left(\mathbb{J}_n^{(\alpha)}\right)^2$ between 0 and $\mathbb{N}_n^{(\alpha)}(1)$ decreases whereas the relative area between all higher zeros increases.

Figure 5 shows the spectral decomposition of a few representative examples: $f(x) = \sin \pi x$, $f(x) = \frac{1}{2} - \left|x - \frac{1}{2}\right|$ (a triangle waveform), and $f(x) = \mathbb{J}_1^{(\frac{1}{2})}(x)$. The first two of these functions are symmetric about $x = \frac{1}{2}$. Expectedly, their spectral decompositions show a decreasing amount of a_1 and an increasing amount of a_2 as α decreases because the basis functions are becoming more skewed to the left. The insets in the figures show how a truncated series compares to $f(x)$.

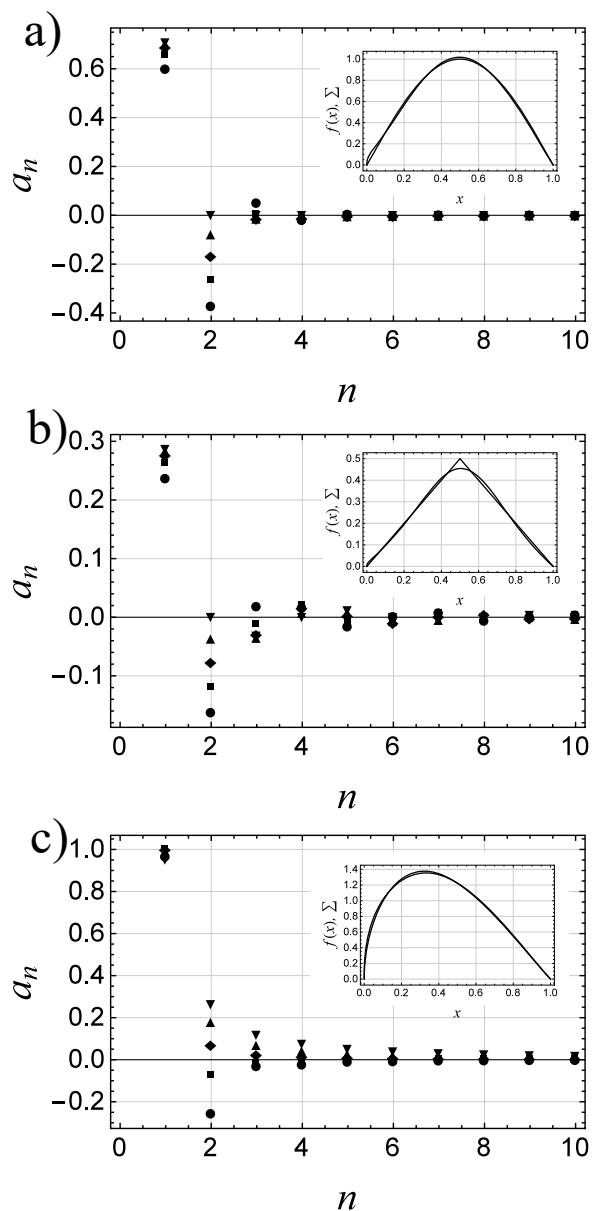


FIGURE 5. Generalized harmonic spectral decomposition series for representative functions, (a) $f(x) = \sin \pi x$, (b) $f(x) = \frac{1}{2} - |x - \frac{1}{2}|$ and (c) $f(x) = \mathbb{J}_1^{(1/2)}(x)$. $\alpha = 1/5$ (\bullet), $2/5$ (\blacksquare), $3/5$ (\blacklozenge), $4/5$ (\blacktriangle), 1 (\blacktriangledown). The insets show the representative function and a truncated series. For (a) and (c) one 2 terms in the series were used while 5 terms were used for (b). If more terms are used the partial sum function becomes visually identical to the representative function (except for near the sharp point in (b)).

One can use the definition of $\mathbb{J}_n^{(\alpha)}$ (Eq. (32)) in Eq. (41) to obtain a relation to the Fourier-Bessel series,

$$f(x) = \sum_{n=1}^{\infty} a_n N_n \sqrt{x^\alpha} J_\eta \left(n_\eta (x^\alpha)^{\frac{1}{2\eta}} \right), \quad (43)$$

where $N_n = 1/\sqrt{(\eta-1)J_{\eta-1}(n_\eta)J_{\eta+1}(n_\eta)}$. Letting $c_n \equiv a_n N_n$ and $z = (x^\alpha)^{\frac{1}{2\eta}} = x^{\frac{1+\alpha}{2}}$ yields

$$f(z) = z^\eta \sum_{n=1}^{\infty} c_n J_\eta (n_\eta z), \quad (44)$$

where the summation factor is recognized as the well-known Fourier-Bessel series [42]. Expressing

$$\frac{f(z)}{z^\eta} = g(z) = \sum_{n=1}^{\infty} c_n J_\eta (n_\eta z) \quad (45)$$

where

$$c_n = \int_0^1 z g(z) J_\eta (n_\eta z) dz. \quad (46)$$

Often the c_n needs to be calculated numerically but several important cases do yield analytic representations. First, consider the case when $g(z) = 1$. In this situation

$$\begin{aligned} c_n &= \int_0^1 z J_\eta (n_\eta z) dz \\ &= \frac{n_\eta^\eta {}_1F_2 \left(\frac{\eta}{2} + 1; \frac{\eta}{2} + 2, \eta + 1; -\frac{1}{4} n_\eta^2 \right)}{2^\eta (\eta + 2) \Gamma(\eta + 1)}, \end{aligned} \quad (47)$$

where ${}_1F_2$ is the hypergeometric function and Γ is the gamma function. Substitution back for x and a_n gives.

$$\sqrt{x^\alpha} = \sum_{n=1}^{\infty} \frac{c_n}{N_n} \mathbb{J}_n^{(\alpha)}(x). \quad (48)$$

More generally consider the case when $g(z) = z^{\gamma-1}$ where $\gamma > -1$. Then,

$$\begin{aligned} c_n &= \int_0^1 z^\gamma J_\eta (n_\eta z) dz \\ &= \frac{n_\eta^\eta {}_1F_2 \left(\frac{1}{2}(\gamma + \eta + 1); \frac{1}{2}(\gamma + \eta + 3), \eta + 1; -\frac{1}{4} n_\eta^2 \right)}{2^\eta \Gamma \left(\frac{1}{2}(\gamma + \eta + 2) \right) \Gamma(\eta + 1)}. \end{aligned} \quad (49)$$

The particular values of $\gamma = m + 1 - \frac{\alpha}{2}$, where m is an integer, gives the monomials, $f(x) = x^m$.

3.4. Relation of $\mathbb{J}_n^{(\alpha)}$ to the confluent hypergeometric functions. It is well-known that the Bessel functions are related to the confluent hypergeometric (or Kummer) functions ${}_1F_1(a; b; x)$ [2]. It turns out that expressing $\mathbb{J}_n^{(\alpha)}$ in terms of ${}_1F_1(a; b; x)$ can be done but leads to a fairly complicated function. It is perhaps better to use the confluent hypergeometric limit function [43, 50] ${}_0F_1(; b; x)$ relation to the Bessel functions. One can employ

$$J_\eta(z) = \frac{z^2 {}_0F_1 \left(; \eta + 1; -\frac{1}{4} z^2 \right)}{2^2 \Gamma(\eta + 1)}. \quad (50)$$

Substitution of this into Eq. (32) expresses $\mathbb{J}_n^{(\alpha)}$ in terms of ${}_0F_1$ as

$$\mathbb{J}_n^{(\alpha)} = \frac{n_\eta^\eta x^\alpha {}_0F_1\left(\cdot; \eta + 1; -\frac{n_\eta^2}{4} x^{\alpha+1}\right)}{2^2 \Gamma(\eta + 1) \sqrt{(\eta - 1) J_{\eta-1}(n_\eta) J_{\eta+1}(n_\eta)}}. \tag{51}$$

3.5. Conformable Sturm-Liouville systems. Consider the operator

$$\hat{S} = \frac{d}{dx} f(x) \frac{d}{dx}, \tag{52}$$

which can be expanded as

$$\hat{S} = f(x) \frac{d^2}{dx^2} + f'(x) \frac{d}{dx}. \tag{53}$$

Now consider the conformable version of \hat{S} ,

$$\hat{S}_{\alpha/\beta} = D^\beta f(x) D^\alpha. \tag{54}$$

The question is how $\hat{S}_{\alpha/\beta}$ relates to \hat{S} . Expanding with $D^\alpha = x^{1-\alpha} D^1$ we see,

$$\begin{aligned} \hat{S}_{\alpha/\beta} &= x^{1-\beta} D^1 [f x^{1-\alpha} D^1] \\ &= x^{1-\beta} ((x^{1-\alpha} f D^2 + ((1 - \alpha)x^{-\alpha} f + x^{1-\alpha} f') D^1) \\ &= x^{2-\alpha-\beta} f D^2 + ((1 - \alpha)x^{1-\alpha-\beta} f + x^{2-\alpha-\beta} f') D^1. \end{aligned} \tag{55}$$

This operator is not in Sturm-Liouville form but can be made so. Define,

$$\begin{aligned} h &= \frac{1}{x^{2-\alpha-\beta} f} \exp \left[\int^x \frac{(1 - \alpha) u^{1-\alpha-\beta} f + u^{2-\alpha-\beta} f'}{u^{2-\alpha-\beta} f} du \right] \\ &= \frac{1}{x^{2-\alpha-\beta} f} \exp \left[\int^x \frac{(1 - \alpha)}{u} du \right] \exp \left[\int^x \frac{f'}{f} du \right] \\ &= \frac{1}{x^{2-\alpha-\beta} f} x^{1-\alpha} f \\ &= x^{\beta-1}. \end{aligned} \tag{56}$$

So,

$$\begin{aligned} \widehat{\mathbb{S}}_{\alpha/\beta} &= h \hat{S}_{\alpha/\beta} \\ &= x^{1-\alpha} f D^2 + ((1 - \alpha)x^{-\alpha} f + x^{1-\alpha} f') D^1 \\ &= \frac{d}{dx} x^{1-\alpha} f \frac{d}{dx}. \end{aligned} \tag{57}$$

Thus a self-adjoint conformable Sturm-Liouville operator is of the form

$$\widehat{\mathbb{S}}_{\alpha/\beta} = x^{\beta-1} D^\beta f(x) D^\alpha. \tag{58}$$

Examples

Case 1: $\beta = \alpha$ and $f(x) = 1$. Then,

$$\widehat{\mathbb{S}}_{\alpha/\alpha} = x^{\alpha-1} D^\alpha D^\alpha = \hat{A}_{2\alpha} \tag{59}$$

where $\hat{A}_{2\alpha}$ is from Eq. (30).

Case 2: $\beta = \alpha$ and $f(x) = x^n$, $n \in \mathbb{Z}^+$ (the positive integers). Then,

$$\widehat{\mathbb{S}}_{\alpha/\alpha} = \frac{d}{dx} x^{n+1-\alpha} \frac{d}{dx}. \tag{60}$$

Some differential equations and their solutions are

$$\widehat{\mathbb{S}}_{\alpha/\alpha}y = 0 \quad (61)$$

giving

$$y = A \frac{x^{\alpha-n}}{\alpha-n} + B, \quad (62)$$

and

$$\widehat{\mathbb{S}}_{\alpha/\alpha}y = \Lambda \quad (63)$$

giving

$$y = \frac{\Lambda x^{\alpha-n+1}}{\alpha-n+1} + A \frac{x^{\alpha-n}}{\alpha-n} + B. \quad (64)$$

When $n \rightarrow 0$ in each of these equations we get the results from acting with $\widehat{A}_{2\alpha}$. This eigenvalue equation,

$$\widehat{\mathbb{S}}_{\alpha/\alpha}y = \Lambda y \quad (65)$$

does not solve except for $n = 1$ which gives

$$\begin{aligned} y &= A(x^\alpha)^{\frac{\alpha-1}{2\alpha}} J_{\frac{\alpha-1}{\alpha}} \left(\frac{2\sqrt{\Lambda x^\alpha}}{\alpha} \right) \\ &+ B(x^\alpha)^{\frac{\alpha-1}{2\alpha}} J_{\frac{1-\alpha}{\alpha}} \left(\frac{2\sqrt{\Lambda x^\alpha}}{\alpha} \right). \end{aligned} \quad (66)$$

Case 3: $\beta = \alpha$ and $f(x) = x^\alpha$. Then

$$\widehat{\mathbb{S}}_{\alpha/\alpha} = \frac{d}{dx} x \frac{d}{dx}. \quad (67)$$

