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# CERTAIN SUBCLASSES OF UNIFORMLY STARLIKE, CONVEX AND SPIRALIKE FUNCTION DEFINED BY SALAGEAN DERIVATIVE OPERATOR

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Abstract. In this paper, the author studies certain new subclasses of uniformly starlike, convex and spiralike functions with negative coefficients. Some properties of functions belonging to these subclasses such as, coefficient estimates, extreme points, growth and distortion bonds are critically investigated.

## 1. Introduction

Let S denote the class of all functions having the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic and univalent in the unit disk $E = \{z : |z| < 1\}$ and by $ST$ and $CV$ the subclasses of $S$ that are starlike and convex respectively. A function $f(z)$ of the form (1) is said to be uniformly convex $(UCV)$ or uniformly starlike $(UST)$ in E if $f(z)$ is in $CV$ or $ST$ and has the property that for every circular arc $\epsilon$ contained in $E$, with centre $\psi$ , the arc $f(\epsilon)$ is convex or starlike with respect to $f(\psi)$ . It is well known that

$$f \in UCV \Leftrightarrow \left| \frac{zf''(z)}{f'(z)} \right| \le Re\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\},$$

$$f \in S_p \Leftrightarrow \left| \frac{zf'(z)}{f(z)} - 1 \right| \le Re\left\{ \frac{zf'(z)}{f(z)} \right\}$$

and

$$f \in S_p(\alpha) \Leftrightarrow \left| \frac{zf'(z)}{f(z)} - 1 \right| \le Re\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\}$$

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see ([3], [9], [11]). Now, suppose that we pose an index $\alpha$ ($\alpha$ is real) on (1) such that

$$f(z)^\alpha = \left(z + \sum_{k=j+1}^{\infty} a_k z^k\right)^\alpha \quad .$$

Then, we can re-write the fractional analytic function $f(z)^\alpha$as

$$f(z)^\alpha = z^\alpha + \sum_{k=j+1}^{\infty} a_k(\alpha) z^{\alpha+k-1} \, . \tag{2}$$

Interested reader may refer to ([4]), ([8]), ([7]) and ([15]) for more study on the kind of fractional analytic function $f(z)^\alpha$ defined in (2). Using Salagean derivative operator ([12]) on (2) we can write for function $f(z)^\alpha$ of the form (2) that

$$D^m f(z)^\alpha = \alpha^m z^\alpha + \sum_{k=j+1}^{\infty} (a+k-1)^m a_k(\alpha) z^{a+k-1} \tag{3}$$

$$\alpha > 0, \quad m \in N_0 \ \ and \ \ z \in E \ .$$

Motivated by the work of ([9]) we give the following definition.
**Definition 1:** Let$f(z)^\alpha \in S\left(\lambda, \gamma, \beta, \alpha, \theta, m, j\right)$. Then,

$$Re\left\{\frac{e^{i\theta} z(D^m f(z)^\alpha)' + (2\lambda^2-\lambda) z^2 (D^m f(z)^\alpha)''}{4(\lambda-\lambda^2) z^\alpha + (2\lambda^2-\lambda) z (D^m f(z)^\alpha)' + (2\lambda^2-3\lambda+1) D^m f(z)^\alpha} - \gamma\right\}$$

$$> \left|\frac{e^{i\theta} z(D^m f(z)^\alpha)' + (2\lambda^2-\lambda) z^2 (D^m f(z)^\alpha)''}{4(\lambda-\lambda^2) z^\alpha + (2\lambda^2-\lambda) z (D^m f(z)^\alpha)' + (2\lambda^2-3\lambda+1) D^m f(z)^\alpha} - 1\right| \tag{4}$$

$$\alpha > 0, \quad m \in N_0, \ \ 0 \le \lambda < 1, \ \ \beta \ge 0, \ \ -1 \le \gamma < 1, \ \ |\theta| < \pi/2 \ \ and \ \ z \in E \ .$$

We also, let $TS\left(\lambda, \gamma, \beta, \alpha, \theta, m, j\right) = S\left(\lambda, \gamma, \beta, \alpha, \theta, m, j\right) \bigcap T$where $T$ is the subclass of S consisting of functions of the form:

$$f(z)^\alpha = z^\alpha - \sum_{k=j+1}^{\infty} a_k(\alpha) z^{a+k+1} \tag{5}$$

For various choices of the parameters involved, we obtained the following subclasses investigated by different authors.
1. $TS\left(\lambda, \gamma, \beta, 1, 0, 0, j\right) = TS\left(\lambda, \gamma, \beta, j\right)$ see ([9]).
2. $TS\left(0, \gamma, 0, 1, 0, 0, 1\right) = T^*\left(\gamma\right)$and $TS\left(1, \gamma, 0, 1, 0, 0, 1\right) = K\left(\gamma\right)$([14]).
3. $TS\left(0, \gamma, 0, 1, 0, 0, j\right) = T^*\left(\gamma, j\right)$and$TS\left(1, \gamma, 0, 1, 0, 0, j\right) = K\left(\gamma, j\right)$ ([16]).
4. $TS\left(0, \gamma, \beta, 1, 0, 0, 1\right) = TS\left(\gamma, \beta\right)$ and $TS\left(1, \gamma, \beta, 1, 0, 0, 1\right) = UCV\left(\gamma, \beta\right)$([2]).
5. $TS\left(0, 0, \beta, 1, 0, 0, 1\right) = TS_p\left(\beta\right)$ ([18]).
6. $TS\left(1, 0, \beta, 1, 0, 0, j\right) = UCV\left(\beta\right)$ ([19]).
7. $TS\left(\lambda, \gamma, 0, 1, 0, 0, j\right) = B\left(\lambda, \gamma, j\right)$([5], [6]).
8. $TS\left(1/2, \gamma, 0, 1, 0, 0, 1\right) = P\left(\gamma\right)$ ([1]) and ([13]).
9. $TS\left(1/2, \gamma, \beta, 1, 0, 0, 1\right) = TR\left(\gamma, \beta\right)$ ([17]).
Next we obtain a necessary and sufficient condition for functions$f(z)^\alpha \in TS\left(\lambda, \gamma, \beta, \alpha, \theta, m, j\right)$.

