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NONEXISTENCE OF GLOBAL SOLUTIONS TO FRACTIONAL NONLINEAR EQUATIONS

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Abstract. The object of this article is to study a class of non local non linear fractional differential equation. In this work, we present sufficient conditions for non existence of global solutions to a non local non linear ultra-parabolic equation. Then, we extend our results to a system of two equations.

1. Introduction

This paper aims to extend the recent results of Kerbal and Kirane [13] by looking at fractional in time and space nonlinear ultra-parabolic equations instead of classical ones. Indeed we will provide the result of blow-up.

$$D^{\alpha_1}_{0|t_1}(u-u_2)+D^{\alpha_2}_{0|t_2}(u-u_1)+(-\Delta)^{\frac{\alpha}{2}}(|u|^m)=h\,|u|^p\,,\ h=t_1^s t_2^l\,|x|^r\,, \tag{1}$$

for

$$(t_1,t_2,x)\in Q=\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^N,\quad N\in\mathbb{N}$$

.

Here $p>m>1$ are real numbers, $0<\alpha_1<\alpha_2<1$ and $D^{\alpha_1}_{0|t_1}$, $D^{\alpha_2}_{0|t_2}$ are the fractional derivatives in the sense of Riemann-Liouville. Then, we extend our results to the system of two equations

$$D^{\alpha_1}_{0|t_1}(u-u_2)+D^{\alpha_2}_{0|t_2}(u-u_1)+(-\Delta)^{\frac{\alpha}{2}}(|u|^m)=k_1\,|v|^p\,,\ k_1=t_1^{s_1}t_2^{l_1}\,|x|^{r_1}\,, \tag{2}$$

$$D^{\beta_1}_{0|t_1}(v-v_2)+D^{\beta_2}_{0|t_2}(v-v_1)+(-\Delta)^{\frac{\beta}{2}}(|v|^n)=k_2\,|u|^q\,,\ k_2=t_1^{s_2}t_2^{l_2}\,|x|^{r_2}\,, \tag{3}$$

for $(t_1,t_2,x)\in Q=\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^N$, and supplemented with the initial conditions

$$u(t_1,0,x)=u_1(t_1,x)\quad,\quad u(0,t_2,x)=u_2(t_2,x)\,, \tag{4}$$

$$v(t_1,0,x)=v_1(t_1,x)\quad,\quad v(0,t_2,x)=v_2(t_2,x)\,. \tag{5}$$

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Here p, q are positive real numbers.
The nonlocal operator $D^{\alpha}_{0|t}$ is defined, for a continuous function $g : \mathbb{R}_+ \longrightarrow \mathbb{R}$, by

$$\left(D^{\alpha}_{0|t}\right) g\left(t\right) = \frac{1}{\Gamma\left(1-\alpha\right)} \frac{d}{dt} \int_0^t \frac{g\left(\tau\right)}{\left(t-\tau\right)^{\alpha}} d\tau$$

where $\Gamma\left(\alpha\right) = \displaystyle\int_0^{\infty} r^{\alpha-1} e^{-r} dr$ is the Euler gamma function.The fractional power of the Laplacian $\left(-\Delta\right)^{\frac{\alpha}{2}}$ $(0 < \alpha \leq 2)$ stands for diffusion in media with impurities and is defined as

$$\left(-\Delta\right)^{\frac{\alpha}{2}} v\left(x\right) = \mathcal{F}^{-1}\left(\left|\zeta\right|^{\alpha} \mathcal{F}\left(v\right)\left(\zeta\right)\right)\left(x\right),$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse, In addition, it satisfies for any v of our space $S(\mathbb{R}^N)^1$ we have $\left(-\Delta\right)^{\frac{\alpha}{2}} v \in L^{\frac{p}{p-m}}(\mathbb{R}^N)$.

The operator $D^{\alpha}_{0|t}$ counts for the abnormal diffusion, a recently very much studied topic in probability, physics, chemistry, biology, image processing, etc, see for instance [7],[8], and their references. Classical multi-time or ultra-parabolic problems have received a special interest and attention by authors due to their applications in real life problems, see for example [2],[5],[9],[13], while the fractional analog are in their preliminary steps.
Now we will write some preliminaries:
The right-sided Riemann-Liouville derivatives of order $0 < \alpha < 1$ for a continuous function $g : \mathbb{R}_+ \rightarrow \mathbb{R}?$ is by:

$$\left(D^{\alpha}_{t|T}\right) g\left(t\right) = -\frac{1}{\Gamma\left(1-\alpha\right)} \frac{d}{dt} \int_t^T \frac{g\left(\tau\right)}{\left(\tau-t\right)^{\alpha}} d\tau.$$

Note that for a differentiable function g, the left-sided Caputö derivatives of order α? is defined as:

$$D^{\alpha}_{0|t}\left(g\right)\left(t\right) = \frac{1}{\Gamma\left(1-\alpha\right)} \int_0^t \frac{g^{'}\left(\tau\right)}{\left(\tau-t\right)^{\alpha}} d\tau.$$

Finally,we recall the following integration by parts formula:

$$\int_0^T f\left(t\right) D^{\alpha}_{0|t} g\left(t\right) dt = \int_0^T D^{\alpha}_{t|T} f\left(t\right) g\left(t\right) dt$$

for functions $f, g \in C\left([0,T]\right)$, with $D^{\alpha}_{0|t} g$, $D^{\alpha}_{t|T} f$ exist at every point in $[0,T]$ and are continuous.
We also need some preparatory lemmas based on the function

$$\phi\left(t\right) = \begin{cases} \left(1-\frac{t}{T}\right)^{\lambda}, & 0 \leq t \leq T, \\ 0, & t > T, \end{cases} \tag{6}$$

where $\lambda \geq 2$.

Lemma 1. Let ϕ be as in (6). We have

$$\int_0^T D^{\alpha}_{t|T} \phi\left(t\right) dt = C_{\alpha,\lambda} T^{1-\alpha}, \tag{7}$$

where

$$C_{\alpha,\lambda} = \frac{\lambda \Gamma\left(\lambda-\alpha\right)}{(\lambda-\alpha+1)\Gamma\left(\lambda-2\alpha+1\right)}.$$

[1] $S(\mathbb{R}^N)$ is Schwartz space

For a proof of the above lemma, see [12],[13].

Lemma 2. Let ϕ be as in (6) and $p > 1$.Then for $p < \lambda + 1$,

$$\int_0^T \phi^{1-p}(t)\,|\phi^{'}(t)\,|^p = C_p T^{1-p},$$

where

$$C_p = \frac{\lambda^p}{1+\lambda-p}.$$

For $\lambda > \alpha p - 1$

$$\int_0^T \phi(t)^{1-p}\,|D_{t|T}^{\alpha}\phi(t)\,|^p dt = C_{p,\alpha} T^{1-\alpha p},$$

where

$$C_{p,\alpha} = \frac{\lambda^p}{(\lambda+1-p\alpha)}\left[\frac{\Gamma(\lambda-\alpha)}{\Gamma(\lambda-2\alpha+1)}\right]^p.$$

For a proof of the above lemma, see [12],[13].