Now,

$$\widehat{\mathbb{S}}_{\alpha/\alpha}y = \Lambda y \quad (68)$$

is simply

$$y = AJ_0(2\sqrt{\Lambda x}) + BY_0(2\sqrt{\Lambda x}). \quad (69)$$

Case 4: $\beta = \alpha$ and $f(x) = x^p$ where p is a rational fraction. MATHEMATICA can not solve this case generally but, a solution for

$$\widehat{\mathbb{S}}_{\alpha/\alpha}y = \Lambda y \quad (70)$$

can be discerned to be

$$y = A(x^\alpha)^{\left(\frac{\alpha-p}{2\alpha}\right)} J_{\frac{p-\alpha}{\kappa}} \left(\frac{2\sqrt{\Lambda}}{\kappa} (x^\alpha)^{\left(\frac{\kappa}{2\alpha}\right)} \right) + B(x^\alpha)^{\left(\frac{\alpha-p}{2\alpha}\right)} J_{\frac{\alpha-p}{\kappa}} \left(\frac{2\sqrt{\Lambda}}{\kappa} (x^\alpha)^{\left(\frac{\kappa}{2\alpha}\right)} \right), \quad (71)$$

where $\kappa = 1 + \alpha - p$. This does simplify further as

$$y = Ax^{\frac{\alpha-p}{2}} J_{\frac{p-\alpha}{\kappa}} \left(\frac{2\sqrt{\Lambda}}{\kappa} x^{\frac{\kappa}{2\alpha}} \right) + BAx^{\frac{\alpha-p}{2}} J_{\frac{\alpha-p}{\kappa}} \left(\frac{2\sqrt{\Lambda}}{\kappa} x^{\frac{\kappa}{2\alpha}} \right). \quad (72)$$

For the case of boundary conditions at $y(0) = y(1) = 0$,

$$y = Bx^{\frac{\alpha-p}{2}} J_{\frac{\alpha-p}{\kappa}} \left(n_{\frac{\alpha-p}{\kappa}} x^{\frac{\kappa}{2\alpha}} \right), \quad (73)$$

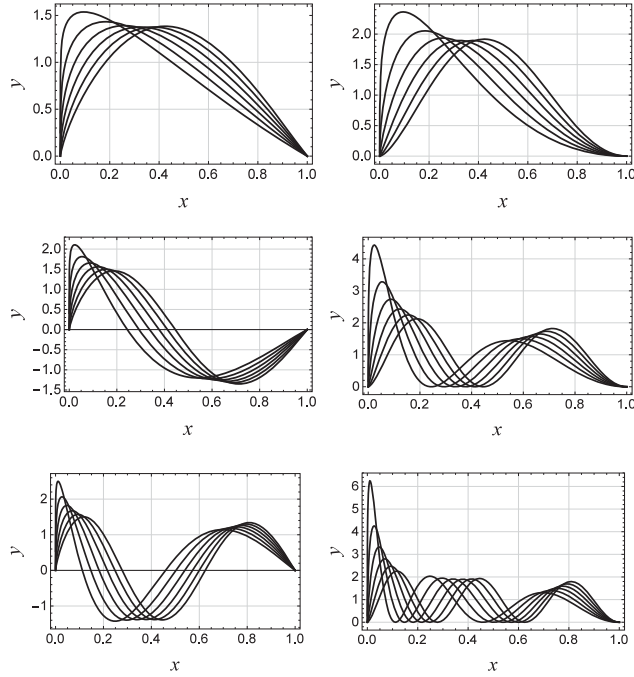


FIGURE 6. y (left column) and y^2 (right column) from Eq. (73) for the case of $\alpha = 3/4$ and $p = 0, 1/8, 1/4, 3/8, 1/2, 5/8$ for the first three eigenstates. As p approaches α the graphs compress towards $x = 0$.

where $n_{\frac{p-\alpha}{\kappa}}$ is the n^{th} zero of $J_{\frac{\alpha-p}{\kappa}}$. The normalization constant is

$$B = \frac{1}{\sqrt{\left(\frac{-1}{1+\alpha-p}\right) J_{\frac{\alpha-p}{\kappa}-1}\left(n_{\frac{\alpha-p}{\kappa}}\right) J_{\frac{\alpha-p}{\kappa}+1}\left(n_{\frac{\alpha-p}{\kappa}}\right)}}. \tag{74}$$

Figure 6 shows y (left column) and y^2 (right column) for the case of $\alpha = 3/4$ and $p = 0, 1/8, 1/4, 3/8, 1/2, 5/8$ for the first three eigenstates. As p approaches α the graphs compress towards $x = 0$. Figure 7 shows the case where $p = 1/4$ and $\alpha = 3/8, 1/2, 5/8, 3/4, 7/8$ and 1 for y for the first and second eigenfunctions.

Case 4 brings up an interesting characteristic in that the solutions depend only on $\alpha - p \equiv \lambda$ thus what matters for the shape of the curve of y is how far p is from

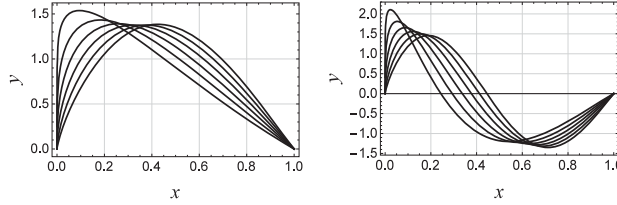


FIGURE 7. The case where $p = 1/4$ and $\alpha = 3/8, 1/2, 5/8, 3/4, 7/8$ and 1 for y for the first and second eigenfunctions for Eq. (73).

α not the absolute values of either. Stepping back further; if $f(x) = x^p g(x)$ then

$$\begin{aligned} \frac{d}{dx} x^{1-\alpha} f \frac{d}{dx} &= \frac{d}{dx} x^{1-\alpha} x^p g(x) \frac{d}{dx} \\ &= \frac{d}{dx} x^{1-\alpha+p} g(x) \frac{d}{dx} \\ &= \frac{d}{dx} x^{1-\lambda} g(x) \frac{d}{dx} x^{\beta-1} D^\beta g(x) D^\lambda. \end{aligned} \quad (75)$$

Here p is not restricted to being a rational fraction nor less than α but λ must still be less than 1.

Case 5: Now consider the case when

$$\begin{aligned} x^{1-\alpha} f' + (1-\alpha)x^{-\alpha} f &= 0 \\ f' + (1-\alpha)x^{-1} f &= 0 \end{aligned} \quad (76)$$

this occurs when $f = x^{\alpha-1}$. So, consider $\beta = \alpha$ and $f(x) = x^{\alpha-1}$. This solves the eigenvalue equation with the form

$$y = A \cos(\sqrt{\Lambda}x) + B \sin(\sqrt{\Lambda}x). \quad (77)$$

Taken together cases 1–5 suggest the conjecture that Eq. (73) and (74) is the solution even when p is expanded from a rational fraction to the reals in which $0 < p \leq \alpha$. And perhaps even when $0 < p \leq \infty$.

Often times one encounters operators of the form

$$\hat{S} = \frac{1}{f(x)} \frac{d}{dx} f(x) \frac{d}{dx}, \quad (78)$$

which, when made conformable, becomes

$$\frac{1}{f(x)} \widehat{\mathcal{S}}_{\alpha/\beta}. \quad (79)$$

We can consider a special case of this type of operator.

Case 6: $f = x^r$ where r is any real number greater than zero. The solutions to

$$\frac{1}{x^r} \widehat{\mathcal{S}}_{\alpha/\beta} y = \Lambda y \quad (80)$$

are

$$y = A(x^\alpha)^{\frac{\alpha-r}{2\alpha}} J_{\frac{\alpha-r}{1+\alpha}} \left(\frac{2\sqrt{\Lambda}(x^\alpha)^{\frac{\alpha+1}{2\alpha}}}{1+\alpha} \right) + B(x^\alpha)^{\frac{\alpha-r}{2\alpha}} J_{\frac{r-\alpha}{1+\alpha}} \left(\frac{2\sqrt{\Lambda}(x^\alpha)^{\frac{\alpha+1}{2\alpha}}}{1+\alpha} \right). \quad (81)$$

4. INTEGRAL TRANSFORMS

Any new definition of a fractional/conformable derivative leads naturally to the consideration of fractional/conformable differential equations and, subsequently, the use of fractional/conformable Laplace transforms to solve them. Indeed, numerous versions of fractional/conformable Laplace transforms have appeared in the literature. Some of these look very much like a regular Laplace transform [1, 30, 25, 44], while others look quite different [45, 15, 47]. The k -Laplace transforms [47] look a bit more like Mellin transforms, while the definitions used by Sharma [45], Deshmukh and Gudadhe [15], and Gorty [18] involve cotangents and cosecants in the exponential Laplace kernel. To be sure, the regular Laplace transform has also been used to tackle fractional differential equations, often resulting in a Mittag-Leffler expansion solution [39].

In this work, the choice was made to actually explore a conformable formulation of a Fourier transform, whose conventional counterpart is

$$\mathfrak{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{i\omega t} dt. \quad (82)$$

This is trivially related to the Laplace transform with $s = -i\omega$ and limiting the integration to $t \geq 0$. The reason for working with the Fourier transform is twofold. First, the inverse transform involves integration on the real axis of the transform variable rather than along the Bromwich contour in the complex s -plane as is done with the inverse Laplace transform. Second, the Fourier transform treats the forward and inverse transforms more symmetrically, and with potential applications to quantum mechanics in mind, this serves as a more natural path. The conformable Laplace transform is explicitly given at the end of subsection 4.1.5.

4.1. The conformable Fourier transform. The Fourier transform is developed here in the context of the definition of the conformable derivative given in Eq. (2), the D^α operator given in Eq. (3) and its corresponding inverse,

$$I^\alpha = \int^t \frac{(\cdot)}{\tau^{1-\alpha}} d\tau. \quad (83)$$

That is, for $\tau = (\alpha x)^{\frac{1}{\alpha}} \implies d\tau = (\alpha x)^{\frac{1}{\alpha}-1} dx$ and endpoints $t_0 \mapsto (\alpha t_0)^{\frac{1}{\alpha}} = t'_0$, $t \mapsto (\alpha t)^{\frac{1}{\alpha}} = t'$, we have

$$I^\alpha [f(t)] = \int_{t_0}^t f(\tau)\tau^{\alpha-1} d\tau = \int_{t'_0}^{t'} f \left[(\alpha x)^{\frac{1}{\alpha}} \right] dx. \quad (84)$$

Abdeljawad has recently explored a Laplace transform in this context.[1] (Note: similar to the change of variables technique used in creating conformable differential equations from ordinary differential equations, one may apply Eq (84) to create a conformable integral equation from an ordinary integral equation.)

In the reverse direction, if one has any integral in the form $\int_{t'_0}^{t'} f(x)dx$ one may create a conformable integral from it via $x = \frac{\tau^\alpha}{\alpha} \implies dx = \tau^{\alpha-1}d\tau$ and endpoints $t_0 = \frac{(t'_0)^\alpha}{\alpha}$ and $t = \frac{(t')^\alpha}{\alpha}$. That is

$$\int_{t'_0}^{t'} f(x)dx = \int_{t_0}^t f\left(\frac{\tau^\alpha}{\alpha}\right) \tau^{\alpha-1}d\tau = I^\alpha [g(t)] \quad (85)$$

for $g(\tau) = f\left(\frac{\tau^\alpha}{\alpha}\right)$.

We do precisely this to recover a conformable Fourier transform. That is,

$$\begin{aligned} \mathbb{F}\left[f\left[(\alpha t)^{\frac{1}{\alpha}}\right]\right] &= \int_{-\infty}^{\infty} f\left[(\alpha t)^{\frac{1}{\alpha}}\right] e^{-ist} dt \\ &= \int_{-\infty}^0 f\left[(\alpha t)^{\frac{1}{\alpha}}\right] e^{-ist} dt + \int_0^{\infty} f\left[(\alpha t)^{\frac{1}{\alpha}}\right] e^{-ist} dt \\ &= \int_{\Gamma^\alpha}^0 f(\tau) e^{-is\frac{\tau^\alpha}{\alpha}} \tau^{\alpha-1} d\tau + \int_0^{\infty} f(\tau) e^{-is\frac{\tau^\alpha}{\alpha}} \tau^{\alpha-1} d\tau \\ &= \mathbb{F}_\alpha[f(\tau)] \tilde{f}_\alpha(s), \end{aligned} \quad (86)$$

where the notation $\int_{\Gamma^\alpha}^0$ indicates integration along the complex ray $re^{i\pi/\alpha}$ where $r \in (0, \infty)$ (see Fig. 8). In this manner, one may compute a conformable Fourier transform from a special case of the regular Fourier transform.

The conformable Fourier transform carries with it some difficulties when viewed in the context of the change of variable because as $t \rightarrow t^\alpha/\alpha$ one no longer can integrate over the negative real values of t . However the same change of variable suggests a suitable integration contour as depicted in Fig. 8. When viewed in the complex plane, one can avoid the branch cut created along the negative real axis by bending the integration contour in Eq. (82) to that along the ray formed by $re^{i\pi/\alpha}$. The symbol \int_{Γ^α} is used to represent integration along the contour shown in Fig. 8.

