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NUMERICAL SOLUTION OF FRACTIONAL ORDER IMMUNOLOGY AND AIDS MODEL VIA LAPLACE TRANSFORM ADOMIAN DECOMPOSITION METHOD

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Abstract. AIDS is one of the major causes of health problems all over the world. In this article the dynamics of Immunology and AIDS model of fractional order is considered. With the help of Laplace transform coupled with the Adomain decomposition method, we develop an analytical scheme to obtain numerical solution for the considered model. The convergent of the series solution is also provided to demonstrate the procedure. Also taking specific values for the parameters involved in the model, the corresponding plots against different order of the differentiations are plotted. Further more the concerned results are compared with that of their fractional order.

1. Introduction

Immune system is the host defense against threats of invasion by pathogens such as microorganisms. All substances that are identified as foreign and particularly, if they are perceived as dangerous stimulate an immune response, which comprise of a series of integrated and complex biological processes within the organism. The specific immune response of the body then attempts to eradicate invaders. Further, malfunctions or failure of the immune system to respond may be lethal, as seen in HIV/AIDS epidemics. For normal functioning of the body, the immune system must detect a range of pathogens and distinguish them from the organism's own healthy tissues, destroy and keep a record of the invading pathogens. These foreign substances or invaders to which the immune system responds are known as antigens. In most cases the antigens are destroyed either by proteins referred to as antibodies, or through cytotoxic or killer cells. Both are modified lymphocytes or white blood cells and are particular to the pathogen encountered. Given the range of invading organisms no doubt, the recognition of invaders pose a serious problem to the immune system. Further, pathogens evolve and adapt rapidly, and thereby avoid detection and neutralization by the immune system. Nonetheless, strains of viruses may recombine to produce more virulent strains, as was the case seen in the 1918 influenza pandemic. High mutation rate of the pathogens like the

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influenza virus, which mutates so quickly that they are not recognizable on the second encounter and another defense system has to be mounted against them. Still, multiple defense mechanisms have also evolved and in organisms like humans, have more sophisticated defense mechanisms, including the ability to adapt over time to recognize specific pathogens. This adaptive immunity creates immunological memory after an initial response to a specific pathogen, leading to an enhanced response to subsequent encounters with that same pathogen. This ability of an antibody to recognize closely related antigens provides the basis for vaccine development. The area of the mathematical models of infectious diseases is an active area of research which was initiated by D. Bernoulli in 1766. Now the area is become very vast and plenty of papers, books, monographs have been written on this area, for detail see[1, 2, 3, 4, 5]. The classical order model of AIDS is given by

$$\begin{cases} V'(t) = \alpha Y(t) - \beta V(t), \\ X'(t) = c - dX(t) - \gamma X(t)V(t), \\ Y'(t) = \gamma X(t)V(t) - \delta Y(t), \end{cases} \tag{1}$$

with given initial condition $V(0) = A_0,\ X(0) = B_0,\ Y(0) = C_0$. Where V be the number of virion in an organism and X be the number of uninfected target cells and Y the number of infected cells. Further, β is the death rate of virus, the production rate of uninfected cell is c which die at specific rate d and become infected at a rate γV and Infected cell die at a specific rate $\delta = \gamma + d$, where γ is the additional death rate due to the infection and d is the natural death rate. The detailed analysis of the above model was carried out in [6].

In last few decades, it has been proved that derivatives and integral of non integer order provided excellent tools for mathematical modeling of infectious disease as compared to classical order derivatives and integrals. Because, the models of fractional order derivatives are more realistic and accurate as compared to classical order models, see for detail [7, 8, 9, 10, 11].

Motivated by the above background, in this article, we consider the following fractional order extension of model (1). The fractional order show the realistic biphasic decline behavior of infection of disease with a slow rate. Thus the new fractional model is given by

$$\begin{cases} {}^cD^qV(t) = \alpha Y(t) - \beta V(t), \\ {}^cD^qX(t) = c - dX(t) - \gamma X(t)V(t), \\ {}^cD^qY(t) = \gamma X(t)V(t) - \delta Y(t), \end{cases} \tag{2}$$

with given initial condition $V(0) = A_0, X(0) = B_0, Y(0) = C_0$, where ${}^cD^q$, for $0 < q < 1$ is the Caputo's derivative of fractional order q show fractional order derivative.