We define the regular function ψ:

$$\psi(\xi) = \begin{cases} 1 & \text{if } \ 0 \le \xi \le 1, \\ \text{decreasing} & \text{if } \ 1 \le \xi \le 2, \\ 0 & \text{if } \ \xi \ge 2, \end{cases} \tag{8}$$

which will be used hereafter.

2. Results of non-existence of solutions

Solutions to (1) subject to conditions;

$$u(t_1, 0, x) = u_1(t_1, x) \quad , \quad u(0, t_2, x) = u_2(t_2, x), \tag{$*$}$$

are meant in the following weak sense.

Definition 1. A function $u \in L^m(Q) \cap L^p(Q)$ is called a weak solution to (1) if

$$\begin{aligned} &\int_Q h|u|^p \varphi dP + \int_Q u_2 D_{t_1|T}^{\alpha_1}\varphi dP + \int_Q u_1 D_{t_2|T}^{\alpha_2}\varphi dP \\ &= \int_Q u D_{t_1|T}^{\alpha_1}\varphi dP + \int_Q u D_{t_2|T}^{\alpha_2}\varphi dP + \int_Q |u|^m (-\Delta)^{\frac{\alpha}{2}} \varphi dP \end{aligned} \tag{9}$$

for any test function φ, such that $P = (t_1, t_2, x)$, $\varphi(T, t_2, x) = \varphi(t_1, T, x) = 0$, and $D_{t_1|T}^{\alpha_1}\varphi < +\infty$, $D_{t_2|T}^{\alpha_2}\varphi < +\infty$,

Note that every weak solution is a classical solution near the points (t_1, t_2, x) where $u(t_1, t_2, x)$ is positive. Our main result dealing with equation (1) subject to initial condition $(*)$ is given by the following theorem.

Theorem 1. Assume that

$$\int_Q u_2 D_{t_1|T}^{\alpha_1} dP > 0, \int_Q u_1 D_{t_2|T}^{\alpha_2} dP > 0.$$

If $1 < p < \min\left\{1 + \frac{s+l+r+\alpha_1}{2+N-\alpha_1} \ , \ 1 + \frac{s+l+r+\alpha_2}{2+N-\alpha_2} \ , \ m\left(1 + \frac{s+l+r+\alpha}{2+N-\alpha}\right)\right\}$, then problem (1) does not admit global weak solutions.

For the proof we need the following proposition .

Proposition 1. Suppose that $\delta \in [0,2], \beta+1 \geq 0$, and $\theta \in C_0^{\infty}\left(\mathbb{R}^N\right)$. Then, the following point-wise inequality holds:

$$|\theta(x)|^{\beta}\theta(x)(-\Delta)^{\frac{\delta}{2}}\theta(x) \geq \frac{1}{\beta+2}(-\Delta)^{\frac{\delta}{2}}|\theta(x)|^{\beta+2}.$$

Proof of theorem 1. Our strategy of proof is to use the weak formulation of the problem with a suitable choice of the test function . We assume that the solution is nontrivial and global. We choose the test function $\varphi\left(t_1,t_2,x\right)$ in the form

$$\varphi\left(t_1,t_2,x\right)=\varphi_1\left(t_1\right)\varphi_2\left(t_2\right)\varphi_3\left(x\right). \tag{10}$$

where $\varphi_1\left(t_1\right)=\left(1-t_1/T\right)_+^{\lambda}$, [2] , $\varphi_2\left(t_2\right)=\left(1-t_2/T\right)_+^{\lambda}$ and $\varphi_3\left(x\right)=\psi\left(|x|^2/T^2\right)$.

Now, replacing φ by φ^{μ} in (Definition 1), we estimate $\int_{Q_T} uD_{t_1|T}^{\alpha_1}\varphi^{\mu}dP$ using the $\varepsilon-$ Young inequality as follows

$$\int_Q uD_{t_1|T}^{\alpha_1}\varphi^{\mu}dP \leq \varepsilon\int_Q h|u|^p\varphi^{\mu}dP + C(\varepsilon)\int_Q h^{\frac{-1}{p-1}}|D_{t_1|T}^{\alpha_1}\varphi^{\mu}|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP. \tag{11}$$

We are going to estimate

$$\int_Q uD_{t_1|T}^{\alpha_1}\varphi^{\mu}dP \ , \ \int_Q uD_{t_2|T}^{\alpha_2}\varphi^{\mu}dP \ , \ \int_Q |u|^m\left(-\Delta\right)^{\frac{\alpha}{2}}\varphi^{\mu}dP.$$

- for $\int_Q uD_{t_1|T}^{\alpha_1}\varphi^{\mu}dP$, we have

$$uD_{t_1|T}^{\alpha_1}\varphi^{\mu} \leq |u||D_{t_1|T}^{\alpha_1}\varphi^{\mu}| = h^{\frac{1}{p}}h^{\frac{-1}{p}}\varphi^{\frac{\mu}{p}}\varphi^{\frac{-\mu}{p}}|u||D_{t_1|T}^{\alpha_1}\varphi^{\mu}|$$

if we set

$$a=h^{\frac{1}{p}}|u|\varphi^{\frac{\mu}{p}}$$

and

$$b=h^{\frac{-1}{p}}|D_{t_1|T}^{\alpha_1}\varphi^{\mu}|\varphi^{\frac{-\mu}{p}}$$

in the ε - Young inequality $ab \leq \varepsilon a^p + C(\varepsilon)b^q$ where $p+q=pq$ and $a,b\in\mathbb{R}^+$ then

$$uD_{t_1|T}^{\alpha_1}\varphi^{\mu} \leq \varepsilon\left(h^{\frac{1}{p}}|u|\varphi^{\frac{\mu}{p}}\right)^p + C(\varepsilon)\left(h^{\frac{-1}{p}}|D_{t_1|T}^{\alpha_1}\varphi^{\mu}|\varphi^{\frac{-\mu}{p}}\right)^{\frac{p}{p-1}}$$

as $q=\dfrac{p}{p-1}$.

Finally,

$$uD_{t_1|T}^{\alpha_1}\varphi^{\mu} \leq \varepsilon\left(h|u|^p\varphi^{\mu}\right) + C(\varepsilon)\left(h^{\frac{-1}{p-1}}|D_{t_1|T}^{\alpha_1}\varphi^{\mu}|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}\right)$$

and then

$$\int_Q uD_{t_1|T}^{\alpha_1}\varphi^{\mu}dP \leq \varepsilon\int_Q h|u|^p\varphi^{\mu}dP + C(\varepsilon)\int_Q h^{\frac{-1}{p-1}}|D_{t_1|T}^{\alpha_1}\varphi^{\mu}|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP. \quad [3]$$

---

[2] $\left(1-\frac{t_1}{T}\right)_+^{\lambda} = \sup\left\{0\ ,\ \left(1-\frac{t_1}{T}\right)^{\lambda}\right\}$

[3] This integration can be simplified using the lemmas, or it's estimate using change of variable , as we shall see later.