Looking at the inverse transform, the same pattern emerges, giving

$$\begin{aligned} 2\pi\mathbb{F}^{-1}\left[\tilde{f}_\alpha\left[(\beta s)^{\frac{1}{\alpha}}\right]\right] &= \int_{-\infty}^{\infty} \tilde{f}_\alpha\left[(\beta s)^{\frac{1}{\alpha}}\right] e^{-ist} ds \\ &= \int_{-\infty}^0 \tilde{f}_\alpha\left[(\beta s)^{\frac{1}{\alpha}}\right] e^{-ist} ds + \int_0^{\infty} \tilde{f}_\alpha\left[(\beta s)^{\frac{1}{\alpha}}\right] e^{-ist} ds \\ &= \int_{\Gamma^\beta}^0 \tilde{f}_\alpha(\omega) e^{-i\frac{\omega^\beta}{\beta}t} \omega^{\beta-1} d\omega + \int_0^{\infty} \tilde{f}_\alpha(\omega) e^{-i\frac{\omega^\beta}{\beta}t} \omega^{\beta-1} d\omega \\ &= 2\pi\mathbb{F}^{-1}\left[\tilde{f}_\alpha(\omega)\right] = 2\pi f_{\alpha/\beta}(t). \end{aligned} \quad (87)$$

Thus the conformable Fourier transform pair is

$$\begin{aligned} \mathbb{F}_{\alpha/\beta}[f(t)] &= \tilde{f}_{\alpha/\beta}(\omega) \\ &= \int_{\Gamma^\alpha}^{\infty} f(t) e^{\frac{i}{\alpha\beta}\omega^\beta t^\alpha} dt^\alpha \\ &= \int_0^{\infty} f\left(\frac{t^\alpha}{\alpha}\right) e^{\frac{i}{\alpha\beta}\omega^\beta t^\alpha} t^{\alpha-1} dt + \int_0^{\infty} f\left(-\frac{t^\alpha}{\alpha}\right) e^{\frac{-i}{\alpha\beta}\omega^\beta t^\alpha} t^{\alpha-1} dt \end{aligned} \quad (88)$$

4.1.1. *Conformable Dirac δ -function.* The Dirac δ -function plays an important role in this analysis and it is useful to define a fractional version of it. Define the conformable Dirac δ -function to be

$$\delta(x^\alpha - x_0^\alpha) \equiv \frac{1}{2\pi} \int_{\Gamma^\beta}^\infty e^{\frac{i}{\alpha\beta} y^\beta (x^\alpha - x_0^\alpha)} dy^\beta. \quad (91)$$

One can investigate the important case of $\delta(f(x^\alpha))$. With the substitution $v = x^\alpha$, one is able to employ the analogous property for the regular δ -function. Namely,

$$\delta(f(v)) = \sum_{\text{roots}} \frac{\delta(v - v_0)}{|f'(v_0)|}, \quad (92)$$

where each root, v_0 , of f provides a term in the summation. Substituting back for x gives

$$\begin{aligned} \delta(f(x^\alpha)) &= \sum_{\text{roots}} \frac{\delta(x^\alpha - v_0)}{|f'(v_0)|} \\ &= \sum_{\text{roots}} \frac{\delta(x - v_0^{1/\alpha})}{\alpha (v_0^{1/\alpha})^{\alpha-1} |f'(v_0)|}. \end{aligned} \quad (93)$$

Two important results that will be used subsequently follow. First,

$$\delta(x^\alpha - x_0^\alpha) = \frac{\delta(x - x_0)}{\alpha x_0^{\alpha-1}}. \quad (94)$$

Second, $\delta(x^\alpha - (x_1^\alpha - x_2^\alpha))$ can be obtained using Eq. (93) to be

$$\delta(x^\alpha - (x_1^\alpha - x_2^\alpha)) = \frac{\delta(x - (x_1^\alpha - x_2^\alpha)^{1/\alpha})}{\alpha ((x_1^\alpha - x_2^\alpha)^{1/\alpha})^{\alpha-1}}. \quad (95)$$

4.1.2. *Inversion pair.* Verifying the inversion pair

$$\begin{aligned} \mathbb{F}_{\alpha/\beta}^{-1}[\mathbb{F}_{\alpha/\beta}[f(t)]] &= \frac{1}{2\pi} \int_{\Gamma^\beta}^\infty \int_{\Gamma_1^\alpha}^\infty f(t_1) e^{\frac{i}{\alpha\beta} \omega^\beta t^\alpha} t_1^{\alpha-1} dt_1 e^{-\frac{i}{\alpha\beta} \omega^\beta t^\alpha} d\omega^\beta \\ &= \int_{\Gamma_1^\alpha}^\infty f(t_1) t_1^{\alpha-1} \frac{1}{2\pi} \int_{\Gamma^\beta}^\infty e^{\frac{i}{\alpha\beta} \omega^\beta (t_1^\alpha - t^\alpha)} d\omega^\beta dt_1. \\ &= \alpha \int_{\Gamma_1^\alpha}^\infty f(t_1) t_1^{\alpha-1} \delta(t_1^\alpha - t^\alpha) dt_1. \end{aligned} \quad (96)$$

Using Eq. (94)

$$\begin{aligned} \mathbb{F}_{\alpha/\beta}^{-1}[\mathbb{F}_{\alpha/\beta}[f(t)]] &= \alpha \int_{\Gamma_1^\alpha}^\infty f(t_1) t_1^{\alpha-1} \frac{\delta(t_1 - t)}{\alpha t_1^{\alpha-1}} dt_1 \\ &= \int_{-\infty}^\infty f(t_1) \delta(t_1 - t) dt_1 \\ &= f(t). \end{aligned} \quad (97)$$

Indeed one recovers the original function. For completeness one can also verify that

$$\mathbb{F}_{\alpha/\beta}[\mathbb{F}_{\alpha/\beta}^{-1}[\tilde{f}_{\alpha/\beta}(\omega)]] = \tilde{f}_{\alpha/\beta}(\omega) \quad (98)$$

in a similar way.

4.1.3. *The derivative and the transform-space derivative.* Consider the conformable Fourier transform of the conformable derivative

$$\begin{aligned}\mathbb{F}_{\alpha/\beta}[D^\alpha[f]] &= \int_0^\infty D^\alpha[f] e^{\frac{i}{\alpha\beta}\omega^\beta t^\alpha} dt^\alpha \\ &= \int_0^\infty t^{1-\alpha} \frac{df}{dt} e^{\frac{i}{\alpha\beta}\omega^\beta t^\alpha} t^{\alpha-1} dt \\ &= \int_0^\infty \frac{df}{dt} e^{\frac{i}{\alpha\beta}\omega^\beta t^\alpha} dt.\end{aligned}\quad (99)$$

Integration by parts gives

$$\begin{aligned}\mathbb{F}_{\alpha/\beta}[D^\alpha[f]] &= -i \frac{\omega^\beta}{\beta} \int_{\Gamma_\alpha} f e^{\frac{i}{\alpha\beta}\omega^\beta t^\alpha} t^{\alpha-1} dt \\ &= -i \frac{\omega^\beta}{\beta} \int_{\Gamma_\alpha} f e^{\frac{i}{\alpha\beta}\omega^\beta t^\alpha} dt^\alpha \\ &= -i \frac{\omega^\beta}{\beta} \tilde{f}_{\alpha/\beta}(\omega)\end{aligned}\quad (100)$$

and one recovers the usual formula with $\frac{\omega^\beta}{\beta}$ replacing ω . Consider the case where $1 < \kappa \leq 2$. Now let $\alpha = \kappa - 1$,

$$\begin{aligned}\mathbb{F}_{\alpha/\beta}[D^\kappa[f]] &= \mathbb{F}_{\alpha/\beta}[D^\alpha D^1[f]] \\ &= \int_{\Gamma_\alpha} D^\alpha f' e^{\frac{i}{\alpha\beta}\omega^\beta t^\alpha} dt^\alpha,\end{aligned}\quad (101)$$

which is just like Eq. (100) but with f replaced by f' . Thus

$$\mathbb{F}_{\alpha/\beta}[D^\kappa[f]] = -i \frac{\omega^\beta}{\beta} \int_{\Gamma_\alpha} f' e^{\frac{i}{\alpha\beta}\omega^\beta t^\alpha} dt^\alpha. \quad (102)$$

The second term is just the conformable Fourier transform of the conformable derivative so, ultimately,

$$\mathbb{F}_{\alpha/\beta}[D^\kappa[f]] = \left(\frac{\omega^\beta}{\beta}\right)^2 \tilde{f}_{\alpha/\beta}(\omega). \quad (103)$$

This same process could be carried out for $\kappa > 2$.

Turning now to the conformable Fourier transform of the function $\frac{t^\alpha}{\alpha} f$ one sees,

$$\begin{aligned}\mathbb{F}_{\alpha/\beta}\left[\frac{t^\alpha}{\alpha} f\right] &= \int_{\Gamma_\alpha} \frac{t^\alpha}{\alpha} f e^{\frac{i}{\alpha\beta}\omega^\beta t^\alpha} dt^\alpha \\ &= \int_{\Gamma_\alpha} D_\omega^\alpha \left[f e^{\frac{i}{\alpha\beta}\omega^\beta t^\alpha} \right] dt^\alpha \\ &= D_\omega^\alpha \left[\int_{\Gamma_\alpha} f e^{\frac{i}{\alpha\beta}\omega^\beta t^\alpha} dt^\alpha \right] \\ &= D_\omega^\alpha [\mathbb{F}_{\alpha/\beta}[f]],\end{aligned}\quad (104)$$

where D_ω^α means the conformable derivative in ω space.

4.1.4. *Conformable convolution and the conformable convolution theorem.* Consider the inverse Fourier transform of the product, $\tilde{f}_{\alpha/\beta}(\omega)\tilde{g}_{\alpha/\beta}(\omega)$. Here,

$$\begin{aligned}\mathbb{F}_{\alpha/\beta}^{-1} \left[\tilde{f}_{\alpha/\beta} \tilde{g}_{\alpha/\beta} \right] &= \frac{1}{2\pi} \int_{\Gamma^\beta} \int_{\Gamma_1^\alpha} f(t_1) e^{\frac{i}{\alpha\beta} \omega^\beta t_1^\alpha} dt_1^\alpha \int_{\Gamma_2^\alpha} g(t_2) e^{\frac{i}{\alpha\beta} \omega^\beta t_2^\alpha} dt_2^\alpha e^{-\frac{i}{\alpha\beta} \omega^\beta t^\alpha} d\omega^\beta \\ &= \int_{\Gamma_1^\alpha} f(t_1) \int_{\Gamma_2^\alpha} g(t_2) \frac{1}{2\pi} \int_{\Gamma^\beta} e^{-\frac{i}{\alpha\beta} \omega^\beta (t^\alpha - t_1^\alpha - t_2^\alpha)} d\omega^\beta dt_2^\alpha dt_1^\alpha \\ &= \alpha \int_{\Gamma_1^\alpha} f(t_1) \int_{\Gamma_2^\alpha} g(t_2) \delta(t_2^\alpha + t_1^\alpha - t^\alpha). \end{aligned} \quad (105)$$

Using Eq. (95),

$$\begin{aligned}\mathbb{F}_{\alpha/\beta}^{-1} \left[\tilde{f}_{\alpha/\beta} \tilde{g}_{\alpha/\beta} \right] &= \alpha \int_{\Gamma_1^\alpha} f(t_1) \int_{\Gamma_2^\alpha} g(t_2) \frac{\delta(t_2 - (t^\alpha - t_1^\alpha)^{1/\alpha})}{\alpha \left((t^\alpha - t_1^\alpha)^{1/\alpha} \right)^{\alpha-1}} t_2^{\alpha-1} dt_2 dt_1^\alpha \\ &= \int_{\Gamma_1^\alpha} f(t_1) g \left((t^\alpha - t_1^\alpha)^{1/\alpha} \right) \frac{\left((t^\alpha - t_1^\alpha)^{1/\alpha} \right)^{\alpha-1}}{\left((t^\alpha - t_1^\alpha)^{1/\alpha} \right)^{\alpha-1}} dt_1^\alpha \\ &= \int_{\Gamma_1^\alpha} f(t_1) g \left((t^\alpha - t_1^\alpha)^{1/\alpha} \right) dt_1^\alpha. \end{aligned} \quad (106)$$

At this point make a substitution $v = \frac{t_1^\alpha}{\alpha}$,

$$\mathbb{F}_{\alpha/\beta}^{-1} \left[\tilde{f}_{\alpha/\beta} \tilde{g}_{\alpha/\beta} \right] = \int_{-\infty}^{\infty} f \left((\alpha v)^{1/\alpha} \right) g \left(\left(\frac{\alpha t^\alpha}{\alpha} - \alpha v \right)^{1/\alpha} \right) dv. \quad (107)$$

For convenience one can recast the functions as

$$\begin{aligned}f(x) &= F \left(\frac{x^\alpha}{\alpha} \right) \\ g(x) &= G \left(\frac{x^\alpha}{\alpha} \right). \end{aligned} \quad (108)$$

Doing so gives

$$\mathbb{F}_{\alpha/\beta}^{-1} \left[\tilde{f}_{\alpha/\beta} \tilde{g}_{\alpha/\beta} \right] = \int_{-\infty}^{\infty} F(v) G \left(\frac{t^\alpha}{\alpha} - v \right) dv, \quad (109)$$

which upon replacing $v = \frac{t_1^\alpha}{\alpha}$ results in

$$\mathbb{F}_{\alpha/\beta}^{-1} \left[\tilde{f}_{\alpha/\beta} \tilde{g}_{\alpha/\beta} \right] = \int_{\Gamma_1^\alpha} F \left(\frac{t_1^\alpha}{\alpha} \right) G \left(\frac{t^\alpha}{\alpha} - \frac{t_1^\alpha}{\alpha} \right) dt_1^\alpha. \quad (110)$$

Noting that the integrand is active for $t_1 > t$, this then serves as a basis for a definition of a conformable convolution:

$$f * g = \int_{\Gamma_1^\alpha} F \left(\frac{t_1^\alpha}{\alpha} \right) G \left(\frac{t^\alpha}{\alpha} - \frac{t_1^\alpha}{\alpha} \right) dt_1^\alpha, \quad (111)$$

such that

$$\mathbb{F}_{\alpha/\beta} [f * g] = \tilde{f}_{\alpha/\beta}(\omega) \tilde{g}_{\alpha/\beta}(\omega). \quad (112)$$

4.1.5. *Special cases.* The transform pair defined in Eqs. (88) and (89) carries with it several special cases. First, $\alpha = \beta = 1$,

$$\mathbb{F}_{1/1}[f] = \int_{-\infty}^{\infty} f(t)e^{i\omega t} dt. \tag{113}$$

$\mathbb{F}_{1/1}$ is just the regular Fourier transform.