In model (2) the initial conditions are independent on each other and satisfy the relation $N(0) = V(t) + X(t) + Y(t)$ where N is the total number of the individuals in the population.

Recently, the area devoted to the numerical solutions of fractional and classical order model is an active area of research. The concerned solutions are obtained via using various technique like Homotopy analysis method(HAM), Homotopy perturbation method(HPM), Variation iteration method(VIM), etc, see [12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. In 1980, Adomian decomposition method

(ADM) was introduced by Adomian, which is an effective method for finding numerical and explicit solution of a wide class of differential equations representing physical problems. This method works efficiently for both initial value problems as well as for boundary value problem, for partial and ordinary differential equations, for linear and non-linear equations and also for stochastic system as well. In this method no perturbation or liberalization is required. ADM has been done extensive work to provide analytical solution of nonlinear classical differential as well as solving fractional order differential equations. In this paper we operate Laplace transform, which is a powerful techniques in engineering and applied mathematics. Adomain decomposition method coupled with Laplace transform is an powerful tools to obtained approximate solutions to mathematical models of both classical and fractional order, for detail see [22, 23, 24]. Therefore, in this paper, we solved the considered model (2) by using Adomian decomposition method coupled with Laplace transform known as Laplace Adomain Decomposition Method (LADM). For the verification of our procedure results, we assigned random values to the initial conditions and parameters.
With the help of this method, we transform fractional differential equations into algebraic equations, then solved this algebraic equations by ADM. The convergence of the proposed method is also provided by extending the idea discusses in [25, 26].

2. Preliminaries

In this portion of research work authors presented some basic definition and results from fractional calculus, for further study, we recommend [13, 14, 15, 16, 25, 26].

Definition 2.1. *"The fractional integral of Riemann-Liouville type of order $q \in (0,1)$ of a function $v \in L^1([0,\infty),\mathbb{R})$" is defined as*

$$I_{0+}^{q} v(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} v(s)\, ds,$$

"provided that integral on the right is point wise defined on $(0,\infty)$".

Definition 2.2. *"The Caputo fractional order derivative of a function v on the interval $[0,\infty)$" is defined by*

$${}^{c}D_{0+}^{q} v(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} v^{(n)}(s)\, ds,$$

where $n = [q]+1$ and $[q]$ represents the integer part of q.

Lemma 2.3. *The following result holds for "fractional differential equations*

$$I^{\alpha}[{}^{c}D^{q} v](t) = v(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

for arbitrary $c_i \in \mathbb{R}$, $\quad i = 0, 1, 2, ..., n-1$, where $n = [q]+1$ and $[q]$ represents the integer part of q".

Definition 2.4. *We recall the definition of "Laplace transform of Caputo derivative as:*

$$\mathcal{L}\{{}^{c}D^{q} v(t)\} = s^{q} v(s) - \sum_{k=0}^{n-1} s^{q-k-1} v^{(k)}(0),\ n-1 < q < n, n \in N.$$

where $n = [q]+1$ and $[q]$ represents the integer part of q".