• Estimate of $\int_Q uD^{\alpha_2}_{t_2|T}\varphi^\mu dP$.
Similarly, we have

$$\int_Q uD^{\alpha_2}_{t_2|T}\varphi^\mu dP \le \varepsilon \int_Q h|u|^p\varphi^\mu dP + C(\varepsilon)\int_Q h^{\frac{-1}{p-1}}|D^{\alpha_2}_{t_2|T}\varphi^\mu|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP. \tag{12}$$

Observe that

$$\int_Q u_2 D^{\alpha_1}_{t_1|T}\varphi^\mu dP = \left(\int_0^T D^{\alpha_1}_{t_1|T}\varphi_1^\mu(t_1)\,dt_1\right)\int_S u(0,t_2,x)\,\varphi_2^\mu(t_2)\,\varphi_3^\mu(x)\,dP_2. \tag{13}$$

With the help of Lemma 1 one can rewrite equation (13) as

$$\int_Q u_2 D^{\alpha_1}_{t_1|T}\varphi^\mu dP = \mathrm{C}_{\alpha_1,\lambda\mu}T^{1-\alpha_1}\int_S u(0,t_2,x)\,\varphi_2^\mu(t_2)\,\varphi_3^\mu(x)\,dP_2. \tag{14}$$

where $\quad S = \mathbb{R}_+ \times \mathbb{R}^N$
Similarly, we have

$$\int_Q u_1 D^{\alpha_2}_{t_2|T}\varphi^\mu dP = \mathrm{C}_{\alpha_2,\lambda\mu}T^{1-\alpha_2}\int_S u(t_1,0,x)\,\varphi_1^\mu(t_1)\,\varphi_3^\mu(x)\,dP_1, \tag{15}$$

where $\quad P_1 = (t_1, x), \quad P_2 = (t_2, x).$
• Estimate of $\int_Q |u|^m(-\Delta)^{\frac{\alpha}{2}}\varphi^\mu dP$.
Using the convexity inequality in proposition 1,

$$(-\Delta)^{\frac{\alpha}{2}}\varphi^\mu \le \mu\varphi^{\mu-1}(-\Delta)^{\frac{\alpha}{2}}\varphi,$$

So

$$\int_Q |u|^m(-\Delta)^{\frac{\alpha}{2}}\varphi^\mu dP \le \int_Q \mu\varphi^{\mu-1}|u|^m(-\Delta)^{\frac{\alpha}{2}}\varphi dP.$$

Using the $\varepsilon-$ Young inequality, we obtain the estimate,

$$\begin{aligned}\int_Q \mu\varphi^{\mu-1}|u|^m(-\Delta)^{\frac{\alpha}{2}}\varphi dP &\le \varepsilon\int_Q h|u|^p\varphi^\mu dP\\ &+ C(\varepsilon)\int_Q h^{\frac{-m}{p-m}}|(-\Delta)^{\frac{\alpha}{2}}\varphi|^{\frac{p}{p-m}}\varphi^{(\mu-1-\frac{m\mu}{p})\frac{p}{p-m}}dP. \end{aligned}\tag{16}$$

Now, using (11), (12), (13), and (16), we obtain

$$\begin{aligned}&\int_Q h|u|^p\varphi^\mu dP + \mathrm{C}_{\alpha_1,\lambda\mu}T^{1-\alpha_1}\int_S u(0,t_2,x)\,\varphi_2^\mu(t_2)\,\varphi_3^\mu(x)\,dP_2\\ &\quad + \mathrm{C}_{\alpha_2,\lambda\mu}T^{1-\alpha_2}\int_S u(t_1,0,x)\,\varphi_1^\mu(t_1)\,\varphi_3^\mu(x)\,dP_1 \qquad (17)\\ &\le 3\varepsilon\int_Q h|u|^p\varphi^\mu dP + C'(\varepsilon)\left(\int_Q h^{\frac{-1}{p-1}}|D^{\alpha_1}_{t_1|T}\varphi^\mu|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP\right.\\ &\left. + \int_Q h^{\frac{-1}{p-1}}|D^{\alpha_2}_{t_2|T}\varphi^\mu|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP + \int_Q h^{\frac{-m}{p-m}}|(-\Delta)^{\frac{\alpha}{2}}\varphi|^{\frac{p}{p-m}}\varphi^{(\mu-1-\frac{m\mu}{p})\frac{p}{p-m}}dP\right).\end{aligned}$$

If we choose $\varepsilon = \frac{1}{6}$ (for example), then we obtain the estimate

$$\int_Q h|u|^p\varphi^\mu dP + 2\mathrm{C}_{\alpha_1,\lambda\mu}T^{1-\alpha_1}\int_S u(0,t_2,x)\,\varphi_2^\mu(t_2)\,\varphi_3^\mu(x)\,dP_2$$

$$+2\mathrm{C}_{\alpha_2,\lambda\mu}T^{1-\alpha_2}\int_S u\left(t_1,0,x\right)\varphi_1^{\mu}\left(t_1\right)\varphi_3^{\mu}\left(x\right)dP_1$$

$$\leq C\left(\int_Q h^{\frac{-1}{p-1}}|D^{\alpha_1}_{t_1|T}\varphi^{\mu}|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP\right. \tag{18}$$

$$\left.+\int_Q h^{\frac{-1}{p-1}}|D^{\alpha_2}_{t_2|T}\varphi^{\mu}|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP+\int_Q h^{\frac{-m}{p-m}}|(-\Delta)^{\frac{\alpha}{2}}\varphi|^{\frac{p}{p-m}}\varphi^{(\mu-1-\frac{m\mu}{p})\frac{p}{p-m}}dP\right).$$

for some positive constant C. The right hand side of (18) is now free of the unknown function u. now passing to the new variables

$$\tau_1=T^{-1}t_1,\quad \tau_2=T^{-1}t_2,\quad y=T^{-1}x, \tag{19}$$

we have

$$\int_Q h^{\frac{-1}{p-1}}|D^{\alpha_1}_{t_1|T}\varphi^{\mu}|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP$$

$$=\left(\int_S t_2^{\frac{-l}{p-1}}|x|^{\frac{-r}{p-1}}\varphi_2^{\mu}\varphi_3^{\mu}dP_2\right)\left(\int_0^T t_1^{\frac{-s}{p-1}}|D^{\alpha_1}_{t_1|T}\varphi_1^{\mu}|^{\frac{p}{p-1}}\varphi_1^{\frac{-\mu}{p-1}}dt_1\right)$$

$$\leq C_1T^{2+N-\frac{s+l+r+\alpha_1p}{p-1}} \tag{20}$$

where $C_1=\left(\int_{\Omega_2}\tau_2^{\frac{-l}{p-1}}|y|^{\frac{-r}{p-1}}\varphi_2^{\mu}\varphi_3^{\mu}dP_{\tau_2}\right)\left(\int_0^1\tau_1^{\frac{-s}{p-1}}|D^{\alpha_1}_{\tau_1|1}\varphi_1^{\mu}|^{\frac{p}{p-1}}\varphi_1^{\frac{-\mu}{p-1}}d\tau_1\right)$,