## 2. Coefficient Estimates

**Theorem 2.1**: Let $f(z)^\alpha$ be a function of the form (2). If $f(z)^\alpha \in TS(\lambda,\gamma,\beta,\alpha,\theta,m,j)$, then

$$\frac{\sum_{k=j+1}^{\infty}(a+k-1)^m\left[(1+\beta)M_k-(\beta-\gamma)F_k\right]|a_k(\alpha)|}{(2+\beta-\gamma)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right]-(1+\beta)\alpha^{m+1}\left[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)\right]}\le 1 \tag{6}$$

where $\alpha>0, m\in N\bigcup\{0\}, 0\le\gamma<1, 1/2\le\lambda\le 1, \beta\ge 0, |\theta|<1/2$ and

$$\left.\begin{aligned} M_k &= (\alpha+k-1)\left[e^{i\theta}+\left(2\lambda^2-\lambda\right)(\alpha+k-2\right] \\ F_k &= (2\lambda^2-\lambda)(\alpha+k-1)+(2\lambda^2-3\lambda+1)\end{aligned}\right\}. \tag{7}$$

**Proof** : It is sufficient to show that

$$\beta\left|\frac{e^{i\theta}z(D^mf(z)^\alpha)'+(2\lambda^2-\lambda)z^2(D^mf(z)^\alpha)''}{4(\lambda-\lambda^2)z^\alpha+(2\lambda^2-\lambda)z(D^mf(z)^\alpha)'+(2\lambda^2-3\lambda+1)D^mf(z)^\alpha}-1\right|$$

$$-Re\left\{\frac{e^{i\theta}z(D^mf(z)^\alpha)'+(2\lambda^2-\lambda)z^2(D^mf(z)^\alpha)''}{4(\lambda-\lambda^2)z^\alpha+(2\lambda^2-\lambda)z(D^mf(z)^\alpha)'+(2\lambda^2-3\lambda+1)D^mf(z)^\alpha}-1\right\}\le 1-\gamma\,.$$

Also, we can write that

$$\beta\left|\frac{e^{i\theta}z(D^mf(z)^\alpha)'+(2\lambda^2-\lambda)z^2(D^mf(z)^\alpha)''}{4(\lambda-\lambda^2)z^\alpha+(2\lambda^2-\lambda)z(D^mf(z)^\alpha)'+(2\lambda^2-3\lambda+1)D^mf(z)^\alpha}-1\right|$$

$$-Re\left\{\frac{e^{i\theta}z(D^mf(z)^\alpha)'+(2\lambda^2-\lambda)z^2(D^mf(z)^\alpha)''}{4(\lambda-\lambda^2)z^\alpha+(2\lambda^2-\lambda)z(D^mf(z)^\alpha)'+(2\lambda^2-3\lambda+1)D^mf(z)^\alpha}-1\right\}$$

$$\le(1+\beta)\left|\frac{e^{i\theta}z(D^mf(z)^\alpha)'+(2\lambda^2-\lambda)z^2(D^mf(z)^\alpha)''}{4(\lambda-\lambda^2)z^\alpha+(2\lambda^2-\lambda)z(D^mf(z)^\alpha)'+(2\lambda^2-3\lambda+1)D^mf(z)^\alpha}-1\right|$$

$$\le(1+\beta)\frac{\left\{\begin{array}{l}\left[\alpha^{m+1}[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)]-\ [4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m]\right]\\ +\sum_{k=j+1}(\alpha+k-1)^m\left(M_k-F_k\right)|a_k(\alpha)|\end{array}\right\}}{[4(\lambda-\lambda^2)-(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m]-\sum_{k=j+1}(\alpha+k-1)^mF_k\,|a_k(\alpha)|}.$$

where

$M_k=(\alpha+k-1)\left[e^{i\theta}+(2\lambda^2-\lambda)(\alpha+k-2)\right]$ *and* $F_k=(2\lambda^2-\lambda)(\alpha+k-1)+(2\lambda^2-3\lambda+1)$.

Now, this last expression is bounded above by $1-\gamma$ if

$$\sum_{k=j+1}(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]\ |a_k(\alpha)|$$

$$\le(2+\beta-\gamma)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right]-(1+\beta)\alpha^{m+1}\left[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)\right]$$

and this ends the proofs.

**Corollary 2.2:** The function $f(z)^\alpha$ of the form (2) is said to be in the class $S(\lambda,\gamma,\beta,\alpha,0,m,j)$, if and only if

$$\sum_{k=j+1}(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]\ |a_k(\alpha)|$$

$$\le(2+\beta-\gamma)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right]-(1+\beta)\alpha^{m+1}\left[1+(\alpha-1)(2\lambda^2-\lambda)\right]$$

where $M_k = (\alpha + k - 1)\Big[1 + (2\lambda^2 - \lambda)(\alpha + k - 2)\Big]$ and $F_k$ is as defined earlier.