3. The Laplace Adomian Decomposition Method

In this section, we discuss the general procedure of the model (2) with given initial conditions. Applying Laplace transform on both side of the model (2) as

$$\begin{cases} \mathcal{L}\{{}^cD^qV(t)\} = \mathcal{L}\{\alpha Y(t) - \beta V(t)\}, \\ \mathcal{L}\{{}^cD^qX(t)\} = \mathcal{L}\{c - dX(t) - \gamma X(t)V(t)\}, \\ \mathcal{L}\{{}^cD^qY(t)\} = \mathcal{L}\{\gamma X(t)V(t) - \delta Y(t)\}, \end{cases} \tag{3}$$

which implies that

$$\begin{cases} s^q\mathcal{L}\{V(t)\} - s^{q-1}V(0) = \mathcal{L}\{\alpha Y(t) - \beta V(t)\}, \\ s^q\mathcal{L}\{X(t)\} - s^{q-1}X(0) = \mathcal{L}\{c - dX(t) - \gamma X(t)V(t)\}, \\ s^q\mathcal{L}\{Y(t)\} - s^{q-1} = \mathcal{L}\{\gamma X(t)V(t) - \delta Y(t)\}, \end{cases} \tag{4}$$

Now using initial conditions and taking inverse inverse Laplace transform to system (4), we have

$$\begin{cases} V(t) = A_0 + \mathcal{L}^{-1}\left[\dfrac{1}{s^q}\mathcal{L}\{\alpha Y(t) - \beta V(t)\}\right], \\ X(t) = B_0 + \mathcal{L}^{-1}\left[\dfrac{1}{s^q}\mathcal{L}\{c - dX(t) - \gamma X(t)V(t)\}\right], \\ Y(t) = C_0 + \mathcal{L}^{-1}\left[\dfrac{1}{s^q}\mathcal{L}\{\gamma X(t)V(t) - \delta Y(t)\}\right]. \end{cases} \tag{5}$$

Assuming that the solutions, $V(t), X(t), Y(t)$ in the form of infinite series given by

$$V(t) = \sum_{i=0}^{\infty}\varepsilon^i V_i,\ X(t) = \sum_{i=0}^{\infty}\varepsilon^i X_i,\ Y(t) = \sum_{i=0}^{\infty}\varepsilon^i Y_i. \tag{6}$$

and the nonlinear terms involved in the model are $X(t)V(t)$ are decompose by Adomian polynomial as

$$X(t)V(t) = \sum_{i=0}^{\infty}P_n. \tag{7}$$

Where P_n are Adomian polynomials defined as

$$P_n = \frac{1}{\Gamma(n+1)}\frac{d^n}{d\eta^n}\left[\sum_{i=0}^{n}\eta^i x_i \sum_{i=0}^{n}\eta^i v_i\right]|_{\eta=0}, \tag{8}$$

using (6),(7) in model (5), we get

$$
\begin{aligned}
&\mathcal{L}(V_0) = \frac{A_0}{s},\ \mathcal{L}(X_0) = \frac{B_0}{s},\ \mathcal{L}(Y_0) = \frac{C_0}{s},\\
&V_1(t) = \mathcal{L}^{-1}\left[\frac{1}{s^q}\mathcal{L}\{\alpha Y_0(t) - \beta V_0(t)\}\right],\\
&X_1(t) = \mathcal{L}^{-1}\left[\frac{1}{s^q}\mathcal{L}\{c - dX_0(t) - \gamma P_0(t)\}\right],\\
&Y_1(t) = \mathcal{L}^{-1}\left[\frac{1}{s^q}\mathcal{L}\{\gamma P_0 - \delta Y_0(t)\}\right],\\
&V_2(t) = \mathcal{L}^{-1}\left[\frac{1}{s^q}\mathcal{L}\{\alpha Y_1(t) - \beta V_1(t)\}\right],\\
&X_2(t) = \mathcal{L}^{-1}\left[\frac{1}{s^q}\mathcal{L}\{-dX_1(t) - \gamma P_1(t)\}\right],\\
&Y_2(t) = \mathcal{L}^{-1}\left[\frac{1}{s^q}\mathcal{L}\{\gamma P_1 - \delta Y_1(t)\}\right],\\
&V_3(t) = \mathcal{L}^{-1}\left[\frac{1}{s^q}\mathcal{L}\{\alpha Y_2(t) - \beta V_2(t)\}\right],\\
&X_3(t) = \mathcal{L}^{-1}\left[\frac{1}{s^q}\mathcal{L}\{-dX_2(t) - \gamma P_2(t)\}\right],\\
&Y_3(t) = \mathcal{L}^{-1}\left[\frac{1}{s^q}\mathcal{L}\{\gamma P_2(t) - \delta Y_2(t)\}\right],\\
&\vdots\\
&V_n(t) = \mathcal{L}^{-1}\left[\frac{1}{s^q}\mathcal{L}\{\alpha Y_{n-1}(t) - \beta V_{n-1}(t)\}\right],\\
&X_n(t) = \mathcal{L}^{-1}\left[\frac{1}{s^q}\mathcal{L}\{-dX_{n-1}(t) - \gamma P_{n-1}(t)\}\right],\\
&Y_n(t) = \mathcal{L}^{-1}\left[\frac{1}{s^q}\mathcal{L}\{\gamma P_{n-1}(t) - \delta Y_{n-1}(t)\}\right],\ n \geq 0.
\end{aligned}
\tag{9}
$$