The second special case is when $\beta = \alpha$. Then,

$$\mathbb{F}_{\alpha/\beta}[f] = \int_{\Gamma^\alpha} f(t)e^{\frac{i}{\alpha^2}\omega^\alpha t^\alpha} dt^\alpha. \tag{114}$$

This is similar to, but not exactly, the definition used by Jumarie.[25] In that work the exponential is placed by a Mittag-Leffler function of $\omega^\alpha t^\alpha$.

A third case arises when $\beta = 1$,

$$\mathbb{F}_{\alpha/1}[f] = \int_{\Gamma^\alpha} f(t)e^{\frac{i}{\alpha}\omega t^\alpha} dt^\alpha, \tag{115}$$

which is the transform used by Abdeljawad [1].

The conversion to a conformable Laplace transform is trivially achieved by replacing $i\frac{\omega^\beta}{\beta}$ with $-\frac{s^\beta}{\beta}$ in Eq. (88). Then,

$$\mathbb{L}_{\alpha/\beta}[f(t)] = \bar{f}_{\alpha/\beta}(s) = \int_0^\infty f(t)e^{-\frac{1}{\alpha\beta}s^\beta t^\alpha} dt^\alpha. \tag{116}$$

Presented here are some applications of Eqs. (88) and (89) to particular functions. In some cases one can recover relatively simple expressions but in other cases software such as MATHEMATICA can compute a (complicated) solution that can be plotted.

It is of importance to consider functions that are explicit in $\frac{t^\alpha}{\alpha}$. In this case

$$\mathbb{F}_{\alpha/\beta} \left[f \left(\frac{t^\alpha}{\alpha} \right) \right] = \int_{\Gamma^\alpha} f \left(\frac{t^\alpha}{\alpha} \right) e^{\frac{i}{\alpha\beta}\omega^\beta t^\alpha} t^{\alpha-1} dt, \tag{117}$$

which upon the change of variables $u = \frac{t^\alpha}{\alpha}$ becomes

$$\mathbb{F}_{\alpha/\beta} \left[f \left(\frac{t^\alpha}{\alpha} \right) \right] = \int_{-\infty}^{\infty} f(u)e^{i\frac{\omega^\beta}{\beta}u} du = \tilde{f} \left(\frac{\omega^\beta}{\beta} \right), \tag{118}$$

where \tilde{f} (with no subscript α/β) is simply the regular Fourier transform. Likewise,

$$\mathbb{L}_{\alpha/\beta} \left[f \left(\frac{t^\alpha}{\alpha} \right) \right] = \bar{f} \left(\frac{s^\beta}{\beta} \right), \tag{119}$$

where \bar{f} is the regular Laplace transform. Consequently, the conformable Laplace transform can be read directly from standard tables.[29] Table I contains a number of common transforms.

Of course, a function not explicit in $\frac{t^\alpha}{\alpha}$ can be made so, $f(t) \Rightarrow F \left(\frac{t^\alpha}{\alpha} \right)$. In principle this should be of practical utility in evaluating conformable Laplace transforms; in practice, however, one often is still confronted with integrals that are not known. One important example are the monomials, $f(t) = t^n$. Placed in $\frac{t^\alpha}{\alpha}$ form this becomes

$$f(t) = \alpha^{\frac{n}{\alpha}} \left(\frac{t^\alpha}{\alpha} \right)^{\frac{n}{\alpha}}. \tag{120}$$

So,

$$\begin{aligned} \mathbb{L}_{\alpha/\beta} \left[\alpha^{\frac{n}{\alpha}} \left(\frac{t^\alpha}{\alpha} \right)^{\frac{n}{\alpha}} \right] &= \mathfrak{L} [\alpha^{\frac{n}{\alpha}} u^{\frac{n}{\alpha}}] \\ &= \alpha^{\frac{n}{\alpha}} \beta^{\left(\frac{n}{\alpha}+1\right)} \frac{\Gamma\left(\frac{n}{\alpha}+1\right)}{(s^\beta)^{\left(\frac{n}{\alpha}+1\right)}}, \end{aligned} \quad (121)$$

where \mathfrak{L} is the regular Laplace transform and Γ is the gamma function. Table I shows this and several other examples.

Table I: Several Laplace ($\bar{f}_{\alpha/\beta}(s)$) and Fourier ($\tilde{f}_{\alpha/\beta}(\omega)$) transforms. Here $n = 1, 2, \dots, p > 1$, Γ is the gamma function, and Erfc is the complimentary error function. For the Laplace transforms, it is assumed that there is an appropriate Bromwich contour for the inverse operation. The Fourier transforms require functions that $f(t)$ vanishes sufficiently rapidly in the limit as $t \rightarrow \infty$.

$f(t)$	$\bar{f}_{\alpha/\beta}(s)$	$\tilde{f}_{\alpha/\beta}(\omega)$
1	$\frac{\beta}{s^\beta}$	$2\pi\delta(s^\beta)$
t	$\alpha^{\frac{1}{\alpha}} \beta^{\left(\frac{1}{\alpha}+1\right)} \frac{\Gamma\left(\frac{1}{\alpha}+1\right)}{(s^\beta)^{\frac{1}{\alpha}+1}}$	–
t^n	$\alpha^{\frac{n}{\alpha}} \beta^{\left(\frac{n}{\alpha}+1\right)} \frac{\Gamma\left(\frac{n}{\alpha}+1\right)}{(s^\beta)^{\frac{n}{\alpha}+1}}$	–
$\frac{t^\alpha}{\alpha}$	$\left(\frac{\beta}{s^\beta}\right)^2$	–
$\left(\frac{t^\alpha}{\alpha}\right)^p$	$\left(\frac{\beta}{s^\beta}\right)^{p+1} \Gamma(p+1)$	–
$e^{-k\frac{t^\alpha}{\alpha}}$	$\frac{\beta}{\beta k + s^\beta}$	$\frac{\beta}{\beta k - i\omega^\beta}$
$\left(\frac{t^\alpha}{\alpha}\right)^p e^{-k\frac{t^\alpha}{\alpha}}$	$\left(\frac{\beta}{\beta k + s^\beta}\right)^{p+1} \Gamma(p+1)$	$\left(\frac{\beta}{\beta k - i\omega^\beta}\right)^{p+1} \Gamma(p+1)$
$\cos q\frac{t^\alpha}{\alpha}$	$\frac{\beta s^\beta}{s^\beta + \beta^2 q^2}$	–
$\sin q\frac{t^\alpha}{\alpha}$	$\frac{\beta^2 q}{s^{2\beta} + \beta^2 q^2}$	–
$e^{-k\frac{t^\alpha}{\alpha}} \cos q\frac{t^\alpha}{\alpha}$	$\frac{(s^\beta + \beta k)}{(s^\beta + \beta k)^2 + \beta^2 q^2}$	$\frac{(\beta k - i\omega^\beta)}{(\beta k - i\omega^\beta)^2 + \beta^2 q^2}$
$e^{-\sigma^2\left(\frac{t^\alpha}{\alpha}\right)^2}$	$\frac{\sqrt{\pi}}{2\sigma} e^{-\frac{s^\beta}{4\sigma^2\beta^2}} \text{Erfc}\left[\frac{s^\beta}{2\sigma\beta}\right]$	$\frac{1}{\sqrt{2}\sigma} e^{-\frac{1}{4\sigma^2}\left(\frac{\omega^\beta}{\beta}\right)^2}$

4.1.6. *Physical interpretation of the transform spaces.* The Fourier transform pair of Eqs. (88) and (89) can generalize the concept of complimentary transform spaces. One sees $\frac{t^\alpha}{\alpha}$ -space is transformed to $\frac{\omega^\beta}{\beta}$ -space and vice versa in the same way the regular Fourier transform connects t -space and ω -space. It is illustrative to use the applications of transforms and their spaces in physics to help glean some insight into conformable transforms. In physical systems time space is related to frequency space via the Fourier transform. On the one hand, one can consider t to carry units (like seconds) and ω to carry the inverse units. If this is the case then β must equal α such that the argument of the kernel in Eq. (88) is unitless. Thus the $\frac{t^\alpha}{\alpha}$ -space/ $\frac{\omega^\beta}{\beta}$ -space connection is restricted to $\frac{t^\alpha}{\alpha}$ and $\frac{\omega^\alpha}{\alpha}$. Attempting to develop some (albeit artificial) intuition one sees a “conformable second” in effect acting to dilate time as time goes on. That is, a function is getting stretched out for larger values of t . This is consistent with recent work on a conformable quantum

particle-in-a-box[4, 9] (where instead of time, space is the independent variable) and a classical harmonic oscillator [39].

On the other hand, one can start with the necessity of $t^\alpha \omega^\beta$ to be unitless but t not necessarily having units of time. Now the ω^β is carrying the inverse units of t^α . Or, the units on t are related to that of ω as

$$t = \omega^{-\frac{\beta}{\alpha}}. \quad (122)$$

This opens up a wider relationship because β need not equal α thereby connecting t to a range of transform spaces. To see this, let t carry conformable “units” of u^a , so that t^α has units of $u^{a\alpha}$. Likewise let ω carry units of u^{-b} , so that ω^β has units of $u^{-b\beta}$. The requirement that $t^\alpha \omega^\beta$ be unitless means $\beta = \frac{a}{b}\alpha$. Thus a scaling relationship exists between α and β : $\beta = \lambda\alpha$. When $\lambda = 1$, the case discussed above ($\beta = \alpha$) is recovered and t and ω have inverse units.

As an illustrative example consider the partner functions

$$f(t) = e^{-\frac{t^\alpha}{\alpha}} \overset{\mathbb{F}_{\alpha/\beta}}{\underset{\mathbb{F}_{\alpha/\beta}^{-1}}{\longleftrightarrow}} f_{\alpha/\beta}(\omega) = \frac{\beta}{\beta - i\omega^\beta}. \quad (123)$$

Without a connection between α and β this transform relationship is of little utility. Consider, though, the case where $\beta = \alpha$, which is plotted in Figs. 9 and 10 for $\alpha = 1/4, 1/2, 3/4$, and 1. When $\alpha = 1$, one sees the familiar Lorentzian curve for $\text{Re}[f_{\alpha/\alpha}(\omega)]$ and dispersion curve for $\text{Im}[f_{\alpha/\alpha}(\omega)]$. As α is decreased both curves sharpen up at low values of ω , with very significant compression of the curves occurring for values of $\alpha < 1/2$. Conversely, the tails of both curves for large values of ω fall away slower for decreasing values of α .

Now consider the case where λ varies. Here,

$$f_{\alpha/\lambda\alpha}(\omega) = \frac{\lambda\alpha}{\lambda\alpha - i\omega^{\lambda\alpha}}. \quad (124)$$

Figure 11 shows the case where $\alpha = 1$ and $\lambda = 1/4, 1/2, 1, 2, 4$. Here one sees a flattening of the Lorentzian curve for $\text{Re}[f_{\alpha/\lambda\alpha}(\omega)]$ and shifting and narrowing of the dispersion curve for $\text{Im}[f_{\alpha/\lambda\alpha}(\omega)]$ for values of $\lambda > 1$. For values of $\lambda < 1$, the same behavior as in Fig. 10 is seen.

5. APPLICATIONS TO QUANTUM MECHANICS

As concrete fodder for physical applications and, more importantly, interpretation of the conformable derivative we use some examples from quantum mechanics. We consider the conformable particle in a box and use it to investigate conformable perturbation theory, and conformable supersymmetry.