**Corollary 2.3:** The function $f(z)^\alpha$ of the form (2) is said to be in the class $S(\lambda, \gamma, \beta, 1, 0, m, j)$, if and only if

$$\sum_{k=j+1} k^m \left[(1+\beta)M_k - (\beta+\gamma)F_k\right] |\alpha_k(1)| \leq 1 - \gamma$$

where $M_k = k\left[1 + (2\lambda^2 - \lambda)(k-1)\right]$ and $F_k = (2\lambda^2 - \lambda)k + (2\lambda^2 - 3\lambda + 1)$.

**Corollary 2.4:** The function $f(z)^\alpha$ of the form (2) is said to be in the class $S(\lambda, \gamma, \beta, 1, 0, 0, j)$, if and only if

$$\sum_{k=j+1} \left[(1+\beta)M_k - (\beta+\gamma)F_k\right] |\alpha_k(1)| \leq 1 - \gamma$$

where $M_k = k\left[1 + (2\lambda^2 - \lambda)(k-1)\right]$ and $F_k = (2\lambda^2 - \lambda)k + (2\lambda^2 - 3\lambda + 1)$. Incidentally, this result coincides with that of ([9]).

**Corollary 2.5:** The function $f(z)^\alpha$ of the form (2) is said to be in the class $S(\lambda,\ \gamma,\ \beta,\ 1,\ 0,\ 1,\ j)$ if and only if

$$\sum_{k=j+1} k\left[(1+\beta)M_k - (\beta+\gamma)F_k\right] |\alpha_k(\alpha)| \leq 1 - \lambda$$

where $M_k = k\big[1 + (2\lambda^2 - \lambda)(k-1)\big]$ and $F_k = k(2\lambda^2 - \lambda) + (2\lambda^2 - 3\lambda + 1)$.

**Theorem 2.6**: The necessary and sufficient condition for function $f(z)^\alpha$ of the form (5) to be in the class $TS(\lambda, \gamma, \beta, \alpha, \theta, m, j)$ is that

$$\sum_{k=j+1}^{\infty} (a + k - 1)^m \left[(1+\beta)M_k - (\beta-\gamma)F_k\right] |a_k(\alpha)|$$

$$\leq (2+\beta-\gamma)\left[4(\lambda-\lambda^2) + (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2 - 3\lambda+1)\alpha^m\right] - (1+\beta)\alpha^{m+1}\left[e^{i\theta} + (\alpha-1)\left(2\lambda^2-\lambda\right)\right] \tag{8}$$

**Proof**. In view of theorem 2.1, we only need to show the necessity. That is, if $f(z)^\alpha \in TS(\lambda, \gamma, \beta, \alpha, \theta, m, j)$ and z is real, then

$$\frac{\alpha^{m+1}[e^{i\theta} + (\alpha-1)(2\lambda^2-\lambda)] - \sum_{k=j+1}(\alpha+k-1)^m M_k a_k(\alpha)\, z^{k-1}}{[4(\lambda-\lambda^2) - (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2-3\lambda+1)\alpha^m] - \sum_{k=j+1}(\alpha+k-1)^m F_k\, a_k(\alpha) z^{k-1}} - \gamma$$

$$\leq \beta \frac{\left\{\begin{array}{c}[\alpha^{m+1}[e^{i\theta} + (\alpha-1)(2\lambda^2-\lambda)] - \ [4(\lambda-\lambda^2) + (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2-3\lambda+1)\alpha^m]] \\ + \sum_{k=j+1}(\alpha+k-1)^m \left[M_k - F_k\right] a_k(\alpha)\, z^{k-1}\end{array}\right\}}{[4(\lambda-\lambda^2) - (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2-3\lambda+1)\alpha^m] - \sum_{k=j+1}(\alpha+k-1)^m F_k a_k(\alpha) z^{k-1}}.$$

Letting $z \to 1$ along the real axis, then the inequality in (8) follows. In conclusion, the function $f(z)^\alpha$ given below is the extremal function.

$$f(z)^\alpha = z^\alpha$$

$$-\frac{(2+\beta-\gamma)\left[4(\lambda-\lambda^2) - (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2-3\lambda+1)\alpha^m\right] - (1+\beta)\alpha^{m+1}[e^{i\theta} + (\alpha-1)(2\lambda^2-\lambda)]}{(\alpha+j)^m\left[(1+\beta)M_{j+1} - (\beta+\gamma)F_{j+1}\right]} z^{\alpha+j} \tag{9}$$

where

$$M_k = (\alpha+j)\Big[e^{i\theta} + \left(2\lambda^2-\lambda\right)(\alpha+j-1)\Big] \ \ \textit{and} \ \ F_k = (2\lambda^2-\lambda)(\alpha+j) + (2\lambda^2-3\lambda+1)$$

**Corollary 2.7:** Let the function $f(z)^{\alpha}$ of the form (5) be in the class $TS(\lambda,\gamma,\beta,\alpha,\theta,m,j)$, then

$$a_k \leq \frac{(2+\beta-\gamma)\left[4(\lambda-\lambda^2)-(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right]-(1+\beta)\alpha^{m+1}[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)]}{(\alpha+j)^m[(1+\beta)M_{j+1}-(\beta+\gamma)F_{j+1}]}. \quad (10)$$

This equality in (10) is attained for the function $f(z)^{\alpha}$ of the form (9).