After using the inverse Laplace transform in (9), we have

$$
\begin{aligned}
&V_0 = A_0,\ X_0 = B_0,\ Y_0 = C_0,\ P_0 = B_0A_0,\\
&V_1 = \frac{(\alpha C_0 - \beta A_0)t^q}{\Gamma(q+1)},\ X_1 = \frac{(c - dB_0 - \gamma P_0)t^q}{\Gamma(q+1)},\ Y_1 = \frac{(\gamma P_0 - \delta C_0)t^q}{\Gamma(q+1)},\\
&V_2 = \left[\alpha(\gamma P_0 - \delta C_0) - \beta(\alpha C_0 - \beta A_0)\right]\frac{t^{2q}}{\Gamma(2q+1)},\\
&X_2 = -\left[(\gamma A_0 + d)(c - dB_0 - \gamma P_0) + \gamma B_0(\alpha C_0 - \beta A_0)\right]\frac{t^{2q}}{\Gamma(2q+1)},\\
&Y_2 = \left[\gamma B_0(\alpha C_0 - \beta A_0) + \gamma A_0(c - dB_0 - \gamma P_0) - \delta(\gamma P_0 - \delta C_0)\right]\frac{t^{2q}}{\Gamma(2q+1)},\\
&V_3 = \Big[\alpha\gamma B_0(\alpha C_0 - \beta A_0) + \alpha\gamma A_0(c - dB_0 - P_0\gamma) - \alpha\delta(P_0\gamma - C_0\delta)\\
&- \beta\alpha(P_0\gamma - C_0\delta) + \beta^2(\alpha C_0 - \beta A_0)\Big]\frac{t^{3q}}{\Gamma(3q+1)},\\
&X_3 = \Big[d^2(c - dB_0 - P_0\gamma) + d\gamma B_0(\alpha C_0 - \beta A_0) + d\gamma A_0(c - dB_0 - \gamma P_0)\\
&- \gamma\alpha B_0(\gamma P_0 - \delta C_0) + \gamma\beta B_0(\alpha C_0 - \beta A_0)\\
&- \frac{\gamma(c - dB_0 - \gamma P_0)(\alpha C_0 - \beta A_0)\Gamma(2q+1)}{(\Gamma(q+1))^2}\\
&+ \gamma A_0 d(c - dB_0 + \gamma P_0) + \gamma^2 A_0 B_0(\alpha C_0 - \beta A_0) + \gamma^2 A_0^2(c - dB_0 - \gamma P_0)\Big]\frac{t^{3q}}{\Gamma(3q+1)},\\
&Y_3 = \Big[\gamma\alpha B_0(\gamma P_0 - \delta C_0) - \gamma\beta B_0(\alpha C_0 - \beta A_0)\\
&+ \frac{\gamma(c - dB_0 - \gamma P_0)(\alpha C_0 - \beta A_0)\Gamma(2q+1)}{(\Gamma(q+1))^2}\\
&- \gamma A_0 d(c - dB_0 - \gamma P_0) - \gamma^2 A_0 B_0(\alpha C_0 - \beta A_0) - \gamma^2 A_0^2(c - dB_0 - \gamma P_0)\\
&- \delta\gamma B_0(\alpha C_0 - \beta A_0) - \delta\gamma A_0(c - dB_0 - \gamma P_0) + \gamma\delta^2(\gamma P_0 - deltaC_0)\Big]\frac{t^{3q}}{\Gamma(3q+1)}.
\end{aligned}
\tag{10}
$$