with $\mu>\dfrac{p}{p-1}$, and $P_{\tau_2}=(\tau_2,y)$, $\Omega_2=\{1\leq\tau_2+|y|\leq 2\}$,

and

$$\int_{\Omega_2}\tau_2^{\frac{-l}{p-1}}|y|^{\frac{-r}{p-1}}\varphi_2^{\mu}\varphi_3^{\mu}dP_{\tau_2}=\int_0^1\tau_2^{\frac{-l}{p-1}}\left(1-\tau_2\right)^{\lambda\mu}\left(\int_{\{1-\tau_2\leq|y|\leq 2-\tau_2\}}|y|^{\frac{-r}{p-1}}\psi^{\mu}\left(|y|^2\right)dy\right)d\tau_2$$

as $|y|^{\frac{-r}{p-1}}\leq\left(1-\tau_2\right)^{\frac{-r}{p-1}}$, and $\psi^{\mu}\left(|y|^2\right)\leq 1,\quad\forall\left(\tau_2,y\right)\in\Omega_2.$

Then

$$\int_{\Omega_2}\tau_2^{\frac{-l}{p-1}}|y|^{\frac{-r}{p-1}}\varphi_2^{\mu}\varphi_3^{\mu}dP_{\tau_2}\leq\int_0^1\tau_2^{\frac{-l}{p-1}}\left(1-\tau_2\right)^{\lambda\mu-\frac{r}{p-1}}\left(\int_{\{1-\tau_2\leq|y|\leq 2-\tau_2\}}dy\right)d\tau_2$$

$$\leq V.\mathbf{B}\left(1-\frac{l}{p-1},1-\frac{r}{p-1}+\lambda\mu\right)$$

where $V=\int_{\{0\leq|y|\leq 2\}}dy<+\infty$, and

$$\int_0^1\tau_2^{\frac{-l}{p-1}}\left(1-\tau_2\right)^{\lambda\mu-\frac{r}{p-1}}d\tau_2=\mathbf{B}\left(1-\frac{l}{p-1},1-\frac{r}{p-1}+\lambda\mu\right)$$

$$=\frac{\Gamma(1-\frac{l}{p-1})\Gamma(1-\frac{r}{p-1}+\lambda\mu)}{\Gamma(2-\frac{l}{p-1}-\frac{r}{p-1}+\lambda\mu)}<+\infty.$$ [4]

For

$$I=\int_0^1\tau_1^{\frac{-s}{p-1}}|D^{\alpha_1}_{\tau_1|1}\varphi_1^{\mu}|^{\frac{p}{p-1}}\varphi_1^{\frac{-\mu}{p-1}}d\tau_1=\int_0^1\tau_1^{\frac{-s}{p-1}}|D^{\alpha_1}_{\tau_1|1}(1-\tau_1)^{\lambda\mu}|^{\frac{p}{p-1}}(1-\tau_1)^{\frac{-\mu}{p-1}}d\tau_1$$

[4] Beta function

With

$$D^{\alpha_1}_{\tau_1|1}(1-\tau_1)^{\lambda\mu} = -\frac{1}{\Gamma(1-\alpha_1)}\frac{d}{d\tau_1}\int_{\tau_1}^{1}\frac{(1-\sigma)^{\lambda\mu}}{(\sigma-\tau_1)^{\alpha_1}}d\sigma$$

$$= -\frac{1}{\Gamma(1-\alpha_1)}\frac{d}{d\tau_1}\int_{\tau_1}^{1}\frac{\frac{(1-\sigma)^{\lambda\mu}}{(1-\tau_1)^{\lambda\mu}}\frac{1}{(1-\tau_1)^{\alpha_1-\lambda\mu}}}{\frac{(\sigma-\tau_1)^{\alpha_1}}{(1-\tau_1)^{\alpha_1}}}d\sigma$$

We put $\sigma = \tau_1 + \eta(1-\tau_1)$,
Then

$$D^{\alpha_1}_{\tau_1|1}(1-\tau_1)^{\lambda\mu} = -\frac{1}{\Gamma(1-\alpha_1)}\frac{d}{d\tau_1}\left((1-\tau_1)^{\lambda\mu-\alpha_1+1}\right)\int_0^1 \eta^{-\alpha_1}(1-\eta)^{\lambda\mu}d\eta$$

$$= -\frac{1}{\Gamma(1-\alpha_1)}\mathbf{B}\left(1-\alpha_1\ ,\ \lambda\mu+1\right)\frac{d}{d\tau_1}(1-\tau_1)^{\lambda\mu-\alpha_1+1}$$

$$= -\frac{1}{\Gamma(1-\alpha_1)}\frac{\Gamma(1-\alpha_1).\Gamma(\lambda\mu+1)}{\Gamma(\lambda\mu-\alpha_1+2)}\left(-(\lambda\mu-\alpha_1+1)\right)(1-\tau_1)^{\lambda\mu-\alpha_1}$$

$$= \frac{\Gamma(\lambda\mu+1)}{\Gamma(\lambda\mu-\alpha_1+1)}(1-\tau_1)^{\lambda\mu-\alpha_1}$$

So

$$I = \int_0^1 \tau_1^{\frac{-s}{p-1}}\left[\frac{\Gamma(\lambda\mu+1)}{\Gamma(\lambda\mu-\alpha_1+1)}(1-\tau_1)^{\lambda\mu-\alpha_1}\right]^{\frac{p}{p-1}}(1-\tau_1)^{\frac{-\mu}{p-1}}d\tau_1$$

$$= \left[\frac{\Gamma(\lambda\mu+1)}{\Gamma(\lambda\mu-\alpha_1+1)}\right]^{\frac{p}{p-1}}\int_0^1 \tau_1^{\frac{-s}{p-1}}(1-\tau_1)^{(\lambda\mu-\alpha_1)\frac{p}{p-1}-\frac{\mu}{p-1}}d\tau_1$$

$$= \left[\frac{\Gamma(\lambda\mu+1)}{\Gamma(\lambda\mu-\alpha_1+1)}\right]^{\frac{p}{p-1}}\mathbf{B}\left(1-\frac{s}{p-1}\ ,\ (\lambda\mu-\alpha_1)\frac{p}{p-1}-\frac{\mu}{p-1}+1\right) < +\infty$$

Finally $C_1 < +\infty$

Similarly, we obtain

$$\int_Q h^{\frac{-1}{p-1}}|D^{\alpha_2}_{t_2|T}\varphi^{\mu}|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP$$

$$= \left(\int_S t_1^{\frac{-s}{p-1}}|x|^{\frac{-r}{p-1}}\varphi_1^{\mu}\varphi_3^{\mu}dP_1\right)\left(\int_0^T t_2^{\frac{-l}{p-1}}|D^{\alpha_2}_{t_2|T}\varphi_2^{\mu}|^{\frac{p}{p-1}}\varphi_2^{\frac{-\mu}{p-1}}dt_2\right)$$

$$\leq C_2T^{2+N-\frac{s+l+r+\alpha_2p}{p-1}} \tag{21}$$

where $C_2 = \left(\int_{\Omega_1}\tau_1^{\frac{-s}{p-1}}|y|^{\frac{-r}{p-1}}\varphi_1^{\mu}\varphi_3^{\mu}dP_{\tau_1}\right)\left(\int_0^1 \tau_2^{\frac{-l}{p-1}}|D^{\alpha_2}_{\tau_2|1}\varphi_2^{\mu}|^{\frac{p}{p-1}}\varphi_2^{\frac{-\mu}{p-1}}d\tau_2\right) < +\infty$, with $\mu > \dfrac{p}{p-1}$, and $P_{\tau_1} = (\tau_1, y)$, $\Omega_1 = \{1 \leq \tau_1 + |y| \leq 2\}$.