## 3. Growth and Distortion theorem

**Theorem 3.1:** Let the function $f(z)^{\alpha}$ of the form (5) be in the class $TS(\lambda,\gamma,\beta,\alpha,\theta,m,j)$. Then, for $|z| < r = 1$,

$$\begin{aligned} & r^{\alpha} - \frac{(2+\beta-\gamma)[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m]-(1+\beta)\alpha^{m+1}[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)]}{(\alpha+j)^m[(1+\beta)M_{j+1}-(\beta+\gamma)F_{j+1}]} r^{a+j} \\ & \leq |f(z)^{\alpha}| \\ & \leq r^{\alpha} + \frac{(2+\beta-\gamma)[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m]-(1+\beta)\alpha^{m+1}[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)]}{(\alpha+j)^m[(1+\beta)M_{j+1}-(\beta+\gamma)F_{j+1}]} r^{a+j} \end{aligned}. \quad (11)$$

The result (11) is attained for the function $f(z)^{\alpha}$given by (9) for $z = \pm r$.

**Theorem 3.2:** Let the function $f(z)^{\alpha}$of the form (5) be in the class $TS(\lambda,\gamma,\beta,\alpha,\theta,m,j)$. Then, for$|z| < r = 1$,

$$\alpha\gamma^{\alpha-1} - \gamma^{a+j-1}\left\{\frac{(\alpha+j)(2+\beta-\gamma)[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m] \\ -(1+\beta)\alpha^{m+1}[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)]}{(\alpha+j)^m[(1+\beta)M_{j+1}-(\beta+\gamma)F_{j+1}]}\right\}$$

$$\leq |f'(z)^{\alpha}|$$

$$\leq \alpha r^{\alpha-1} + \alpha\lambda^{\alpha+j-1}\left\{\frac{(\alpha+j)(2+\beta-\gamma)[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m] \\ -(1+\beta)\alpha^{m+1}[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)]}{(\alpha+j)^m[(1+\beta)M_{j+1}-(\beta+\gamma)F_{j+1}]}\right\} z^{\alpha+k-1}. \quad (12)$$

**Theorem 3.3:** Let

$$f_j(z)^{\alpha} = z^{\alpha} \quad (13)$$

and

$$\begin{aligned} & f(z)^{\alpha} \\ & = z^{\alpha} - \frac{(2+\beta-\gamma)[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m]-(1+\beta)\alpha^{m+1}[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)]}{(\alpha+k-1)^m[(1+\beta)M_k-(\beta+\gamma)F_k]} z^{\alpha+k-1} \end{aligned} \quad (14)$$

for $k \geq j+1$, $0 \leq \lambda \leq 1$, $\beta \geq 0$, $-1 \leq \gamma < 1$, $m \in N_0$, $\alpha > 0$ and $0 \leq |\theta| < \pi/2$. Then $f(z)^{\alpha}$ in the class $TS(\lambda,\gamma,\beta,\alpha,\theta,m,j)$ if and only if it can be expressed in the form

$$f(z)^{\alpha} = \sum_{k=j}^{\infty} \delta_k f_k(z)^{\alpha} \quad (15)$$

where $\delta_k \geq 0$ $(k \geq j)$ and $\sum_{k=j}^{\infty} \gamma_k = 1$.

**Proof:** Suppose that $f(z)^{\alpha} = \sum_{k=j}^{\infty} \delta_k f_k(z)^{\alpha} = \delta_j f_j(z)^{\alpha} + \sum_{k=j+1}^{\infty} \delta_k f_k(z)^{\alpha}$.

Then

$$f(z)^{\alpha} = \delta_j f_j(z)^{\alpha} + \sum_{k=j+1}^{\infty} \delta_k \left[ z^{\alpha} - \frac{\begin{array}{c}(2+\beta-\gamma)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right] \\ -(1+\beta)\alpha^{m+1}\left[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)\right]\end{array}}{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]} \right] z^{\alpha+k-1}$$

$$= \delta_j f_j(z)^{\alpha} + \sum_{k=j+1}^{\infty} \delta_k z^{\alpha} - \sum_{k=j+1}^{\infty} \delta_k \left[ \frac{\begin{array}{c}(2+\beta-\gamma)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right] \\ -(1+\beta)\alpha^{m+1}\left[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)\right]\end{array}}{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]} \right] z^{\alpha+k-1}.$$

Since $f_j(z)^{\alpha} = z^{\alpha} \Rightarrow \delta_j f_j(z)^{\alpha} + \sum_{k=j+1}^{\infty} \delta_k z^{\alpha} = \delta_j z^{\alpha} + \sum_{k=j+1} \delta_k z^{\alpha} = \left(\delta_j + \sum_{k=j+1} \delta_k\right) z^{\alpha} =$ $\sum_{k=j} \delta_k z^{\alpha} = z^{\alpha}, \quad \left(\sum_{k=j} \delta_k = 1\right)$.

So,

$$f(z)^{\alpha} = z^{\alpha} - \sum_{k=j+1}^{\infty} \delta_k \left\{ \frac{\begin{array}{c}(2+\beta-\gamma)[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m] \\ -(1+\beta)\alpha^{m+1}[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)]\end{array}}{(\alpha+k-1)^m[(1+\beta)M_k-(\beta+\gamma)F_k]} \right\} z^{\alpha+k-1}.$$

Since

$$\sum_{k=j+1}^{\infty} \delta_k \left\{ \frac{\begin{array}{c}(2+\beta-\gamma)\left[\begin{array}{c}4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1} \\ +(2\lambda^2-3\lambda+1)\alpha^m\end{array}\right] \\ -(1+\beta)\alpha^{m+1}\left[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)\right]\end{array}}{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]} \right\} \times \left\{ \frac{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]}{\begin{array}{c}(2+\beta-\gamma)\left[\begin{array}{c}4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1} \\ +(2\lambda^2-3\lambda+1)\alpha^m\end{array}\right] \\ -(1+\beta)\alpha^{m+1}\left[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)\right]\end{array}} \right\}$$

$$= \sum_{k=j+1}^{\infty} \delta_k = \sum_{k=j}^{\infty} \delta_k - \delta_j = 1 - \delta_j \leq 1, \quad \left(\sum_{k=j}^{\infty} \delta_k = 1\right).$$