On the similar way, we can obtain the remaining terms and finally, we get the solution in the form of infinite series as given by

$$
\begin{aligned}
&V(t) = V_0 + V_1 + V_2 + V_3 + \ldots,\ X(t) = X_0 + X_1 + X_2 + X_3 + \ldots,\\
&Y(t) = Y_0 + Y_1 + Y_2 + Y_3 + \ldots.
\end{aligned}
\tag{11}
$$

4. Numerical Simulations and convergence of the method

By using the following value

$$A_0 = 5, B_0 = 10, C_0 = 20, c = 0.6, d = 0.7, \alpha = 0.2, \beta = 0.3, \gamma = 0.4, \delta = 0.5,$$

and after simplification up-to three terms while using few Adomain polynomials for easiness, we get

$$V_1 = \frac{2.5t^q}{\Gamma(q+1)},\ X_1 = -\frac{26.4t^q}{\Gamma(q+1)},\ Y_1 = \frac{10t^q}{\Gamma(q+1)}, V_2 = \frac{1.25t^{2q}}{\Gamma(2q+1)},\ X_2 = \frac{61.28t^{2q}}{\Gamma(2q+1)},$$
$$Y_2 = -\frac{44.800t^{2q}}{\Gamma(2q+1)},\ V_3 = -\frac{9.935t^{3q}}{\Gamma(3q+1)},\ X_3 = \left\{ -170.456 + \frac{26.400\Gamma(2q+1)}{(\Gamma(q+1))^2} \right\} \frac{t^{3q}}{\Gamma(3q+1)},$$
$$Y_3 = \left\{ 149.96 - \frac{26.4\Gamma(2q+1)}{(\Gamma(q+1))^2} \right\} \frac{t^{3q}}{\Gamma(3q+1)}. \tag{12}$$

Thus the solution after three terms becomes

$$V(t) = 5 + \frac{2.5t^q}{\Gamma(q+1)} + \frac{1.25t^{2q}}{\Gamma(2q+1)} - \frac{9.935t^{3q}}{\Gamma(3q+1)},$$
$$X(t) = 10 - \frac{26.4t^q}{\Gamma(q+1)} + \frac{61.28t^{2q}}{\Gamma(2q+1)} + \left\{ -170.456 + \frac{26.400\Gamma(2q+1)}{(\Gamma(q+1))^2} \right\} \frac{t^{3q}}{\Gamma(3q+1)},$$
$$Y(t) = 20 + \frac{10t^q}{\Gamma(q+1)} - \frac{44.800t^{2q}}{\Gamma(2q+1)} + \left\{ 149.96 - \frac{26.4\Gamma(2q+1)}{(\Gamma(q+1))^2} \right\} \frac{t^{3q}}{\Gamma(3q+1)}.$$

We plot by Mathematica the series solution (11) upto first hyndered terms. In numerical plot it is to be noted that for all three Figure $1, 2, 3$, the top curve is against classical order that is $q = 1$, the second curve is $q = 0.9$, and third curve against $q = 0.8$, the lowermost curve has been plotted against $q = 0.7$.