5.1. Conformable quantum particle in a box. The conformable quantum particle in a box has served as a good model system for gaining an understanding of conformable quantum mechanics.[9, 32, 33, 34, 35, 24, 36, 19, 52, 20] It has been studied using the nonlocal formulations of the conformable derivative. This has led to controversy [24, 36, 8] and the suggestion that the results for the solution to these formulations of particle in a box cannot be valid [24, 36]. It also is difficult to solve the problem in correct (observing nonlocality) form, although the nonlocality itself may offer some richness to the conformable Schrödinger equation [36]. The current work does not directly provide input into this on-going discussion. It does however offer an alternative formulation of the conformable quantum particle in a

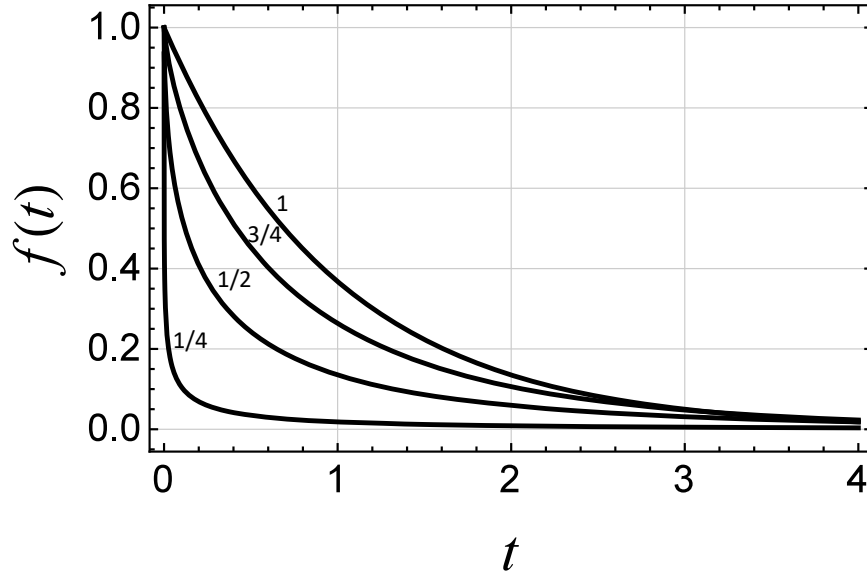


FIGURE 9. The fractional exponential function, $f = e^{-\frac{t^\alpha}{\alpha}}$ for values of $\alpha = 1/4, 1/2, 3/4,$ and 1 . The value of $\alpha = 1$ yields an exponentially decreasing function. As the value of α is decreased, the curve becomes much sharper. The Fourier-Laplace transform of this function is given in Fig. 2.

box that has many appealing features. It is based upon a local formulation of the conformable derivative (Eq. (2)) and it develops via Eq. (30) from a self-adjoint differential equation that, although complicated, is a normal differential equation that can be solved. As such, the solutions form an orthonormal set and the eigenvalues are real. Some issues remain with this formulation of the conformable quantum particle in a box. Most notably the point at $x = 0$ is not a regular point.

This formulation of a conformable quantum particle in a box has been suggested in an earlier work [4]. We explore a few more features of this model here; most importantly the results from perturbation theory. Further, the concept of a “phantom potential energy” is discussed in an effort to provide some physical insight into the model.

5.1.1. *Perturbation theory.* One can use the $\mathbb{J}_n^{(\alpha)}$ functions as a basis for time independent perturbation of the particle in a box potential in the standard way [12]. The unperturbed system is taken to be the conformable particle in a box, with units chosen so mass and Planck’s constant can be suppressed for convenience. Then the unperturbed wavefunction is $\psi_n^{(0)} = \mathbb{J}_n^{(\alpha)}$. Likewise the unperturbed energy, $E_n^{(0)}$, is given by Eq. (33). The full Hamiltonian with perturbation, V_I is then

$$H = A_{2\alpha} + \lambda V_I. \quad (125)$$

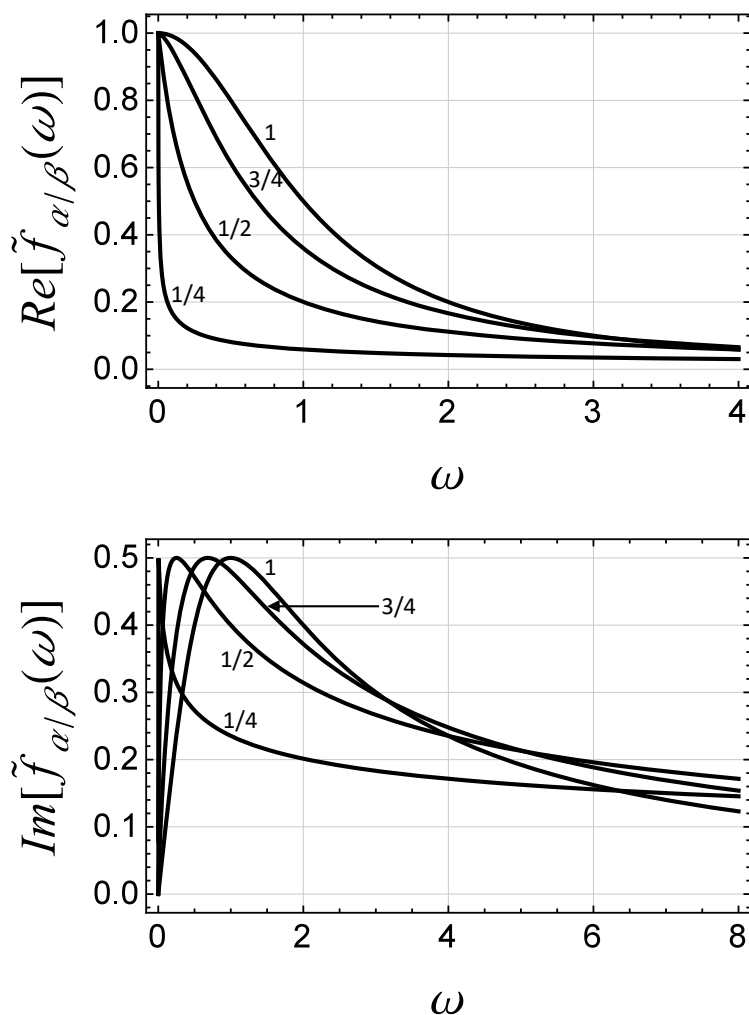


FIGURE 10. The Fourier-Laplace transform $f_{\alpha/\beta}(\omega) = \beta/(\beta - i\omega^\beta)$ of the fractional exponential function, $f = e^{-\frac{t^\alpha}{\alpha}}$ for values of $\beta = 1/4, 1/2, 3/4,$ and 1 . The real (imaginary) part of $f_{\alpha/\beta}(\omega)$ is plotted on the top (bottom) graph. The real part exhibits a Lorentzian lineshape for $\beta = 1$ and sharpens as β decreases. Not shown within this plot range, is the fact that the wings of these functions remain elevated longer for decreasing values of β . The imaginary part exhibits dispersion lineshape for $\beta = 1$ and again sharpens for decreasing β while maintaining higher values for large ω .

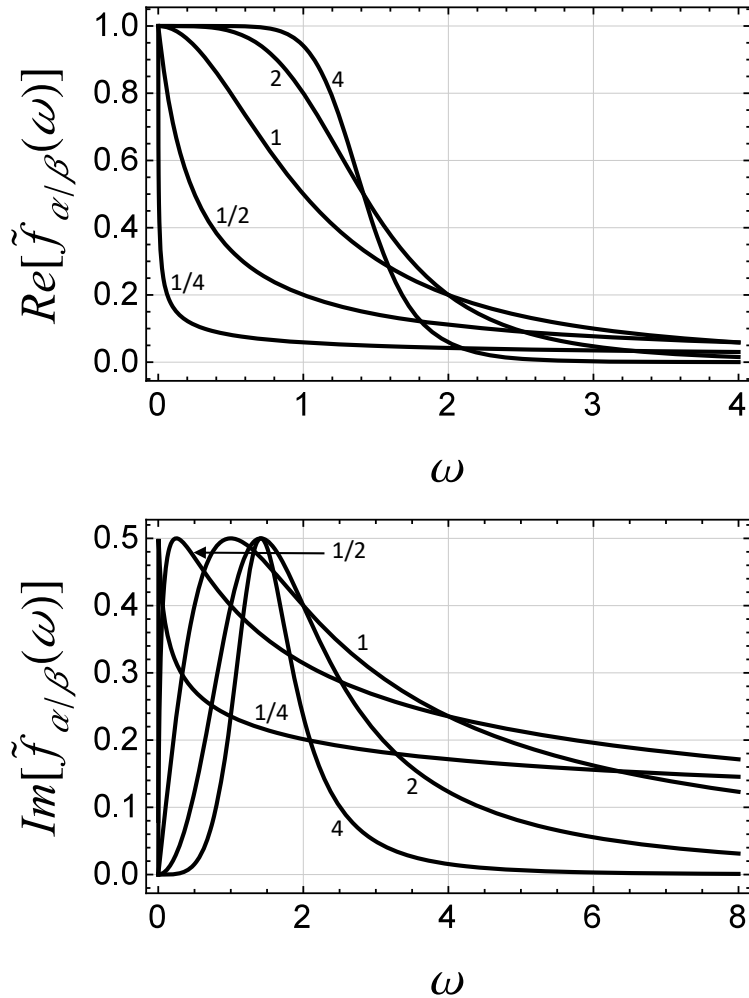


FIGURE 11. The Fourier-Laplace transform $f_{\alpha/\beta}(\omega) = \beta/(\beta - i\omega^\beta)$ of the fractional exponential function when $\alpha = 1$ and $\beta = \lambda\alpha$ for values of $\lambda = 1/4, 1/2, 1, 2,$ and 4 . The real (imaginary) part of $f_{\alpha/\beta}(\omega)$ is plotted on the top (bottom) graph. Consistent with Fig 2. The function sharpens for small values of ω but maintains higher values in the wings when $\lambda < 1$. The converse is true for values of $\lambda > 1$.

Figure 12a shows the wavefunction to first order, $\psi_1^{(1)}$, for the case where $V_I = E_1^{(0)}x$ and λ is set to -1 (solid curve) and 1 (dashed curve) for the case of $\alpha = 1/2$. The dotted curve shows $\psi_1^{(0)}$. Perhaps not surprisingly, when $\lambda < 0$ (the potential energy decreases with increasing x) the wavefunction shifts to the right; it becomes less skewed and more sine-like. The opposite is true when $\lambda > 0$. Figures 12b and 12c

shows the case where

$$V_I = \begin{cases} E_1^{(0)} & 0 \leq x \leq \frac{1}{4} \\ 0 & \frac{1}{4} < x \leq 1 \end{cases}, \quad (126)$$

for $\alpha = 1/2$ and $\alpha = 1/4$ respectively. The perturbation variable λ is set to -1 (solid curve) and 1 (dashed curve). For these cases, the presence of the step ($\lambda > 0$) on the left side of the well pushes the wavefunction to the right and it becomes more sine-like. Conversely the presence of hole ($\lambda < 0$) increases the skewing to the left. Comparing Figs 12b and 12c one sees the impact of the perturbation is more pronounced for $\alpha = 1/4$ than for $\alpha = 1/2$.

Figure 13 considers the effect of the perturbation of Eq. (126) and its mirror image,

$$V_I = \begin{cases} 0 & 0 \leq x \leq \frac{3}{4} \\ E_1^{(0)} & \frac{3}{4} < x \leq 1 \end{cases}, \quad (127)$$

on the first order correction to the ground state energy. The solid diamond symbol represents the data for the case where $\alpha = 1$. Because of the symmetry of the wavefunctions for this case both perturbations (Eq. (126) and Eq. (127)) have the same effect. The ground state energy is increased for a step ($\lambda > 0$) and decreased for a hole ($\lambda < 0$). When one considers the case where $\alpha = 1/2$ the symmetry of the wavefunction about $x = 1/2$ is broken. When the step (or hole) is on the left (data: solid circle), the impact on the ground state energy is more pronounced than when the step (or hole) is on the right side (data: solid square). This makes intuitive sense based on the shape of the wavefunction, but it also points to the idea that the lower values of x carry more weight than higher values (discussed more below). As α decreases from unity, the distinction increases, but interestingly, not in a linear fashion (Fig. 13b). The most pronounced difference between the effect of a step (or hole) on the left versus right occurs at roughly a value of $\alpha = 2/5$.

5.1.2. “Phantom potential energy”. As noted, a distinctive feature of the wavefunctions for $\alpha < 1$ is the skewing towards lower values of x . This suggests a concept of a “phantom potential energy” when viewed within x -space. The probability distribution ($|\psi_n|^2$) is not symmetrically distributed about $x = 1/2$, rather the low values of x are “emphasized” more so than the higher values of x . While the potential energy is zero for $0 \leq x \leq 1$, there is an apparent presence of a “phantom potential energy” pushing the probability distribution towards lower values of x . This “phantom potential energy” is zero for $\alpha = 1$ and increases in effect as $\alpha \rightarrow 0$. This leads to the question of whether or not the effect of factorization of the kinetic energy in a conformable Schrödinger equation can be mapped to an attendant potential energy term in a normal, non-conformable Schrödinger equation. There does not appear to be an analytically realizable solution to a normal Schrödinger equation that produces the $\mathbb{J}_n^{(\alpha)}$ functions. However, one can consider forms of a perturbational potential energy in the normal particle in a box equation which yield corrected wavefunctions similar to the $\mathbb{J}_n^{(\alpha)}$ functions.

Using arbitrary functions to serve as a potential energy, perturbation theory was used to approximate solutions of a non-conformable Schrödinger Equation. As shown in Fig. 14, potential energies of the form $V_I = x$, $V_I = x^\alpha$, and $V_I = x^{\alpha/2}$ all shift the non-conformable wavefunction to the left, showing similar shape to the $\mathbb{J}_n^{(\alpha)}$ wavefunction given by the conformable Schrödinger Equation. For Fig. 14, $\alpha = 1/2$.