It follows from theorem 2.6 that the function$f(z)^{\alpha}$ of the from (5) belong to the class $TS(\lambda, \gamma, \beta, \alpha, \theta, m, j)$. Conversely, let the function $f(z)^{\alpha}$ of the form (5) belongs to the class $TS(\lambda, \gamma, \beta, \alpha, \theta, m, j)$. Then,

$$a_k(\alpha) \leq \frac{(2+\beta-\gamma)[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m]-(1+\beta)\alpha^{m+1}[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)]}{(\alpha+k-1)^m[(1+\beta)M_k-(\beta+\gamma)F_k]}$$

for $k \geq j+1$.

Now, setting

$$\delta_k = \frac{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]}{(2+\beta-\gamma)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right]-(1+\beta)\alpha^{m+1}\left[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)\right]} a_k(\alpha)$$

and

$$\delta_j = 1 - \sum_{k=j+1}^{\infty} \delta_k,$$

we have that

$$f(z)^{\alpha} = z^{\alpha} - \sum_{k=j+1}^{\infty} a_k(\alpha) z^{a+k+1}.$$

Implies that

$$f(z)^\alpha = z^\alpha - \sum_{k=j+1}^{\infty} \delta_k \left\{ \frac{\begin{array}{c}(2+\beta-\gamma)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right] \\ -(1+\beta)\alpha^{m+1}\left[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)\right]\end{array}}{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]} \right\} z^{\alpha+k-1}$$

From (15), we have that

$$f(z)^\alpha = z^\alpha + \sum_{k=j+1}^{\infty}[f_k(z)^\alpha - z^\alpha]\delta_k = z^\alpha - \sum_{k=j+1}^{\infty}\delta_k z^\alpha + \sum_{k=j+1}^{\infty} f_k(z)^\alpha\delta_k = \left(1-\sum_{k=j+1}^{\infty}\delta_k\right)z^\alpha + \sum_{k=j+1}^{\infty} f_k(z)^\alpha\delta_k$$

$$f(z)^\alpha = \delta_j z^\alpha + \sum_{k=j+1}^{\infty} f_k(z)^\alpha\delta_k = \delta_j f_j z^\alpha + \sum_{k=j+1}^{\infty}\delta_k f_k z^\alpha = \sum_{k=j}^{\infty}\delta_k f_k(z)^\alpha$$

hence the proof.

**Corollary 3.4:** The extreme points of the class $TS\,(\lambda,\gamma,\beta,\alpha,\theta,m,j)$ are functions given by (13) and (14).

## 4. Radii of close-to-convexity, starlikeness and convexity

**Theorem 4.1:** Let $f(z)^\alpha \in TS\,(\lambda,\gamma,\beta,\alpha,\theta,m,j)$, the $f(z)^\alpha$ is close-to-convex of order $\psi$ $(0 \le \psi < 1)$in the disk $|z| < r_1$where

$$r_1 = \inf\left\{ \frac{(\alpha-\psi)(\alpha+k-1)^m[(1+\beta)M_k-(\beta+\gamma)F_k]}{\begin{array}{l}(2+\beta-\gamma)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right] \\ -(1+\beta)\alpha^{m+1}\left[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)\right]\end{array}} \right\}^{\frac{1}{k-1}} \quad k \ge j+1\,. \tag{16}$$

The result is sharp with extremal function $f(z)^\alpha$ given by (9).

**Proof**: It is sufficient to show that

$$\left|\frac{f(z)^\alpha}{z^{\alpha-1}} - \alpha\right| < \alpha - \psi, \qquad |z| < r_1\,. \tag{17}$$

It implies that,

$$\left|\frac{f'(z)^\alpha}{z^{\alpha-1}} - \alpha\right| = \left|\sum_{k=j+1}^{\infty}(\alpha+k-1)a_k(\alpha)z^{k-1}\right| < \alpha - \psi.$$

That is

$$\frac{\sum_{k=j+1}^{\infty}(\alpha+k-1)a_k(\alpha)z^{k-1}}{\alpha-\psi} \le 1\ . \tag{18}$$

Now, (18) is proven true if

$$\frac{(\alpha+k-1)|a_k(\alpha)||z|^{k-1}}{\alpha-\psi} \le \frac{(\alpha+k-1)^m[(1+\beta)M_k-(\beta+\gamma)F_k]}{(2+\beta-\gamma)[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m]-(1+\beta)\alpha^{m+1}[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)]}\,.$$

Solving for$|z|^{k-1}$, we have (16).

**Theorem 4.2:** Let $f(z)^\alpha \in TS\,(\lambda,\gamma,\beta,\alpha,\theta,m,j)$, then $f(z)^\alpha$ is starlike of order

$\psi$ $(0 \leq \psi < 1)$ in the disk $|z| < r_2$ where,

$$r_2 = \inf \left\{ \frac{(\alpha-\psi)(\alpha+k-1)^m[(1+\beta)M_k-(\beta+\gamma)F_k]}{\begin{array}{l}(\alpha+k-\psi-1)(2+\beta-\gamma)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right] \\ -(1+\beta)\alpha^{m+1}\left[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)\right]\end{array}} \right\}^{\frac{1}{k-1}} \quad (k \geq j+1). \tag{19}$$

The result is sharp for the extremal function $f(z)^\alpha$ given by (9).