Now , we estimate

$$\int_Q h^{\frac{-m}{p-m}}|(-\Delta)^{\frac{\alpha}{2}}\varphi|^{\frac{p}{p-m}}\varphi^{(\mu-1-\frac{m\mu}{p})\frac{p}{p-m}}dP.$$

We have

$$\int_Q h^{\frac{-m}{p-m}}|(-\Delta)^{\frac{\alpha}{2}}\varphi|^{\frac{p}{p-m}}\varphi^{(\mu-1-\frac{m\mu}{p})\frac{p}{p-m}}dP$$

$$= \left(\int_{\mathbb{R}^N} |x|^{\frac{-mr}{p-m}} |(-\Delta)^{\frac{\alpha}{2}} \varphi_3|^{\frac{p}{p-m}} \varphi_3^{(\mu-1-\frac{m\mu}{p})\frac{p}{p-m}} dx \right) \left(\int_{[0,T]\times[0,T]} t_1^{\frac{-ms}{p-m}} t_2^{\frac{-ml}{p-m}} \varphi_1^\mu \varphi_2^\mu dt_1 dt_2 \right)$$

$$\leq C_3 T^{2+N-\frac{m(s+l+r)+\alpha p}{p-m}}. \qquad (22)$$

where

$$C_3 = \int_{\text{support}\psi} |y|^{\frac{-mr}{p-m}} |(-\Delta)_y^{\frac{\alpha}{2}} \psi|^{\frac{p}{p-m}} \psi^{(\mu-1-\frac{m\mu}{p})\frac{p}{p-m}} dy \int_\Omega \tau_1^{\frac{-ms}{p-m}} \tau_2^{\frac{-ml}{p-m}} (1-\tau_1)^{\lambda\mu}(1-\tau_2)^{\lambda\mu} d\tau_1 d\tau_2$$

As $\text{support}\psi = \{0 \leq |y|^2 \leq 2\}$, and $\psi^{(\mu-1-\frac{m\mu}{p})\frac{p}{p-m}}(|y|^2) \leq 1$ [5], for all $y \in \text{support}\psi$,

Then ,using $\varepsilon-$Young inequality, we have

$$\int_{\text{support}\psi} |y|^{\frac{-mr}{p-m}} |(-\Delta)_y^{\frac{\alpha}{2}} \psi|^{\frac{p}{p-m}} \psi^{(\mu-1-\frac{m\mu}{p})\frac{p}{p-m}} dy \leq \int_{\text{support}\psi} |y|^{\frac{-mr}{p-m}} |(-\Delta)_y^{\frac{\alpha}{2}} \psi|^{\frac{p}{p-m}} dy$$

$$\leq \varepsilon \int_{\{0\leq|y|^2\leq 2\}} |y|^{\frac{-2mr}{p-m}} |(-\Delta)_y^{\frac{\alpha}{2}} \psi\left(|y|^2\right)|^{\frac{2p}{p-m}} dy + C(\varepsilon) \int_{\{0\leq|y|^2\leq 2\}} 1^2 dy$$

For $\varepsilon \to 0$ we find

$$\int_{\text{support}\psi} |y|^{\frac{-mr}{p-m}} |(-\Delta)_y^{\frac{\alpha}{2}} \psi|^{\frac{p}{p-m}} \psi^{(\mu-1-\frac{m\mu}{p})\frac{p}{p-m}} dy \leq C \int_{\{0\leq|y|^2\leq 2\}} dy < +\infty.$$

on the other hand

$$\int_\Omega \tau_1^{\frac{-ms}{p-m}} \tau_2^{\frac{-ml}{p-m}} \varphi_1^\mu \varphi_2^\mu d\tau_1 d\tau_2 = \int_{[0,1]\times[0,1]} \tau_1^{\frac{-ms}{p-m}} \tau_2^{\frac{-ml}{p-m}} (1-\tau_1)^{\lambda\mu}(1-\tau_2)^{\lambda\mu} d\tau_1 d\tau_2$$

.

$$= \int_0^1 \tau_1^{\frac{-ms}{p-m}} (1-\tau_1)^{\lambda\mu} d\tau_1 \int_0^1 \tau_2^{\frac{-ml}{p-m}} (1-\tau_2)^{\lambda\mu} d\tau_2$$

$$= \mathbf{B}\left(1 - \frac{ms}{p-m}\ ,\ 1+\lambda\mu\right) \mathbf{B}\left(1 - \frac{ml}{p-m}\ ,\ 1+\lambda\mu\right) < +\infty$$

with, $\mu > \dfrac{p}{p-m}$, and $\Omega = [0,1]\times[0,1]$.

Finally $C_3 < +\infty$,

By (20) , (21) , (22), we obtain for (18) the following estimate

$$\int_Q h|u|^p \varphi^\mu dP + 2\mathrm{C}_{\alpha_1,\lambda\mu} T^{1-\alpha_1} \int_S u(0,t_2,x)\, \varphi_2^\mu(t_2)\, \varphi_3^\mu(x)\, dP_2$$

$$+2\mathrm{C}_{\alpha_2,\lambda\mu} T^{1-\alpha_2} \int_S u(t_1,0,x)\, \varphi_1^\mu(t_1)\, \varphi_3^\mu(x)\, dP_1$$

$$\leq C\left(T^{2+N-\frac{s+l+r+\alpha_1 p}{p-1}} + T^{2+N-\frac{s+l+r+\alpha_2 p}{p-1}} + T^{2+N-\frac{m(s+l+r)+\alpha p}{p-m}}\right). \qquad (23)$$

for some positive constant C

Now, we require:

(a) $2+N-\frac{s+l+r+\alpha_1 p}{p-1} < 0$ or $1 < p < 1 + \frac{s+l+r+\alpha_1}{2+N-\alpha_1}$.

(b) $2+N-\frac{s+l+r+\alpha_2 p}{p-1} < 0$ or $1 < p < 1 + \frac{s+l+r+\alpha_2}{2+N-\alpha_2}$.

[5] $(\mu-1-\frac{m\mu}{p})\frac{p}{p-m} > 0$, because $\mu > \frac{p}{p-m}$.

