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NOTES ON THE FRACTIONAL TAYLOR'S FORMULA

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Abstract. A new ideal formula for the generalization of the classic Taylor's formula have been described. The fractional derivatives are introduced by a new Caputo like-fractional derivative. Furthermore, some interesting examples are pointed out.

1. Introduction

In recent decades the power series expansion has been widely used in computational science obtaining an easy approximate of a function [1], numerical schemes to integrate a Cauchy problem [3], or gaining knowledge about the singularities of a function by comparing two different Taylor series expansions around different points [4]. This is a fundamental tool to linearize a problem which guaranties easy analysis. Recently, the Fractional calculus has become an important part of various sciences such as mathematics, engineering and physics [2]. It has emerged as an effective and powerful tool for the mathematical modeling of several engineering and scientific phenomena. Therefore, it is necessary to have some mathematical apparatus and tools in order to understand this concept. Also, the fractional power series is becoming an essential tool in the study of elementary functions, particularly in the fractional calculus approach. In the context of the fractional derivatives, Taylor series has been developed for different definitions [5-6]. Recently, many authors [7-11] presented a generalization of the classical Taylor's formula, but in general they consist on a series in powers of $x^{n+\alpha}$which in fact is not a purely fractional series. Others consist on a series in powers of $x^{n\alpha}$,which also in fact is not a purely true. In the following fractional Taylor formula

$$f(x)=\sum_{k=0}^{n}\frac{(D^{k\alpha}f)(x_0)}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}+\frac{(D^{(n+1)\alpha}f)(\xi)}{\Gamma((n+1)\alpha+1)}(x-x_0)^{(n+1)\alpha}$$

$$;\ k=0,\,1,\,...,n\,;\ 0<\alpha\le 1\,;\,a<x_0<\xi<x<b\,\forall x\in(a,\,b)$$

i.e.
$$\begin{aligned}f(x)&=f(x_0)+\tfrac{(x-x_0)^{\alpha}}{\Gamma(\alpha+1)}D^{\alpha}f(x_0)+\tfrac{(x-x_0)^{2\alpha}}{\Gamma(2\alpha+1)}D^{2\alpha}f(x_0)+\tfrac{(x-x_0)^{3\alpha}}{\Gamma(3\alpha+1)}D^{3\alpha}f(x_0)\\&+\tfrac{(x-x_0)^{4\alpha}}{\Gamma(4\alpha+1)}D^{4\alpha}f(x_0)+\tfrac{(x-x_0)^{5\alpha}}{\Gamma(5\alpha+1)}D^{5\alpha}f(x_0)+...\end{aligned}$$

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which is extensively studied, we find that order of fractional derivatives in all terms as0, α, 2α, 3α, 4α, 5α, ..., i.e

$$0,\ (0,\ 1],\ (0,\ 2],\ (0,\ 3], (0,\ 4],\ (0,\ 5],\ ....$$

Hence, we note that all order of fractional derivatives are repeated. i.e. all order of fractional derivatives are repeated in the second term $n-1$ times, in the third term $n-2$ times, in the fourth term $n-3$ times, $\ldots$ and so on.
In the present work a similar study has been made for Caputo fractional derivative defined as

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha}\, Df(t)dt = I_a^{1-\alpha} Df(x), \tag{1}$$

where the fractional integral I_a^{α} is defined as

$$I_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1}\, f(t)dt \tag{2}$$

The use of this definition of the fractional derivative is justified since it has good physical properties [2] as, for example that the derivative of a constant is zero or that Cauchy problems requires initial conditions formulated in terms of integer order derivatives interpreted as initial position, initial velocity, etc.
In order to develop a generalization of fractional Taylor's formula it is necessary to give some interesting new results including definitions and properties of the Caputo fractional derivative.
As it is well-known, elementary manipulations with entire order derivatives as the summation of index, Leibniz rule or the chain rule are not valid for Caputo fractional derivative as also happen with Riemann-Liouville definition. Taking this into account, and with no possibility of confusion it will be used the following convection for the sequential derivative in order to simplify notation

$$D_a^{\alpha_k}, k = 0,\ 1,\ 2,\ ...,\ n;\ \ k-1 < \alpha_k \leq k, \alpha_0 = 0$$