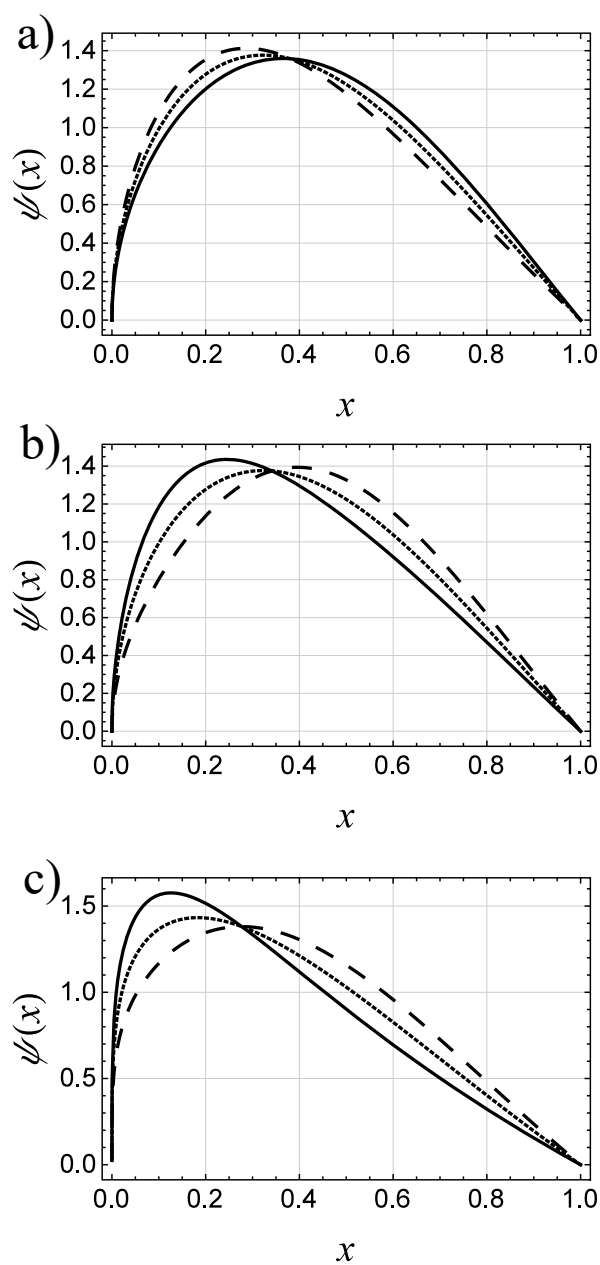


FIGURE 12. Several representative examples of first order wavefunctions for perturbations (a) $V_I = E_1^{(0)}x$ and (b) and (c) Eq. (126). For (a) and (b) $\alpha = 1/2$ and for (c) $\alpha = 1/4$.

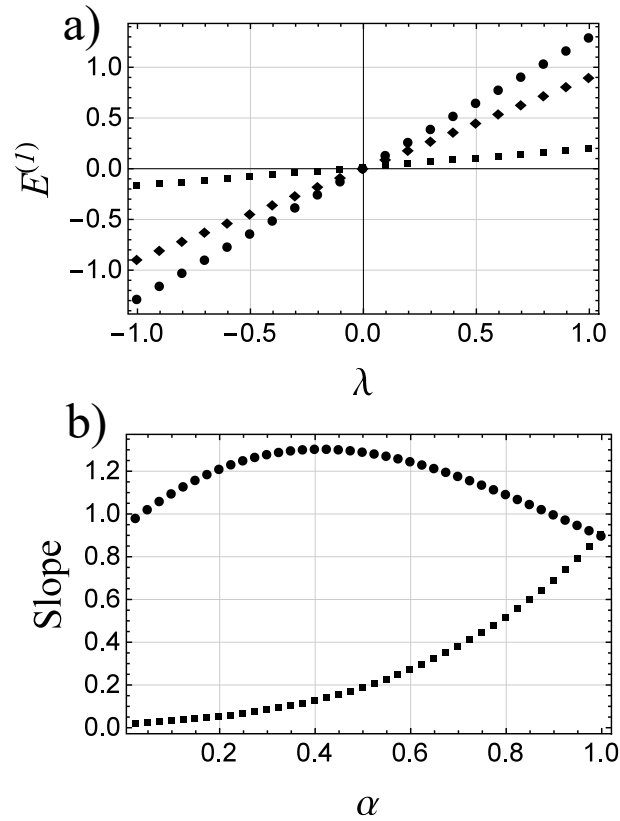


FIGURE 13. A comparison of the effects on the first order energy correction of a wall (or hole) on the left side (Eq. (126)) and right side (127) of the infinite box. (a) The (●) represents the wall (or hole) on the left for $\alpha = 1/2$. The (■) represents the wall (or hole) being on the right for $\alpha = 1/2$. The (◆) represents the comparison case for $\alpha = 1$ which, of course, is the same for the wall (or hole) on either side of the infinite box. (b) The slope of the line through the points in (a) but extended to include a range of α values from 0 to 1. Interestingly, the wall (or hole) has maximum impact on the ground state energy around $\alpha = 2/5$ rather than for $\alpha \rightarrow 0$.

These first-order corrected wavefunctions provide a link between the properties of the conformable and non-conformable Schrödinger equations. In conformable form, the Schrödinger equation creates a first-derivative term, something absent from the non-conformable equation. The goal here was to use a simple function as a perturbation to the normal particle in a box system. Motivation for the choice of x^α resulted from the presence of $x^{-\alpha}$ on the conformable Schrödinger equation's first derivative term. The x and $x^{\alpha/2}$ choices served as natural comparisons. For the particular case depicted in Fig. 14, $V_I = x^{\alpha/2}$ gave the best fit to the target

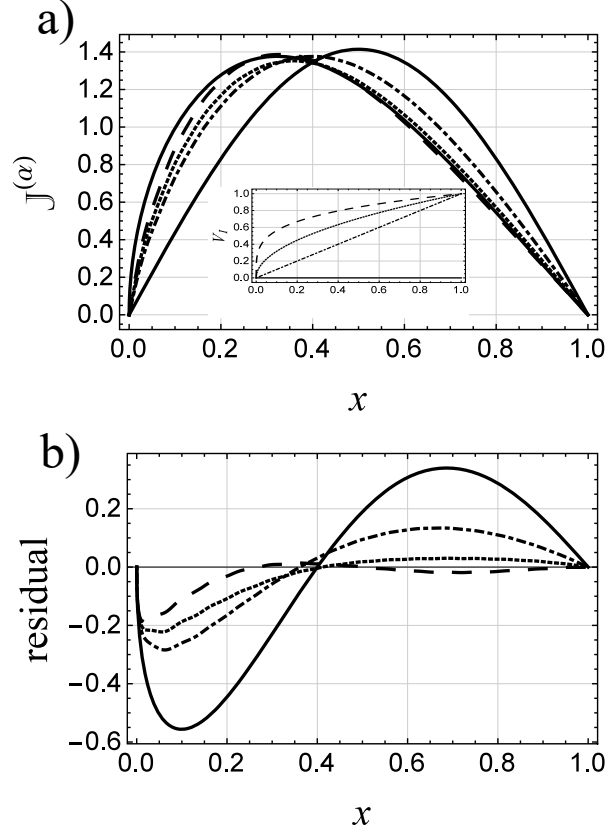


FIGURE 14. (a) First order correction to the regular particle in a box wavefunctions for $V_I = x$ (dash) $V_I = x^\alpha$ (dot), and $V_I = x^{\alpha/2}$ (dash-dot). The two solid curves are unperturbed wavefunction, $\sqrt{2} \sin \pi x$, and the target function, $\mathbb{J}_1^{(1/2)}(x)$. The inset shows V_I . (b) The residual curve, $\psi^{(1)} - \mathbb{J}_1^{(1/2)}(x)$, for each of the curves shown in (a).

function, $\mathbb{J}_n^{1/2}$ as seen in the residual plot, Fig 14b. For $\alpha = 3/4$, $V_I = x$ produces the best match (not shown).

5.2. Shape invariance supersymmetry in conformable Sturm-Liouville systems. The particle in a box model is often used as a pedagogical example in supersymmetric procedures such as SUSY [13, 14]. It is interesting to do that here for the conformable particle in a box model.

5.2.1. *Symmetric differential operators.* Consider the class of generic conformable, symmetric differential operators (note the different font is used to indicate that these operators are not necessarily conformable derivative operators),

$$\hat{p}_S = -i \frac{\mathcal{D}^\alpha + \mathcal{D}^\beta}{2}. \quad (128)$$

Here \mathcal{D}^γ where $0 \leq \gamma \leq 1$ is a conformable differential operator that satisfies the product (and quotient) rule. Note, in general $[\mathcal{D}^\alpha, \mathcal{D}^\beta] \neq 0$ and $\mathcal{D}^\alpha \mathcal{D}^\beta \neq \mathcal{D}^{\alpha+\beta}$. One sees that when $\alpha = \beta = 1$ one recovers the regular differential operator $\hat{p} = -i\mathcal{D}$ (when $D = \frac{d}{dx}$). We now consider the class of conformable Sturm-Liouville systems for the form

$$\begin{aligned} \hat{p}_S^2 \psi(x) + V(x)\psi(x) &= \Lambda \psi(x) \\ \hat{H}\psi &= \Lambda \psi, \end{aligned} \quad (129)$$

where $\hat{H} = \hat{p}_S^2 \psi + V(x)$.

We now consider the case where the set of eigenvalues has the form $\{\Lambda_0 > 0, \Lambda_n \geq \Lambda_{n+1}\}$, $n = 0, 1, 2, \dots$, with corresponding eigenfunctions $\{\psi_n(x)\}$. Following the normal shape invariance/SUSY procedure [13, 14, 17, 16, 10], we define $\hat{H}_1 = \hat{H} - \Lambda_0$ (so $V_1 = V - \Lambda_0$) and consider $\hat{H}_1 \varphi = \Lambda^{(1)} \varphi$ for which $\hat{H}_1 \varphi_0 = 0$ because $\Lambda_0^{(1)} = 0$.

Use of Eq. (128) and expansion gives

$$-\frac{1}{4} (\mathcal{D}^\alpha \mathcal{D}^\alpha + \mathcal{D}^\beta \mathcal{D}^\alpha + \mathcal{D}^\alpha \mathcal{D}^\beta + \mathcal{D}^\beta \mathcal{D}^\beta) \varphi_0 + V_1 \varphi_0 = 0. \quad (130)$$

So,

$$V_1 = \frac{(\mathcal{D}^\alpha \mathcal{D}^\alpha + \mathcal{D}^\beta \mathcal{D}^\alpha + \mathcal{D}^\alpha \mathcal{D}^\beta + \mathcal{D}^\beta \mathcal{D}^\beta) \varphi_0}{4\varphi_0}. \quad (131)$$

We now define the following operators ($W = W(x)$)

$$A = \frac{1}{2} (\mathcal{D}^\alpha + W) \quad (132)$$

$$\bar{A} = \frac{1}{2} (-\mathcal{D}^\alpha + W) \quad (133)$$

$$B = \frac{1}{2} (\mathcal{D}^\beta + W) \quad (134)$$

$$\bar{B} = \frac{1}{2} (-\mathcal{D}^\beta + W), \quad (135)$$

and consider $(\bar{A} + \bar{B})(A + B) = \bar{A}A + \bar{B}A + \bar{A}B + \bar{B}B$. The ordered products can be shown to be

$$\bar{A}A = \frac{1}{4} (-\mathcal{D}^\alpha \mathcal{D}^\alpha - \mathcal{D}^\alpha W + W^2) \quad (136)$$

$$\bar{B}A = \frac{1}{4} (-\mathcal{D}^\beta \mathcal{D}^\alpha + W \mathcal{D}^\alpha - W \mathcal{D}^\beta - \mathcal{D}^\beta W + W^2) \quad (137)$$

$$\bar{A}B = \frac{1}{4} (-\mathcal{D}^\alpha \mathcal{D}^\beta + W \mathcal{D}^\beta - W \mathcal{D}^\alpha - \mathcal{D}^\alpha W + W^2) \quad (138)$$

$$\bar{B}B = \frac{1}{4} (-\mathcal{D}^\beta \mathcal{D}^\beta - \mathcal{D}^\beta W + W^2), \quad (139)$$

where the product rule was employed and $\mathcal{D}^\gamma W f(x)$ means $(\mathcal{D}^\gamma W) f(x)$ as opposed to $\mathcal{D}^\gamma [W f(x)]$.