**Proof:** It is sufficient to show that

$$\left|\frac{f'(z)^\alpha}{z^{\alpha-1}} - \alpha\right| \leq \alpha - \psi \Rightarrow \left|\frac{\sum_{k=j+1}^{\infty}(k-1)a_k(\alpha)z^{k-1}}{1-\sum_{k=j+1}^{\infty}a_k(\alpha)z^{a+k+1}}\right| < \alpha - \psi.$$

That is

$$\frac{\sum_{k=j+1}^{\infty}(\alpha+k-\psi-1)\left|a_k(\alpha)\right|\left|z\right|^{k-1}}{\alpha-\psi} \leq 1. \tag{20}$$

Now, (20) is proven true if

$$\frac{(\alpha+k-\psi-1)|a_k(\alpha)||z|^{k-1}}{\alpha-\psi}$$
$$\leq \frac{(\alpha+k-1)^m[(1+\beta)M_k-(\beta+\gamma)F_k]}{(2+\beta-\gamma)[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m]-(1+\beta)\alpha^{m+1}[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)]}.$$

Solving for $|z|^{k-1}$, we obtain (19).

**Theorem 4.3:** Let $f(z)^\alpha \in TS\,(\lambda,\gamma,\beta,\alpha,\theta,m,j)$, then $f(z)^\alpha$ is starlike of order $\psi$ $(0 \leq \psi < 1)$ in the disk $|z| < r_3$ where,

$$r_3 = \inf \left\{ \frac{\alpha(\alpha-\psi)(\alpha+k-1)^m[(1+\beta)M_k-(\beta+\gamma)F_k]}{\begin{array}{l}(\alpha+k-1)(\alpha+k-\psi-1)(2+\beta-\gamma)[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m] \\ -(1+\beta)\alpha^{m+1}[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)]\end{array}} \right\}^{\frac{1}{k-1}}$$

The result is sharp for the extremal function $f(z)^\alpha$ of the form (9).

**Proof**: It suffices to show that

$$\left|1+\frac{zf''(z)^\alpha}{f'(z)^\alpha}-\alpha\right| < \alpha-\psi.$$

It implies that

$$\left|\frac{\sum_{k=j+1}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)(k-1)a_k(\alpha)z^{k-1}}{1-\sum_{k=j+1}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)a_k(\alpha)z^{k-1}}\right| < \alpha-\psi\ .$$

That is

$$\frac{\sum_{k=j+1}^{\infty}(\alpha+k-1)(\alpha+k-\psi-1)\left|a_k(\alpha)\right|\left|z\right|^{k-1}}{\alpha(\alpha-\psi)} \leq 1. \tag{21}$$

So (21) is proven from if

$$\frac{(\alpha+k-1)(\alpha+k-\psi-1)\left|a_k(\alpha)\right|\left|z\right|^{k-1}}{\alpha(\alpha-\psi)}$$
$$\leq \frac{(\alpha+k-1)^m[(1+\beta)M_k-(\beta+\gamma)F_k]}{(2+\beta-\gamma)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right]-(1+\beta)\alpha^{m+1}\left[e^{i\theta}+(\alpha-1)(2\lambda^2-\lambda)\right]}.$$

Solving for $|z|^{k-1}$, we obtained the desired result.

## 5. Modified Hadamard product

Let the function $f_i(z)^\alpha$ $(i = 1, 2)$ be given

$$f_i(z)^\alpha = z^\alpha - \sum_{k=j+1}^{\infty} a_{k,i}(\alpha) z^{a+k+1}, \qquad a_{k,i} \geq 0; j \in N \tag{22}$$

then we define the modified Hadamard product of $f_1(z)^\alpha$and$f_2(z)^\alpha$by

$$f_1(z)^\alpha * f_2(z)^\alpha = z^\alpha - \sum_{k=j+1}^{\infty} a_{k,1}(\alpha)\, a_{k,2}(\alpha) z^{a+k+1}.$$

**Theorem 5.1:** Let each of the $f_i(z)^\alpha$ $(i = 1, 2)$ defined above be in the class $TS(\lambda, \gamma, \beta, \alpha, \theta, m, j)$. Then

$$\sigma_1 = \frac{(\alpha+k-1)^m \left[(1+\beta)M_k - (\beta+\gamma)F_k\right]^2 \left\{ \begin{array}{l} (2+\beta)\left(4(\lambda-\lambda^2) + (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2-3\lambda+1)\alpha^m\right) \\ -(1+\beta)\alpha^{m+1}\left[e^{i\theta} + (\alpha-1)(2\lambda^2-\lambda)\right] \end{array} \right\} - \left[(1+\beta)M_k - (\beta+\gamma)F_k\right] \left\{ \begin{array}{l} (2+\beta-\gamma)\left(4(\lambda-\lambda^2) + (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2-3\lambda+1)\alpha^m\right) \\ -(1+\beta)\alpha^{m+1}\left(e^{i\theta} + (\alpha-1)(2\lambda^2-\lambda)\right) \end{array} \right\}}{(\alpha+k-1)^m \left(4(\lambda-\lambda^2) + (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2-3\lambda+1)\alpha^m\right) \left[(1+\beta)M_k - (\beta+\gamma)F_k\right]^2 - (2+\beta-\gamma)\left(4(\lambda-\lambda^2) + (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2-3\lambda+1)\alpha^m\right) - (1+\beta)\alpha^{m+1}\left(e^{i\theta} + (\alpha-1)(2\lambda^2-\lambda)\right)^2 F_k}$$

The result is sharp.