(c) $2+N-\frac{m(s+l+r)+\alpha p}{p-m}<0$ or $1<p<m\left(1+\frac{s+l+r+\alpha}{2+N-\alpha}\right)$.

Letting T aproach infinity in (23), we obtain a contradiction as the left hand side is positive while the right hand side goes to zero.

For the second case, we assume the exponents of T in (23) are zeros. Applying Hölder's inequality to the right hand side of inequality (18), we obtain

$$\int_Q |u||D^{\alpha_1}_{t_1|T}\varphi^\mu|dP=\int_Q |u|\left(h\varphi^\mu\right)^{\frac{1}{p}}|D^{\alpha_1}_{t_1|T}\varphi^\mu|\left(h\varphi^\mu\right)^{\frac{-1}{p}}dP$$

$$\leq\left(\int_{C_T} h|u|^p\varphi^\mu dP\right)^{\frac{1}{p}}\left(\int_Q h^{\frac{-1}{p-1}}|D^{\alpha_1}_{t_1|T}\varphi^\mu|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP\right)^{\frac{p-1}{p}},$$

$$\int_Q |u||D^{\alpha_2}_{t_2|T}\varphi^\mu|dP=\int_Q |u|\left(h\varphi^\mu\right)^{\frac{1}{p}}|D^{\alpha_2}_{t_2|T}\varphi^\mu|\left(h\varphi^\mu\right)^{\frac{-1}{p}}dP$$

$$\leq\left(\int_{C_T} h|u|^p\varphi^\mu dP\right)^{\frac{1}{p}}\left(\int_Q h^{\frac{-1}{p-1}}|D^{\alpha_2}_{t_2|T}\varphi^\mu|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP\right)^{\frac{p-1}{p}}$$

and

$$\int_Q |u|^m|\left(-\Delta\right)^{\frac{\alpha}{2}}\varphi^\mu|dP=\int_Q |u|^m\left(h\varphi^\mu\right)^{\frac{m}{p}}|\left(-\Delta\right)^{\frac{\alpha}{2}}\varphi^\mu|\left(h\varphi^\mu\right)^{\frac{-m}{p}}dP$$

$$\leq\mu\left(\int_{C_T} h|u|^p\varphi^\mu dP\right)^{\frac{m}{p}}\left(\int_Q h^{\frac{-m}{p-m}}|\left(-\Delta\right)^{\frac{\alpha}{2}}\varphi|^{\frac{p}{p-m}}\varphi^{\left(\mu-1-\frac{m\mu}{p}\right)\frac{p}{p-m}}dP\right)^{\frac{p-m}{p}}$$

$$\leq\mu C\left(\int_{C_T} h|u|^p\varphi^\mu dP\right)^{\frac{1}{p}}\left(\int_Q h^{\frac{-m}{p-m}}|\left(-\Delta\right)^{\frac{\alpha}{2}}\varphi|^{\frac{p}{p-m}}\varphi^{\left(\mu-1-\frac{m\mu}{p}\right)\frac{p}{p-m}}dP\right)^{\frac{p-m}{p}}$$

because at using the $\varepsilon-$ Young inequality,we have,

$$\left(\int_{C_T} h|u|^p\varphi^\mu dP\right)^{\frac{m}{p}}\leq\varepsilon 1^{\frac{1}{1-m}}+C(\varepsilon)\left(\left(\int_{C_T} h|u|^p\varphi^\mu dP\right)^{\frac{m}{p}}\right)^{\frac{1}{m}}.$$

For $\varepsilon\to 0$ we have,

$$\left(\int_{C_T} h|u|^p\varphi^\mu dP\right)^{\frac{m}{p}}\leq C\left(\int_{C_T} h|u|^p\varphi^\mu dP\right)^{\frac{1}{p}}.$$

Then

$$\int_Q h|u|^p\varphi^\mu dP+2\mathrm{C}_{\alpha_1,\lambda\mu}T^{1-\alpha_1}\int_S u\left(0,t_2,x\right)\varphi_2^\mu\left(t_2\right)\varphi_3^\mu\left(x\right)dP_2$$
$$+2\mathrm{C}_{\alpha_2,\lambda\mu}T^{1-\alpha_2}\int_S u\left(t_1,0,x\right)\varphi_1^\mu\left(t_1\right)\varphi_3^\mu\left(x\right)dP_1$$
$$\leq\left(\int_{C_T} h|u|^p\varphi^\mu dP\right)^{\frac{1}{p}}C\left(\varphi\right). \tag{24}$$

where

$$C\left(\varphi\right)=C\left(\int_Q h^{\frac{-1}{p-1}}|D^{\alpha_1}_{t_1|T}\varphi^\mu|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP\right.$$
$$\left.+\int_Q h^{\frac{-1}{p-1}}|D^{\alpha_2}_{t_2|T}\varphi^\mu|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP+\int_Q h^{\frac{-m}{p-1}}|(-\Delta)^{\frac{\alpha}{2}}\varphi|^{\frac{p}{p-m}}\varphi^{(\mu-1-\frac{m\mu}{p})\frac{p}{p-m}}dP\right).$$

Whereupon, using Lebesgue′s dominated convergence theorem we have,

$$\int_Q h|u|^p\varphi^\mu dP \le C \Longrightarrow \lim_{T\to+\infty}\int_{C_T} h|u|^p dP = 0$$

where $C_T = \{(t_1,t_2,x)\ /\ T \le t_1+t_2+|x| \le 2T\}$.
Then, letting T appraoch infinity in (24), the right-hand side approaches zero, which is again contradiction. □

Finally, we expand our results into system with a two-dimensional fractional time.

We consider

$$D^{\alpha_1}_{0|t_1}(u-u_2)+D^{\alpha_2}_{0|t_2}(u-u_1)+(-\Delta)^{\frac{\alpha}{2}}(|u|^m)=k_1|v|^q,\ k_1=t_1^{s_1}t_2^{l_1}|x|^{r_1} \tag{25}$$

$$D^{\beta_1}_{0|t_1}(v-v_2)+D^{\beta_2}_{0|t_2}(v-v_1)+(-\Delta)^{\frac{\beta}{2}}(|v|^n)=k_2|u|^p,\ k_2=t_1^{s_2}t_2^{l_2}|x|^{r_2} \tag{26}$$

posed for $(t_1,t_2,x)\in Q=\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^N$, and supplemented with the initial conditions

$$u(t_1,0,x)=u_1(t_1,x)\ ,u(0,t_2,x)=u_2(t_2,x) \tag{27}$$

$$v(t_1,0,x)=v_1(t_1,x)\ ,v(0,t_2,x)=v_2(t_2,x) \tag{28}$$

Here p,q are positive real numbers and $0<\alpha_1<\alpha_2<1$, $0<\beta_1<\beta_2<1$,$0<\alpha,\beta\le 2$.
Let us set

$$I_0=\int_Q u_2 D^{\alpha_1}_{t_1|T}\varphi dP+\int_Q u_1 D^{\alpha_2}_{t_2|T}\varphi dP$$

$$J_0=\int_Q v_2 D^{\beta_1}_{t_1|T}\varphi dP+\int_Q v_1 D^{\beta_2}_{t_2|T}\varphi dP$$