Definition 1. We define the Caputo like fractional derivative of $f(x)$of order $\alpha_n > 0$with $a \geq 0$, as

$$D_a^{\alpha_n} f(x) = I_a^{n-\alpha_n} D^n f(x) = \frac{1}{\Gamma(n-\alpha_n)} \int_a^x (x-t)^{n-\alpha_n-1} (D^n f(t))dt \tag{3}$$

with $n-1 < \alpha_n \leq n\,,\ n \in \mathrm{N},\ x \geq a$.
Also,

$$D_a^{\alpha_n} f(x) = \frac{1}{\Gamma(1-\alpha_n)} \int_a^x (x-t)^{-\alpha_n} (Df(t))dt = I_a^{1-\alpha_n} Df(x) \tag{4}$$

where the fractional integral $I_a^{\alpha_n}$ is defined as

$$I_a^{\alpha_n} f(x) = \frac{1}{\Gamma(\alpha_n)} \int_a^x (x-t)^{\alpha_n-1}\, f(t)dt \tag{5}$$

Proposition 1. Let $\alpha_n,\ n-1 < \alpha_n \leq n$ and $f(x) \in C_{\alpha_n}^n$, then we have

$$I_a^{\alpha_n} D_a^{\alpha_n} f(x) = f(x) - f(a) \tag{6}$$

Applying (4), we have

$$I_a^{\alpha_n} D_a^{\alpha_n} f(x) = I_a^1 Df(x)$$

Then (6) obtained directly.

Proposition 2. Let $\alpha_n,\ n-1<\alpha_n\le n$ and $f(x)\in C^n_{\alpha_n}$, then we have

$$f(x)=f(a)+\frac{(x-a)^{\alpha_n}}{\Gamma(\alpha_n+1)}D_a^{\alpha_n}f(\xi) \tag{7}$$

for any $\xi\in(a,\,x)$, being $D_a^{\alpha_n}f(x)$ continuous in $[a,\,x]$.
From (5), we have

$$I_a^{\alpha_n}D_a^{\alpha_n}f(x)=\frac{1}{\Gamma(\alpha_n)}\int_a^x(x-t)^{\alpha_n-1}\,D_a^{\alpha_n}f(t)dt \tag{8}$$

Applying the mean value theorem for the integral, we have

$$I_a^{\alpha_n}D_a^{\alpha_n}f(x)=\frac{D_a^{\alpha_n}f(\xi)}{\Gamma(\alpha_n)}\int_a^x(x-t)^{\alpha_n-1}\,dt=\frac{D_a^{\alpha_n}f(\xi)}{\Gamma(\alpha_n+1)}(x-a)^{\alpha_n} \tag{9}$$

for any $\xi\in(a,\,x)$. From (6), then we have the result.
Definition 2. Let $\alpha_n,\ n-1<\alpha_n\le n$ and $f(x)\in C^n_{\alpha_n}$, we can defined

$$I_a^{\alpha_n}D_a^{\alpha_n}f(x)=f(x)-\sum_{i=0}^{n-1}(D^{\alpha_i}f)(a)\,\frac{(x-a)^{\alpha_i}}{\Gamma(\alpha_i+1)}\ ;\qquad \alpha_0=0 \tag{10}$$

In this article, we present the ideal formula for the generalization of the classic Taylor's formula

2. Generalization of the classical Taylor's formula

Theorem 1. Suppose that $D_a^{\alpha_k}f(x),\ D_a^{\alpha_{k+1}}f(x)\ \in C_\alpha(a,b]$ for $k=0,\,1,\,2,\,...,\,n;\ k-1<\alpha_k\le k,\ \alpha_0=0$, then

$$I_a^{\alpha_k}D_a^{\alpha_k}f(x)-I_a^{\alpha_{k+1}}D_a^{\alpha_{k+1}}f(x)=\frac{(x-a)^{\alpha_k}}{\Gamma(\alpha_k+1)}(D_a^{\alpha_k}f)(a) \tag{11}$$