**Proof**: We need to show the largest $\sigma_1$ such that

$$\sum_{k=j+1}^{\infty} \frac{(\alpha+k-1)^m \left[(1+\beta)M_k - (\beta+\sigma_1)F_k\right]}{(2+\beta-\sigma_1)\left[4(\lambda-\lambda^2) + (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2-3\lambda+1)\alpha^m\right] - G} a_{k,1}a_{k,2} \leq 1$$

where $G = (1+\beta)\alpha^{m+1}\left[e^{i\theta} + (\alpha-1)(2\lambda^2-\lambda)\right]$ and both $M_k$ and $F_k$ are as earlier defined.

From theorem 2.1, one can write that

$$\sum_{k=j+1}^{\infty} \frac{(\alpha+k-1)^m \left[(1+\beta)M_k - (\beta+\gamma)F_k\right]}{(2+\beta-\gamma)\left[4(\lambda-\lambda^2) + (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2-3\lambda+1)\alpha^m\right] - G} a_{k,1} \leq 1$$

and

$$\sum_{k=j+1}^{\infty} \frac{(\alpha+k-1)^m \left[(1+\beta)M_k - (\beta+\gamma)F_k\right]}{(2+\beta-\gamma)\left[4(\lambda-\lambda^2) + (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2-3\lambda+1)\alpha^m\right] - G} a_{k,2} \leq 1$$

where $G$ is as defined above.

By the Cauchy-Schwarz inequality, we have

$$\sum_{k=j+1}^{\infty} \frac{(\alpha+k-1)^m \left[(1+\beta)M_k - (\beta+\gamma)F_k\right]}{-(2+\beta-\gamma)\left[4(\lambda-\lambda^2) + (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2-3\lambda+1)\alpha^m\right] - G} \sqrt{a_{k,1}a_{k,2}} \leq 1.$$

Thus, it is sufficient to show that

$$\sum_{k=j+1}^{\infty} \frac{(\alpha+k-1)^m \left[(1+\beta)M_k - (\beta+\sigma_1)F_k\right]}{(2+\beta-\sigma_1)\left[4(\lambda-\lambda^2) + (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2-3\lambda+1)\alpha^m\right] - G} a_{k,1}a_{k,2}$$

$$\leq \sum_{k=j+1}^{\infty} \frac{(\alpha+k-1)^m \left[(1+\beta)M_k - (\beta+\gamma)F_k\right]}{(2+\beta-\gamma)\left[4(\lambda-\lambda^2) + (2\lambda^2-\lambda)\alpha^{m+1} + (2\lambda^2-3\lambda+1)\alpha^m\right] - G} \sqrt{a_{k,1}\, a_{k,2}}\,.$$

That is

$$\sqrt{a_{k,1}\ a_{k,2}} \leq \frac{[(1+\beta)M_k-(\beta+\gamma)F_k]\left\{\ (2+\beta-\sigma_1)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right]-G\ \right\}}{[(1+\beta)M_k-(\beta+\sigma_1)F_k]\left\{\ (2+\beta-\gamma)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right]-G\ \right\}}.$$

Observe that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(2+\beta-\gamma)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right]-G}{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]}.$$

It implies that

$$\frac{(2+\beta-\gamma)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right]-G}{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]}$$

$$\leq \frac{[(1+\beta)M_k-(\beta+\gamma)F_k]\left\{\ (2+\beta-\sigma_1)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right]-G\ \right\}}{[(1+\beta)M_k-(\beta+\sigma_1)F_k]\left\{\ (2+\beta-\gamma)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right]-G\ \right\}}.$$

Or

$$\sigma_1$$

$$\leq \frac{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\sigma_i)F_k\right]^2\left\{\begin{array}{l}(2+\beta)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right]-G\\ -\left[\ (2+\beta-\gamma)\left[\begin{array}{l}4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}\\+(2\lambda^2-3\lambda+1)\alpha^m\end{array}\right]-G\ \right]^2 H\end{array}\right\}}{\left[(\alpha+k-1)^m\left(\begin{array}{l}4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}\\+(2\lambda^2-3\lambda+1)\alpha^m\end{array}\right)\left[(1+\beta)M_k-(\beta+\sigma_i)F_k\right]^2\right]-IF_k}$$

$$= \Delta_1(k)$$

where $H = [(1+\beta)M_k-(\beta+\sigma_i)F_k]$ and $I = \left[\begin{array}{l}(2+\beta-\gamma)\left[\begin{array}{l}4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}\\+(2\lambda^2-3\lambda+1)\alpha^m\end{array}\right]\\-G\end{array}\right]^2.$

Since $\Delta_1(k)$ is an increasing function of $k$ $(k \geq j+1)$ *and* $j \in N$,letting $k=j+1$ in the inequality above, then

$$\sigma_1 \leq \Delta(j+1) = \frac{Q-W}{J-P}$$

where $J = (\alpha+j)^m\left(4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right)\left[\begin{array}{l}(1+\beta)M_{j+1}\\-(\beta-\gamma)F_{j+1}\end{array}\right]^2,$

$$Q = \left[\begin{array}{l}(\alpha+j)^m\\ \left[\begin{array}{l}(1+\beta)M_{j+1}\\-(\beta-\gamma)F_{j+1}\end{array}\right]^2\end{array}\right]\left[\left((2+\beta)\left(\begin{array}{l}4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}\\+(2\lambda^2-3\lambda+1)\alpha^m\end{array}\right)-G\right)\right],$$

$$W = \left[(2+\beta-\gamma)\left(\begin{array}{l}4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}\\+(2\lambda^2-3\lambda+1)\alpha^m\end{array}\right)-G\right]^2\left[M_{j+1}-(\beta-\gamma)F_{j+1}\right],$$

and

$$P = \left[\begin{array}{l}(2+\beta-\gamma)\left(4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right)\\-G\end{array}\right]^2 F_{j+1}$$

which is the required assertion. The result is sharp for the functions defined by (9).