Definition 2: we say that $(u,v)\in(L^p\cap L^m)\times(L^q\cap L^n)$ is a weak formulation to system (25)-(26) if

$$\int_Q k_1|v|^q\varphi dP+I_0=\int_Q uD^{\alpha_1}_{t_1|T}\varphi dP+\int_Q uD^{\alpha_2}_{t_2|T}\varphi dP+\int_Q |u|^m(-\Delta)^{\frac{\alpha}{2}}\varphi dP.$$

$$\int_Q k_2|u|^p\varphi dP+J_0=\int_Q vD^{\beta_1}_{t_1|T}\varphi dP+\int_Q vD^{\beta_2}_{t_2|T}\varphi dP+\int_Q |v|^n(-\Delta)^{\frac{\beta}{2}}\varphi dP. \tag{29}$$

for any test function $\varphi\in C^\infty$. Now, set

$$\sigma_1=-\frac{q\left[(s_2+l_2+r_2)+\alpha_1 p+\beta_1-(N+2)p\right]+(s_1+l_1+r_1)+(N+2)}{pq-1}$$

$$\sigma_2=-\frac{q\left[(s_2+l_2+r_2)+\alpha_1 p+\beta_2-(N+2)p\right]+(s_1+l_1+r_1)+(N+2)}{pq-1}$$

$$\sigma_3=-\frac{q\left[(s_2+l_2+r_2)+\alpha_1 p+\beta-(N+2)p\right]+n(s_1+l_1+r_1)+n(N+2)}{pq-n}$$

$$\sigma_4=-\frac{q\left[(s_2+l_2+r_2)+\alpha_2 p+\beta_1-(N+2)p\right]+(s_1+l_1+r_1)+(N+2)}{pq-1}$$

$$\sigma_5 = -\frac{q\left[(s_2+l_2+r_2)+\alpha_2 p+\beta_2-(N+2)p\right]+(s_1+l_1+r_1)+(N+2)}{pq-1}$$

$$\sigma_6 = -\frac{q\left[(s_2+l_2+r_2)+\alpha_2 p+\beta-(N+2)p\right]+n(s_1+l_1+r_1)+n(N+2)}{pq-n}$$

$$\sigma_7 = -\frac{q\left[m(s_2+l_2+r_2)+\alpha p+m\beta_1-(N+2)p\right]+m(s_1+l_1+r_1)+m(N+2)}{pq-m}$$

$$\sigma_8 = -\frac{q\left[m(s_2+l_2+r_2)+\alpha p+m\beta_2-(N+2)p\right]+m(s_1+l_1+r_1)+m(N+2)}{pq-m}$$

$$\sigma_9 = -\frac{q\left[m(s_2+l_2+r_2)+\alpha p+m\beta-(N+2)p\right]+nm(s_1+l_1+r_1)+nm(N+2)}{pq-nm}$$

Theorem 2. Let $p>1, q>1, p>m, q>n$ and assume that

$$\int_Q u_2 D^{\alpha_1}_{t_1|T}\varphi^\mu dP>0,\quad \int_Q u_1 D^{\alpha_2}_{t_2|T}\varphi^\mu dP>0,$$

$$\int_Q v_2 D^{\beta_1}_{t_1|T}\varphi^\mu dP>0,\quad \int_Q v_1 D^{\beta_2}_{t_2|T}\varphi^\mu dP>0,$$

then solutions to system (25)-(26) blow-up whenever we have

$$\max\{\sigma_1,\cdots,\sigma_9,\delta_1,\cdots,\delta_9\}\leq 0.$$

Proof. Assume that the solution is nontrivial and global. Next, replacing φ by φ^μ in (29) and then using Hölders inequality to estimate the I_u and I_v (As we shall see later), we obtain the following estimates:

• For $p>1$

$$\int_Q uD^{\alpha_1}_{t_1|T}\varphi^\mu dP \leq \left(\int_Q k_2|u|^p\varphi^\mu dP\right)^{\frac{1}{p}} \times \left(\int_Q k_2^{\frac{-1}{p-1}}|D^{\alpha_1}_{t_1|T}\varphi^\mu|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP\right)^{\frac{p-1}{p}} \tag{30}$$

$$\int_Q uD^{\alpha_2}_{t_2|T}\varphi^\mu dP \leq \left(\int_Q k_2|u|^p\varphi^\mu dP\right)^{\frac{1}{p}} \times \left(\int_Q k_2^{\frac{-1}{p-1}}|D^{\alpha_2}_{t_2|T}\varphi^\mu|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP\right)^{\frac{p-1}{p}} \tag{31}$$

• For $p>m$

$$\int_Q |u|^m(-\Delta)^{\frac{\alpha}{2}}\varphi^\mu dP$$
$$\leq \mu\left(\int_Q k_2|u|^p\varphi^\mu dP\right)^{\frac{m}{p}} \times \left(\int_Q k_2^{\frac{-m}{p-m}}|(-\Delta)^{\frac{\alpha}{2}}\varphi|^{\frac{p}{p-m}}\varphi^{\mu-\frac{p}{p-m}}dP\right)^{\frac{p-m}{p}} \tag{32}$$

Similarly, we have
• For $q > 1$

$$\int_Q vD^{\beta_1}_{t_1|T}\varphi^\mu dP \le \left(\int_Q k_1|v|^q\varphi^\mu dP\right)^{\frac{1}{q}} \times \left(\int_Q k_1^{\frac{-1}{q-1}}|D^{\beta_1}_{t_1|T}\varphi^\mu|^{\frac{q}{q-1}}\varphi^{\frac{-\mu}{q-1}}dP\right)^{\frac{q-1}{q}} \quad (33)$$

$$\int_Q vD^{\beta_2}_{t_2|T}\varphi^\mu dP \le \left(\int_Q k_1|v|^q\varphi^\mu dP\right)^{\frac{1}{q}} \times \left(\int_Q k_1^{\frac{-1}{q-1}}|D^{\beta_2}_{t_2|T}\varphi^\mu|^{\frac{q}{q-1}}\varphi^{\frac{-\mu}{q-1}}dP\right)^{\frac{q-1}{q}} \quad (34)$$

• For $q > n$

$$\int_Q |v|^n\,(-\Delta)^{\frac{\beta}{2}}\,\varphi^\mu dP$$
$$\le \mu\left(\int_Q k_1|v|^q\varphi^\mu dP\right)^{\frac{n}{q}} \times \left(\int_Q k_1^{\frac{-n}{q-n}}|\,(-\Delta)^{\frac{\beta}{2}}\,\varphi|^{\frac{q}{q-n}}\varphi^{\mu-\frac{q}{q-n}}dP\right)^{\frac{q-n}{q}} \quad (35)$$