We can express \hat{H}_1 in terms of these ordered products

$$\begin{aligned} \hat{H}_1 &= \bar{A}A + \bar{B}A + \bar{A}B + \bar{B}B \\ &= \frac{1}{4} (-\Delta + 4W^2 - 2(\mathcal{D}^\alpha W + \mathcal{D}^\beta W)), \end{aligned} \quad (140)$$

where $\Delta \equiv \mathcal{D}^\alpha \mathcal{D}^\alpha + \mathcal{D}^\beta \mathcal{D}^\alpha + \mathcal{D}^\alpha \mathcal{D}^\beta + \mathcal{D}^\beta \mathcal{D}^\beta$. Comparison of Eq. (140) with Eq. (130) reveals

$$V_1 = W^2 - \frac{1}{2}(\mathcal{D}^\alpha W + \mathcal{D}^\beta W). \quad (141)$$

Using Eq. (131) gives a first order conformable first order differential equations for W . Guided by regular SUSY,[13, 14, 17, 16, 10] we make the ansatz that

$$W = -\frac{(\mathcal{D}^\alpha \varphi_0 + \mathcal{D}^\beta \varphi_0)}{2\varphi_0}. \quad (142)$$

To confirm the ansatz we see

$$W^2 = \frac{1}{4\varphi_0^2} (\mathcal{D}^\beta \varphi_0 \mathcal{D}^\beta \varphi_0 + \mathcal{D}^\beta \varphi_0 \mathcal{D}^\alpha \varphi_0 + \mathcal{D}^\alpha \varphi_0 \mathcal{D}^\beta \varphi_0 + \mathcal{D}^\beta \varphi_0 \mathcal{D}^\beta \varphi_0), \quad (143)$$

$$\mathcal{D}^\alpha \varphi_0 = \frac{\mathcal{D}^\alpha \mathcal{D}^\alpha \varphi_0 + \mathcal{D}^\alpha \mathcal{D}^\beta \varphi_0}{2\varphi_0}, \quad (144)$$

and

$$\mathcal{D}^\beta \varphi_0 = \frac{\mathcal{D}^\beta \mathcal{D}^\alpha \varphi_0 + \mathcal{D}^\beta \mathcal{D}^\beta \varphi_0}{2\varphi_0}. \quad (145)$$

So the right hand side of Eq. (141) becomes

$$\begin{aligned} W^2 - \frac{1}{2}(\mathcal{D}^\alpha W + \mathcal{D}^\beta W) &= \frac{(\mathcal{D}^\alpha \mathcal{D}^\alpha + \mathcal{D}^\beta \mathcal{D}^\alpha + \mathcal{D}^\alpha \mathcal{D}^\beta + \mathcal{D}^\beta \mathcal{D}^\beta) \varphi_0}{4\varphi_0} \\ &= V_1 \end{aligned} \quad (146)$$

and the ansatz is confirmed.

Finding analogy with regular SUSY we consider the reverse ordered products $(A + B)(\bar{A} + \bar{B})$ to construct the partner operator,

$$H_2 = A\bar{A} + A\bar{B} + B\bar{A} + B\bar{B}. \quad (147)$$

With some analysis similar to above one obtains

$$H_2 = \frac{1}{4} (-\Delta + 4W^2 + 2(\mathcal{D}^\alpha W + \mathcal{D}^\beta W)). \quad (148)$$

Thus the partner potential is

$$V_2 = W^2 + \frac{(\mathcal{D}^\alpha W + \mathcal{D}^\beta W)}{2}. \quad (149)$$

We note that when $\alpha = \beta = 1$ we recover the regular SUSY partner potentials

$$\left\{ \begin{array}{l} V_1 = W^2 - W' \\ V_2 = W^2 + W' \end{array} \right\}. \quad (150)$$

As with regular SUSY [13, 14], the two systems are isospectral aside from $\Lambda_0^{(1)}$, such that $\Lambda_n^{(2)} = \Lambda_{n+1}^{(1)}$. The wavefunctions for H_2 , $\{\vartheta_n\}$ are related to those for H_1 via

$$\vartheta_n \propto (A + B) \varphi_{n+1} \quad (151)$$

and

$$\varphi_{n+1} \propto (\bar{A} + \bar{B}) \vartheta_n. \quad (152)$$

One can confirm the appropriate anticommutator algebra by defining

$$Q_X = \begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix}, \quad \bar{Q}_{\bar{X}} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}. \quad (153)$$

So, $Q = Q_A + Q_B$ and $\bar{Q} = \bar{Q}_A + \bar{Q}_B$, and

$$\begin{aligned} \mathcal{H} &= \{\bar{Q}, Q\} = \begin{bmatrix} (\bar{A} + \bar{B})(A + B) & 0 \\ 0 & (A + B)(\bar{A} + \bar{B}) \end{bmatrix} \\ &= \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \end{aligned}$$

and $\{\bar{Q}, \bar{Q}\} = \{Q, Q\} = 0$.

5.2.2. *Asymmetric differential operators.* Consider the class of conformable, symmetric differential operators,

$$\hat{p} = -i\mathcal{D}^\gamma. \quad (154)$$

The second order equation becomes

$$-\mathcal{D}^\alpha \mathcal{D}^\beta \psi + V\psi = \Lambda\psi \quad (155)$$

which upon adjusting the potential energy by Λ_0 becomes

$$-\mathcal{D}^\alpha \mathcal{D}^\beta \varphi_0 + V_1 \varphi_0 = 0. \quad (156)$$

Thus

$$V_1 = \frac{\mathcal{D}^\alpha \mathcal{D}^\beta \varphi_0}{\varphi_0}. \quad (157)$$

Now the factorization of the resultant equation is less straightforward. Nonetheless, one can define

$$A = \mathcal{D}^\alpha + W_A \quad (158)$$

$$\bar{A} = -\mathcal{D}^\alpha + \bar{W}_A \quad (159)$$

$$B = \mathcal{D}^\beta + W_B \quad (160)$$

$$\bar{B} = -\mathcal{D}^\beta + \bar{W}_B. \quad (161)$$

The Hamiltonian is constructed as to maintain the $\mathcal{D}^\alpha \mathcal{D}^\beta$ ordering of the operators,

$$H_1 = \bar{A}B \quad (162)$$

$$H_1 = (-\mathcal{D}^\alpha + \bar{W}_A)(\mathcal{D}^\beta + W_B)$$

$$H_1 = -\mathcal{D}^\alpha \mathcal{D}^\beta + \bar{W}_A \mathcal{D}^\beta - W_B \mathcal{D}^\alpha - \mathcal{D}^\alpha W_B + \bar{W}_A W_B,$$

where as before $\mathcal{D}^\alpha W_B$ means $(\mathcal{D}^\alpha W_B)$. The first SUSY partner potential is now more complicated

$$V_1 = \bar{W}_A W_B - \mathcal{D}^\alpha W_B - W_B \mathcal{D}^\alpha + \bar{W}_A \mathcal{D}^\beta. \quad (163)$$

However, one can impose the condition $-W_B \mathcal{D}^\alpha + \bar{W}_A \mathcal{D}^\beta = 0$ which will determine the functional relationship between W_B and \bar{W}_A . This then leaves an expression that is similar to regular SUSY,

$$V_1 = \bar{W}_A W_B - \mathcal{D}^\alpha W_B. \quad (164)$$

The second SUSY partner Hamiltonian is again produced by requiring the $\mathcal{D}^\alpha \mathcal{D}^\beta$ ordering. This means $H_2 = A\bar{B}$ and thus,

$$V_2 = \bar{W}_A W_B - \mathcal{D}_B^\alpha \bar{W}_B - W_A \mathcal{D}^\beta + \bar{W}_B \mathcal{D}^\alpha. \quad (165)$$

One can again require $-W_A \mathcal{D}^\beta + \bar{W}_B \mathcal{D}^\alpha = 0$ to determine the functional relationship between W_A and \bar{W}_B . This gives

$$V_2 = \bar{W}_A W_B - \mathcal{D}_B^\alpha \bar{W}. \quad (166)$$

The asymmetric treatment is convenient for Sturm-Liouville equations. Some examples are given below.

One important case is when $\beta = \alpha < 1$. Here,

$$H_1 = -\mathcal{D}^\alpha \mathcal{D}^\alpha \varphi + V_1 \varphi = \Lambda^{(1)} \varphi \quad (167)$$

and

$$H_2 = -\mathcal{D}^\alpha \mathcal{D}^\alpha \vartheta + V_2 \vartheta = \Lambda^{(2)} \vartheta. \quad (168)$$

Where, like regular SUSY $\Lambda_0^{(2)} = \Lambda_1^{(1)}$, $\Lambda_n^{(2)} = \Lambda_{n+1}^{(1)}$, the partner potentials are

$$\left\{ \begin{array}{l} V_1 = W^2 - \mathcal{D}^\alpha W \\ V_2 = W^2 + \mathcal{D}^\alpha W \end{array} \right\} \quad (169)$$

and one sees an identical structure compared to that of regular SUSY except with the conformable differential operator playing the role of the regular derivative.

5.2.3. *The conformable derivative.* Consider here the case of the conformable derivative. Now $\mathcal{D}^\alpha[f] = D^\alpha[f] = x^{1-\alpha} f'$. Consider the case for $V = 0$ for $0 \leq x \leq 1$ and boundary conditions $\psi(0) = \psi(1) = 0$.

$$H\psi = \Lambda\psi \quad (170)$$

$$-D^\alpha D^\alpha \psi = \Lambda\psi \quad (171)$$

has been studied and has solutions [4]

$$\psi_n = N \sin\left(\sqrt{\Lambda_n} \frac{x^\alpha}{\alpha}\right), \quad (172)$$

where N is a normalization constant and $\Lambda_n = \alpha^2(n+1)^2\pi^2$, $n = 0, 1, 2, \dots$. Thus $\Lambda_n^{(1)} = \alpha^2(n+1)^2\pi^2 - \alpha^2\pi^2 = \alpha^2 n(n+2)\pi^2$ and

$$\varphi_n = N \sin((n+1)\pi x^\alpha). \quad (173)$$

Following regular SUSY, $\varphi_0 = N \sin(\pi x^\alpha)$. Thus from Eq. (142) with $\beta = \alpha$,

$$\begin{aligned} W &= -\frac{D^\alpha \varphi_0}{\varphi_0} = -\frac{x^{1-\alpha} \varphi_0'}{\varphi_0} \\ &= -\frac{N\alpha\pi \cos(\pi x^\alpha)}{N \sin(\pi x^\alpha)} \\ &= -\alpha\pi \cot(\pi x^\alpha). \end{aligned} \quad (174)$$

The partner potential is obtained from Eq. (169) as

$$\begin{aligned} V_2 &= \alpha^2\pi^2 (\cot^2(\pi x^\alpha) + \csc^2(\pi x^\alpha)) \\ &= \alpha^2\pi^2 (2 \csc^2(\pi x^\alpha) - 1). \end{aligned} \quad (175)$$

The ϑ_n are obtained by acting on φ_n with the operator $A + B$ with $\beta = \alpha$

$$\begin{aligned} \vartheta_n &\propto (A + B) \varphi_{n+1} \\ &\propto (D^\alpha + W) \varphi_{n+1} \\ &\propto x^{1-\alpha} \varphi_{n+1}' - \alpha\pi \cot(\pi x^\alpha) \varphi_{n+1} \\ &\propto (n+2)\alpha\pi \cos((n+2)\pi x^\alpha) - \alpha\pi \cot(\pi x^\alpha) \sin((n+2)\pi x^\alpha). \end{aligned} \quad (176)$$

The first couple of eigenstates are (with the use of some trigonometric identities)

$$\vartheta_0 \propto \sin^2(\pi x^\alpha) \quad (177)$$

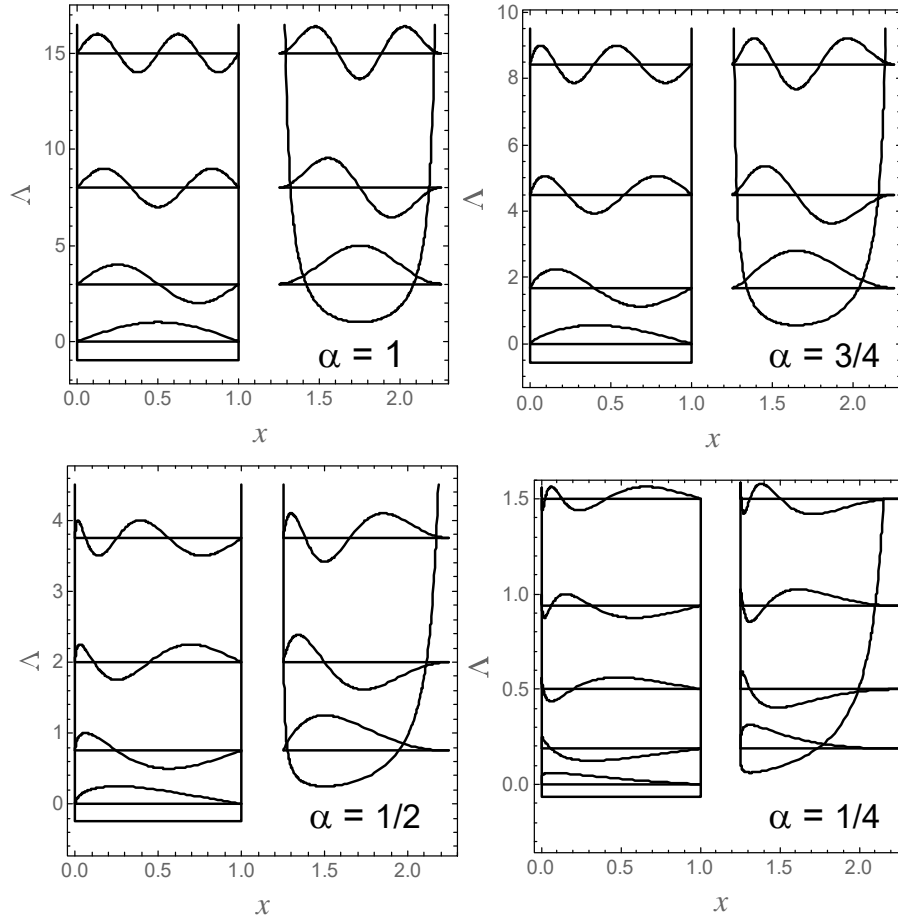


FIGURE 15. Symmetric version of the conformable SUSY plot of the particle in a box system associated with H_1 of Eq. (167) and H_2 of Eq (168) for $\beta = \alpha = 1, 3/4, 1/2, 1/4$. The case when $\alpha = 1$ recovers the regular SUSY partner potential and energy levels for regular SUSY [13, 14]

and

$$\vartheta_1 \propto \sin(\pi x^\alpha) \sin(2\pi x^\alpha). \tag{178}$$

Figure 15 shows the a plot of the eigenvalue/eigenfunction system associated with H_1 and H_2 for $\alpha = 1, 3/4, 1/2, 1/4$.