**Theorem 5.2:** Let each of the $f_i(z)^\alpha$ $(i=1,\,2)$ defined in (22) be in the class

$TS(\lambda,\gamma,\beta,\alpha,\theta,m,j)$. If the sequence $\{(1+\beta)M_k-(\beta+\gamma)F_k\}$ is non-decreasing. Then the function

$$h(z)^\alpha = z^\alpha - \sum_{k=j+1}^{\infty}(a_{k,1}^2+a_{k,2}^2)z^{\alpha+k-1}$$

belongs to the class $TS(\lambda,\gamma,\beta,\alpha,\theta,m,j)$.

**Proof:** From theorem 2, we have for $f_j(z)^\alpha$ $(j=1,\ 2)\in TS(\lambda,\gamma,\beta,\alpha,\theta,m,j)$ that

$$\sum_{k=j+1}^{\infty}\left\{\frac{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]}{(2+\beta-\gamma)\left(4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right)-G}\right\}^2 a_{k,1}^2$$

$$\leq \sum_{k=j+1}^{\infty}\left\{\frac{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]}{(2+\beta-\gamma)\left(4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right)-G}a_{k,1}^2\right\}^2\leq 1 \tag{23}$$

and

$$\sum_{k=j+1}^{\infty}\left\{\frac{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]}{(2+\beta-\gamma)\left(4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right)-G}\right\}^2 a_{k,2}^2$$

$$\leq \sum_{k=j+1}^{\infty}\left\{\frac{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]}{(2+\beta-\gamma)\left(4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right)-G}a_{k,2}\right\}^2\leq 1. \tag{24}$$

It follows from (23) and (24) that

$$\sum_{k=j+1}^{\infty}\frac{1}{2}\left\{\frac{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\gamma)F_k\right]}{(2+\beta-\gamma)\left(4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right)-G}\right\}^2\left(a_{k,1}^2+a_{k,2}^2\right)\leq 1.$$

Therefore, we need to find the largest $\sigma_2$, such that

$$\frac{(\alpha+k-1)[(1+\beta)M_k-(\alpha+\gamma)F_k]}{(2+\beta-\sigma_2)\Big[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\Big]-G}$$

$$\leq\frac{1}{2}\left\{\frac{(\alpha+k-1)\big[(1+\beta)M_k-(\alpha+\gamma)F_k\big]}{(2+\beta-\gamma)\Big[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\Big]-(G}\right\}^2.$$

That is

$$\sigma_2$$

$$\leq\frac{(\alpha+k-1)^m\left[(1+\beta)M_k-(\beta+\sigma_i)F_k\right]^2\left\{\begin{array}{c}(2+\beta)\left[4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right]-G\\ -2\left[\ (2+\beta-\gamma)\left[\begin{array}{c}4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}\\ +(2\lambda^2-3\lambda+1)\alpha^m\end{array}\right]-G\ \right]^2 H\end{array}\right\}}{\left[(\alpha+k-1)^m\left(\begin{array}{c}4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}\\ +(2\lambda^2-3\lambda+1)\alpha^m\end{array}\right)\left[(1+\beta)M_k-(\beta+\sigma_i)F_k\right]^2\right]-2IF_k}$$

$$=\Delta_2(k)$$

where $H=[(1+\beta)M_k-(\beta+\sigma_i)F_k]$ and $I=\left[\begin{array}{l}(2+\beta-\gamma)\left[\begin{array}{l}4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}\\+(2\lambda^2-3\lambda+1)\alpha^m\end{array}\right]\\-G\end{array}\right]^2$.

Since $\Delta_2(k)$ is an increasing function $k\ (k\geq j+1)\ and\ j\in N$,we can have that

$$\sigma_1\leq\Delta(j+1)=\frac{Q-2W}{J-2P}$$

where $J=(\alpha+j)^m\left(4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right)\left[\begin{array}{l}(1+\beta)M_{j+1}\\-(\beta-\gamma)F_{j+1}\end{array}\right]^2$,

$$Q=\left[\begin{array}{l}(\alpha+j)^m\\\left[\begin{array}{l}(1+\beta)M_{j+1}\\-(\beta-\gamma)F_{j+1}\end{array}\right]^2\end{array}\right]\left[\left((2+\beta)\left(\begin{array}{l}4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}\\+(2\lambda^2-3\lambda+1)\alpha^m\end{array}\right)-G\right)\right],$$

$$W=\left[(2+\beta-\gamma)\left(\begin{array}{l}4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}\\+(2\lambda^2-3\lambda+1)\alpha^m\end{array}\right)-G\right]^2[M_{j+1}-(\beta-\gamma)F_{j+1}],$$

and

$$P=\left[\begin{array}{l}(2+\beta-\gamma)\left(4(\lambda-\lambda^2)+(2\lambda^2-\lambda)\alpha^{m+1}+(2\lambda^2-3\lambda+1)\alpha^m\right)\\-G\end{array}\right]^2F_{j+1}$$

which ends the proof of theorem 5.2.

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