If we set

$$I_u = \int_Q k_2|u|^p\varphi^\mu dP\ ,\ I_v = \int_Q k_1|v|^q\varphi^\mu dP$$

$$A(p) = \left(\int_Q k_2^{\frac{-1}{p-1}}|D^{\alpha_1}_{t_1|T}\varphi^\mu|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP\right)^{\frac{p-1}{p}}$$

$$A(q) = \left(\int_Q k_1^{\frac{-1}{q-1}}|D^{\beta_1}_{t_1|T}\varphi^\mu|^{\frac{q}{q-1}}\varphi^{\frac{-\mu}{q-1}}dP\right)^{\frac{q-1}{q}}$$

$$B(p) = \left(\int_Q k_2^{\frac{-1}{p-1}}|D^{\alpha_2}_{t_2|T}\varphi^\mu|^{\frac{p}{p-1}}\varphi^{\frac{-\mu}{p-1}}dP\right)^{\frac{p-1}{p}}$$

$$B(q) = \left(\int_Q k_1^{\frac{-1}{q-1}}|D^{\beta_2}_{t_2|T}\varphi^\mu|^{\frac{q}{q-1}}\varphi^{\frac{-\mu}{q-1}}dP\right)^{\frac{q-1}{q}}$$

$$C(p,m) = \mu\left(\int_Q k_2^{\frac{-m}{p-m}}|\,(-\Delta)^{\frac{\alpha}{2}}\,\varphi|^{\frac{p}{p-m}}\varphi^{\mu-\frac{p}{p-m}}dP\right)^{\frac{p-m}{p}}$$

$$C(q,n) = \mu\left(\int_Q k_1^{\frac{-n}{q-n}}|\,(-\Delta)^{\frac{\beta}{2}}\,\varphi|^{\frac{q}{q-n}}\varphi^{\mu-\frac{q}{q-n}}dP\right)^{\frac{q-n}{q}}$$

$$I_0^\mu = \int_Q u_2D^{\alpha_1}_{t_1|T}\varphi^\mu dP + \int_Q u_1D^{\alpha_2}_{t_2|T}\varphi^\mu dP$$

$$J_0^\mu = \int_Q v_2D^{\beta_1}_{t_1|T}\varphi^\mu dP + \int_Q v_1D^{\beta_2}_{t_2|T}\varphi^\mu dP$$

Then, using estimates (30)-(35), we can write (29) as

$$I_v + I_0^\mu \leq I_u^{\frac{1}{p}} A(p) + I_u^{\frac{1}{p}} B(p) + I_u^{\frac{m}{p}} C(p,m)$$
$$I_u + J_0^\mu \leq I_v^{\frac{1}{q}} A(q) + I_v^{\frac{1}{q}} B(q) + I_v^{\frac{n}{q}} C(q,n)$$

Since $I_0^\mu, J_0^\mu > 0$ we have

$$I_v \leq I_u^{\frac{1}{p}} A(p) + I_u^{\frac{1}{p}} B(p) + I_u^{\frac{m}{p}} C(p,m) \tag{36}$$
$$I_u \leq I_v^{\frac{1}{q}} A(q) + I_v^{\frac{1}{q}} B(q) + I_v^{\frac{n}{q}} C(q,n) \tag{37}$$

Now, from (36) and (37), we have

$$I_v + I_0^\mu \leq \left(I_v^{\frac{1}{pq}} A^{\frac{1}{p}}(q) + I_v^{\frac{1}{pq}} B^{\frac{1}{p}}(q) + I_v^{\frac{n}{pq}} C^{\frac{1}{p}}(q,n) \right) A(p)$$

$$+ \left(I_v^{\frac{1}{pq}} A^{\frac{1}{p}}(q) + I_v^{\frac{1}{pq}} B^{\frac{1}{p}}(q) + I_v^{\frac{n}{pq}} C^{\frac{1}{p}}(q,n) \right) B(p)$$

$$+ \left(I_v^{\frac{m}{pq}} A^{\frac{m}{p}}(q) + I_v^{\frac{m}{pq}} B^{\frac{m}{p}}(q) + I_v^{\frac{mn}{pq}} C^{\frac{m}{p}}(q,n) \right) C(p,m)$$

Then Youngs inequality implies

$$I_v + I_0^\mu \leq K \Bigg[\left(A^{\frac{1}{p}}(q) A(p) \right)^{\frac{pq}{pq-1}} + \left(B^{\frac{1}{p}}(q) A(p) \right)^{\frac{pq}{pq-1}} + \left(C^{\frac{1}{p}}(q,n) A(p) \right)^{\frac{pq}{pq-n}}$$

$$+ \left(A^{\frac{1}{p}}(q) B(p) \right)^{\frac{pq}{pq-1}} + \left(B^{\frac{1}{p}}(q) B(p) \right)^{\frac{pq}{pq-1}} + \left(C^{\frac{1}{p}}(q,n) B(p) \right)^{\frac{pq}{pq-n}}$$

$$+ \left(A^{\frac{m}{p}}(q) C(p,m) \right)^{\frac{pq}{pq-m}} + \left(B^{\frac{m}{p}}(q) C(p,m) \right)^{\frac{pq}{pq-m}} + \left(C^{\frac{m}{p}}(q,n) C(p,m) \right)^{\frac{pq}{pq-nm}} \Bigg]$$

For some positive constant K. Using the scaled variables (10) we obtain

$$A(p) = CT^{-\frac{1}{p}(s_2+l_2+r_2)-\alpha_1+(N+2)(1-\frac{1}{p})}$$

$$A(q) = CT^{-\frac{1}{q}(s_1+l_1+r_1)-\beta_1+(N+2)(1-\frac{1}{q})}$$

$$B(p) = CT^{-\frac{1}{p}(s_2+l_2+r_2)-\alpha_2+(N+2)(1-\frac{1}{p})}$$

$$B(q) = CT^{-\frac{1}{q}(s_1+l_1+r_1)-\beta_2+(N+2)(1-\frac{1}{q})}$$

$$C(p,m) = CT^{-\frac{m}{p}(s_2+l_2+r_2)-\alpha+(N+2)(1-\frac{m}{p})}$$

$$C(q,n) = CT^{-\frac{n}{q}(s_1+l_1+r_1)-\beta+(N+2)(1-\frac{n}{q})}$$

For some positive constant C. Hence, we obtain

$$I_v + I_0^{\mu} \le K\left[T^{\sigma_1} + T^{\sigma_2} + \cdots + T^{\sigma_9}\right]. \tag{38}$$

Similarly, we obtain for I_u the estimate

$$I_u + J_0^{\mu} \le K\left[T^{\delta_1} + T^{\delta_2} + \cdots + T^{\delta_9}\right]. \tag{39}$$

Finally, passing to the limit as $T \to +\infty$, we observe that:
Either $\max\{\sigma_1, \cdots, \sigma_9, \delta_1, \cdots, \delta_9\} < 0$ and in this case, the right hand side tends to zero while the left hand side is strictly positive. Hence, we obtain a contradiction. Or $\max\{\sigma_1, \cdots, \sigma_9, \delta_1, \cdots, \delta_9\} = 0$ and in this case, following the analysis similar as in one equation, we prove a contradiction. □

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