An interesting variant is to consider an asymmetric version

$$\begin{aligned} \mathcal{D}^\alpha &= D^1 = \frac{d}{dx} \\ \mathcal{D}^\beta &= D^\alpha = x^{1-\alpha} \frac{d}{dx} \end{aligned} \tag{179}$$

$$\begin{aligned}
D^\alpha D^\beta &= \frac{d}{dx} \left[x^{1-\alpha} \frac{d}{dx} \right] \\
&= x^{1-\alpha} \frac{d^2}{dx^2} + (1-\alpha)x^{-\alpha} \frac{d}{dx}.
\end{aligned} \tag{180}$$

This is a self-adjoint version of the conformable double-derivative given in Eq. (30). The solution set of

$$-\frac{d}{dx} \left[x^{1-\alpha} \frac{d}{dx} \right] \psi = \Lambda \psi \tag{181}$$

subject to $\psi(0) = \psi(1) = 0$, is (cf., Eq. (32))

$$\psi_n = \mathbb{J}_n^{(\alpha)}(x). \tag{182}$$

The denominator of $\mathbb{J}_n^{(\alpha)}(x)$, will be important. It is defined here as N_n , and is (cf., Eq. (32))

$$N_n = \frac{1}{\sqrt{(\eta-1) J_{\eta-1}(n_\eta) J_{\eta+1}(n_\eta)}}, \tag{183}$$

where n_η is the n^{th} zero of $J_\eta(x)$ for $\eta = \frac{\alpha}{1+\alpha}$. These wavefunctions are purely real. The energy levels are (cf., Eq. (33))

$$\Lambda_n = \frac{(1+\alpha)^2 n_\eta^2}{4}. \tag{184}$$

One must first solve

$$\begin{aligned}
-W_B D^\alpha + \overline{W}_A D^\beta &= 0 \\
-W_B \frac{d}{dx} + \overline{W}_A x^{1-\alpha} \frac{d}{dx} &= 0 \\
-W_B + \overline{W}_A x^{1-\alpha} &= 0
\end{aligned} \tag{185}$$

to obtain

$$\overline{W}_A = x^{\alpha-1} W_B. \tag{186}$$

Considering H_1 such that $\Lambda_0^{(1)} = 0$, leads to

$$\varphi_n = N_{n+1} \sqrt{x^\alpha} J_\eta \left((n+1)_\eta (x^\alpha)^{\frac{1}{2\eta}} \right). \tag{187}$$

From Eqs. (164) and (186),

$$V_1 = x^{\alpha-1} W_B^2 - \frac{dW_B}{dx} \tag{188}$$

and from Eq. (157)

$$V_1 = \frac{\frac{d}{dx} \left[x^{1-\alpha} \frac{d}{dx} \right] \varphi_0}{\varphi_0}. \tag{189}$$

Taking the ansatz,

$$W_B = -\frac{x^{1-\alpha} \varphi_0'}{\varphi_0}, \tag{190}$$

one may verify,

$$\begin{aligned}
V_1 &= x^{\alpha-1} \left(\frac{x^{1-\alpha} \varphi_0'}{\varphi_0} \right)^2 + \frac{-x^{1-\alpha} (\varphi_0')^2 + \varphi_0 (x^{1-\alpha} \varphi_0'' + (1+\alpha) x^\alpha \varphi_0')}{\varphi_0^2} \\
V_1 &= \frac{x^{1-\alpha} \varphi_0'' + (1+\alpha) x^\alpha \varphi_0'}{\varphi_0} \\
V_1 &= \frac{\frac{d}{dx} [x^{1-\alpha} \frac{d}{dx}] \varphi_0}{\varphi_0}. \tag{191}
\end{aligned}$$

Considering the SUSY partner potential, we again first address

$$\begin{aligned}
-W_A D^\beta + \overline{W}_B D^\alpha &= 0 \\
-W_A x^{1-\alpha} \frac{d}{dx} + \overline{W}_B \frac{d}{dx} &= 0 \\
-W_A x^{1-\alpha} + \overline{W}_B &= 0,
\end{aligned}$$

thus

$$W_A = x_B^{\alpha-1} \overline{W}_B \tag{192}$$

so,

$$V_2 = x_B^{\alpha-1} \overline{W}_B^2 + \frac{d\overline{W}_B}{dx}. \tag{193}$$

There is the flexibility with Eqs. (186) and (192) to set

$$\overline{W}_B = W_B = W = -\frac{x^{1-\alpha} \varphi_0'}{\varphi_0}. \tag{194}$$

Consequently,

$$\begin{aligned}
\vartheta_n &\propto B \varphi_{n+1} \tag{195} \\
\vartheta_n &\propto x^{\frac{1-\alpha}{2}} \frac{d\sqrt{x^\alpha} J_\eta \left(n_\eta (x^\alpha)^{\frac{1}{2\eta}} \right)}{d} - \frac{x^{\frac{1-\alpha}{2}} \varphi_0'}{\varphi_0} \sqrt{x^\alpha} J_\eta \left(n_\eta (x^\alpha)^{\frac{1}{2\eta}} \right).
\end{aligned}$$

Both W and ϑ_n are complicated combinations of Bessel functions but can be plotted as shown in Fig. 16 which shows W for $\alpha = 1, 3/4, 1/2, 1/4$ and Fig 17 which is analogous to Fig 15.

6. CONCLUSION

It is hoped this work clearly shows that *the conformable derivative for differentiable functions is equivalent to a simple change of variable*. But it is also hoped that the variety of areas shown in this work suggests that there is value in studying the properties of the conformable derivative.

This work discussed the use of a self-adjoint operator, $\hat{A}_{2\alpha}$, which is built from the conformable derivative. The solution to the eigenvalue problem with boundary conditions $y(0) = y(1) = 0$ leads to the complete orthonormal set of functions $\mathbb{J}_n^{(\alpha)}$ which are parameterized by the α of the conformable derivative used in $\hat{A}_{2\alpha}$. Various properties of the $\mathbb{J}_n^{(\alpha)}$ functions were explored including the nature of the roots of the functions, scaling relations, and areas between zeros. The behavior of the moments of the functions were plotted and discussed. The recasting of $\mathbb{J}_n^{(\alpha)}$ in terms of the confluent hypergeometric limiting functions was done. The $\mathbb{J}_n^{(\alpha)}$ functions form the basis for a generalization of the Fourier series and several example functions were investigated. The relationship to the Fourier-Bessel series was found and, although

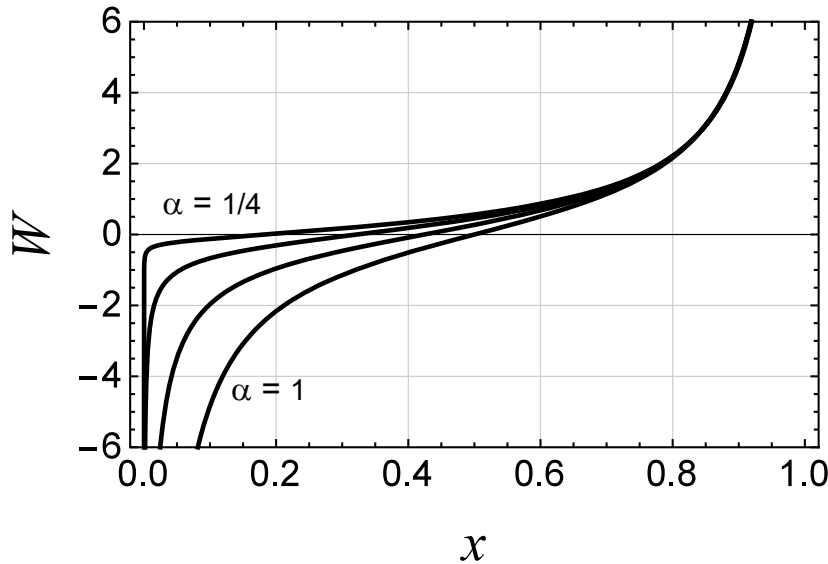


FIGURE 16. Plot of the W from Eq. (194) for an asymmetric version of the form $D^\alpha D^\beta = \frac{d}{dx} [x^{1-\alpha} \frac{d}{dx}]$ with $\alpha = 1, 3/4, 1/2, 1/4$.

most often one would need to resort to numerical integration, several special cases yielded analytic representation.

This work offers a fairly general consideration of the conformable Fourier transform pair that connects $\frac{t^\alpha}{\alpha}$ -space to $\frac{\omega^\beta}{\beta}$ -space in the same way the regular Fourier transform connects t -space to ω -space. This definition was shown to be a one-to-one transform and exhibits many important properties of a regular Fourier transform. These include the convolution theorem, formulas for the derivative, and explicit functions of $\frac{t^\alpha}{\alpha}$ and/or $\frac{\omega^\beta}{\beta}$. Further, it provided a natural framework for an expression for a conformable convolution. It is hoped that insights into the nature of conformable derivatives and in the relationship between $\frac{t^\alpha}{\alpha}$ -space to $\frac{\omega^\beta}{\beta}$ -space can be gleaned from the transform pair. One can envision potential application wherever there is a physical connection between complementary spaces. In particular, in quantum mechanics position and momentum are related to one another via Fourier transformation and the physical operator representing momentum is essentially the derivative with respect to position.

Finally this work discussed several applications in quantum mechanics. Perturbation theory was discussed and the concept of a “phantom potential energy” was developed. As a second application, a simple SUSY calculation was performed for the particle in a box model. Perhaps the use of the conformable derivative will be valuable in forming phenomenological models. Only quantum mechanics was discussed in this work, but one could envision exploring other areas of physics as well.

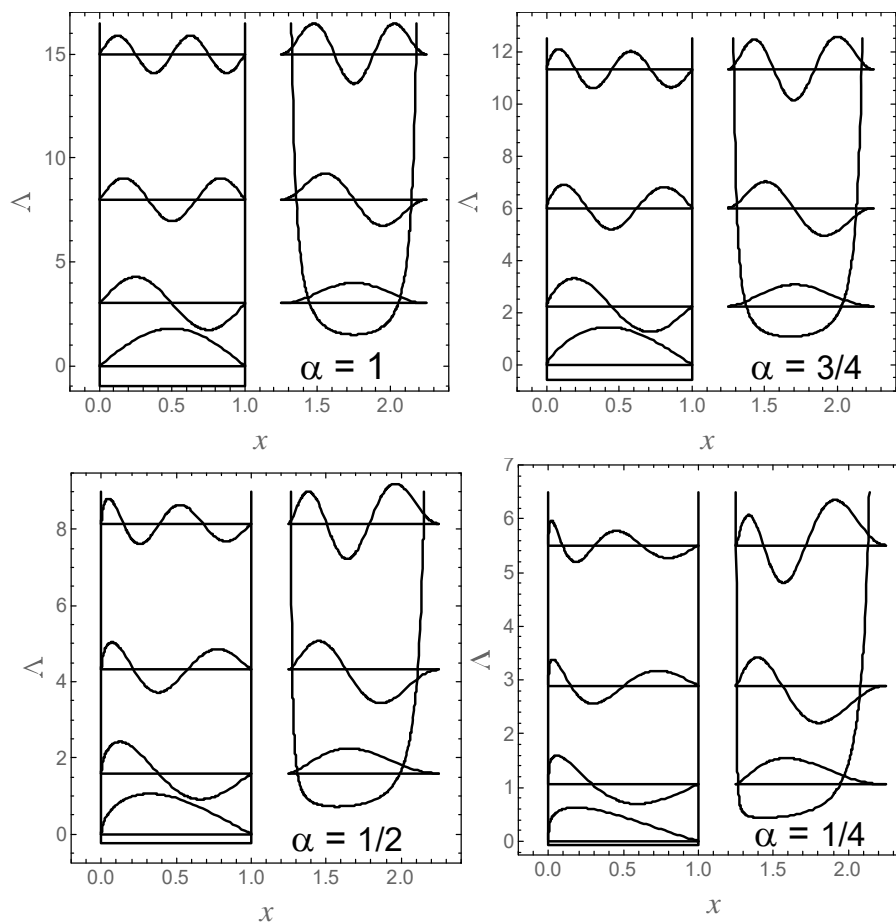


FIGURE 17. Analog of Fig. 15 for an asymmetric version of the form $D^\alpha D^\beta = \frac{d}{dx} [x^{1-\alpha} \frac{d}{dx}]$. Plots are show for $\alpha = 1, 3/4, 1/2, 1/4$. Differences between the plots show in this figure and that of Fig. 15 become more pronounced for smaller values of α .

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