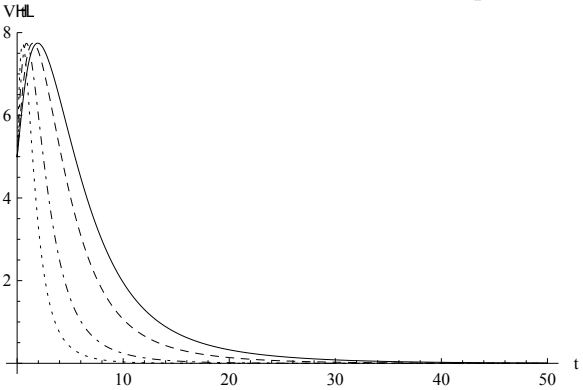

Fig. (1) Numerical plots of $V(t)$ at $q = 0.7, 0.8, 0.9, 1$ using $LADM$ method.

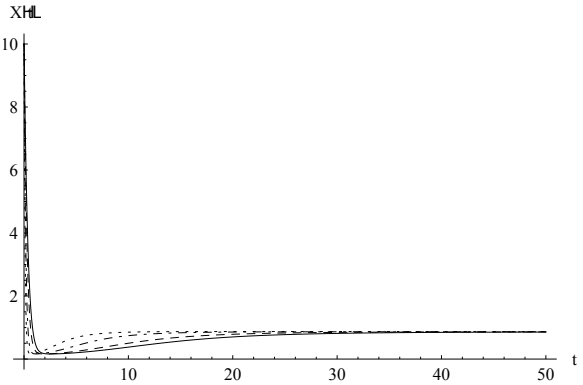

Fig. (2) Numerical plots of $X(t)$ at $q = 0.7, 0.8, 0.9, 1$ using $LADM$ method.

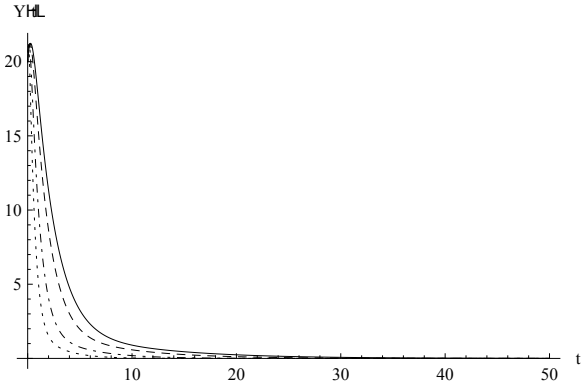

Fig. (3) Numerical plots of $Y(t)$ at $q = 0.7, 0.8, 0.9, 1$ using $LADM$ method.

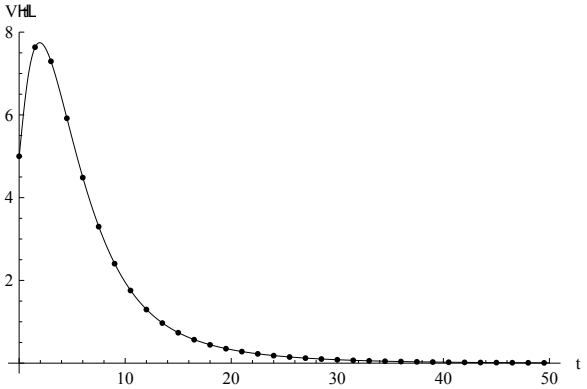

Fig. (4) Comparison of solutions of $V(t)$ using $LADM$ =Solid curve and $RK4 = dots$ at $q = 1$.

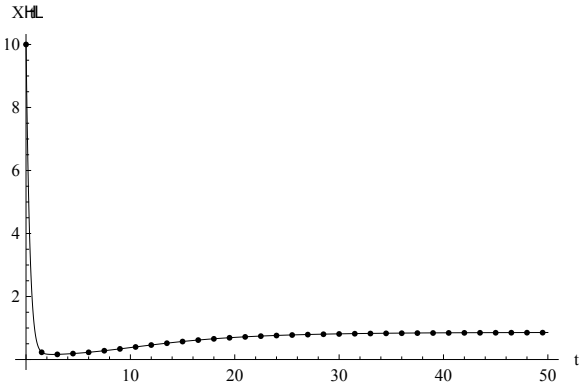

Fig. (5)Comparison of solutions of $X(t)$ using $LADM$ =Solid curve and $RK4 = dots$ at $q = 1$.

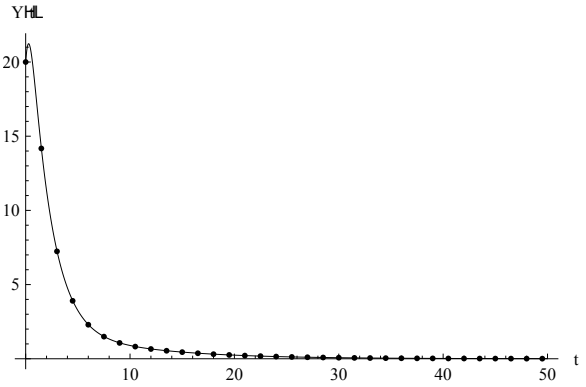

Fig. (6) Comparison of solutions of V(t) using $LADM$ =Solid curve and $RK4 = dots$ at $q = 1$.

From the Figure 1 to 3, we see that fractional order plays important role in the numerical solutions of the proposed model, lesser the order fastest the decaying or growing of the cell and vice versa. From Figure 1, we see that smaller the order of the differentiation, lesser the time interval in which virion cells are vanishing and vice versa. Similar behavior, one can see for other cells. The concerned model has been plotted at $q = 1$ as shown in Figures $4, 5, 6$, by using $RK4$ method. It is obvious that the solutions of both methods are coincided for classical order.

5. Convergence Analysis

The above series solution is a series, which is rapidly convergent series and converge uniformly to the exact solution. To check the convergence of the series (10), we use classical techniques, to give sufficient conditions of convergence of this method

Theorem 5.1. *Let $\mathcal{E}$ be the Banach space and $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ be a contractive nonlinear operator such that for all $w, w^{'} \in \mathcal{E}, ||\mathcal{F}(w) - \mathcal{F}(w^{'})|| \leq k||w - w^{'}||, 0 < k < 1$. Then by Banach contraction principle T has a unique point w such that $\mathcal{F}w = w$, where $w = (v, x, y)$. The series given in* (10) *can be written by applying Adomian Decomposition method as:*

$$w_n = \mathcal{F}w_{n-1}, w_{n-1} = \sum_{i=1}^{n-1} w_i, n = 1, 2, 3, \ldots,$$

and assume that $w_0 = w_0 \in S_r(w)$ where $S_r(w) = \{w^{'} \in X : ||w^{'} - w|| < r\}$, then, we have

$$(i)\ w_n \in S_r(w);$$
$$(ii)\ \lim_{n\to\infty} w_n = w.$$

Proof. For (i), using mathematical induction for $n = 1$, we have

$$||w_0 - w|| = ||\mathcal{F}(w_0) - \mathcal{F}(w)|| \leq k||w_0 - w||.$$

Let the result is true for $n - 1$, then

$$||w_0 - w|| \leq k^{n-1}||w_0 - w||.$$

we have

$$||w_n - w|| = ||\mathcal{F}(w_{n-1}) - \mathcal{F}(w)|| \leq k||w_{n-1} - w|| \leq k^n||w_0 - w||.$$

Hence using (i) we, have

$$||w_n - w|| \leq k^n ||w_0 - w|| \leq k^n r < r$$

which implies that $w_n \in S_r(w)$.
(ii) Since $||w_n - w|| \leq k^n ||w_0 - w||$ and as $\lim_{n\to\infty} k^n = 0$.
So, we have $\lim_{n\to\infty} ||w_n - w|| = 0 \Rightarrow \lim_{n\to\infty} w_n = w$. □

6. Conclusion

We have successfully developed a scheme for numerical solutions of Immunology and AIDS model of fractional order by using a method coupled of Laplace transform and Adomain decomposition method. The solutions obtained via this method is closely agree to that obtained by other method like $RK4$ method shown in Figure $4, 5, 6$. The method is efficient for the solutions of such type of problems. Also the convergence analysis has been provided to demonstrate the efficiency of the method.

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