Proof. Using (10), we have

$$\begin{aligned}I_a^{\alpha_k}D_a^{\alpha_k}f(x)&=f(x)-\textstyle\sum_{i=0}^{k-1}(D_a^{\alpha_i}f)(a)\,\frac{(x-a)^{\alpha_i}}{\Gamma(\alpha_i+1)}\\&=f(x)-f(a)-(D_a^{\alpha_1}f)(a)\,\frac{(x-a)^{\alpha_1}}{\Gamma(\alpha_1+1)}-\ ...-\ (D_a^{\alpha_{k-1}}f)(a)\,\frac{(x-a)^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1}+1)}\end{aligned} \tag{12}$$

and

$$\begin{aligned}I_a^{\alpha_{k+1}}D_a^{\alpha_{k+1}}f(x)&=f(x)-\textstyle\sum_{i=0}^{nk}(D_a^{\alpha_i}f)(a)\,\frac{(x-a)^{\alpha_i}}{\Gamma(\alpha_i+1)}\\&=f(x)-f(a)-(D_a^{\alpha_1}f)(a)\,\frac{(x-a)^{\alpha_1}}{\Gamma(\alpha_1+1)}-\ ...\\&\quad-(D_a^{\alpha_{k-1}}f)(a)\,\frac{(x-a)^{\alpha_{k-1}}}{\Gamma(\alpha_{k-1}+1)}-(D_a^{\alpha_k}f)(a)\,\frac{(x-a)^{\alpha_k}}{\Gamma(\alpha_k+1)}\end{aligned} \tag{13}$$

Then, by subtracting Eq. (13) from Eq.(12), we have the result.
Remark 1. When $k=0$, we have

$$I_a^0(D_a^0f(x))=f(x),$$

by considering

$$I_a^0(D_a^0f(x))-f(x)+f(a)=f(a),$$

Theorem 2. Suppose that $D_a^{\alpha_k}f(x)\ \in C_\alpha(a,b]$ for $k\in$ N; $k-1<\alpha_k\le k,\ \alpha_0=0$, then

$$I_a^{\alpha_k}D_a^{\alpha_i}f(x)=\frac{D_a^{\alpha_i}f(\xi)}{\Gamma(\alpha_k+1)}(x-t)^{\alpha_k}\ \ ,\quad k,\,i\,\in N, \tag{14}$$

for any $\xi\in(a,\,x)$.

Proof. Using (8), we obtain

$$I_a^{\alpha_k} D_a^{\alpha_i} f(x) = \frac{1}{\Gamma(\alpha_k)} \int_a^x (x-t)^{\alpha_k - 1} D_a^{\alpha_i} f(t) dt.$$

Applying the mean value theorem, gives the result.

Theorem 3. Suppose that $D_a^{\alpha_{k+1}} f(x) \in C_\alpha(a,b]$for $k = 0, 1, 2, ..., n$; $k-1 < \alpha_k \leq k$, $\alpha_0 = 0$, then we have the generalized fractional Taylor's formula

$$f(x) = \sum_{k=0}^{n} \frac{(D_a^{\alpha_k} f)(a)}{\Gamma(\alpha_k + 1)} (x-a)^{\alpha_k} + \frac{(D_a^{\alpha_{n+1}} f)(\xi)}{\Gamma(\alpha_{n+1} + 1)} (x-a)^{\alpha_{n+1}}, \tag{15}$$

with $\xi \in (a, x)$, $\forall x \in (a, b)$.

Proof. From (2), we have

$$\sum_{k=0}^{n} (I_a^{\alpha_k} D_a^{\alpha_k} f(x) - I_a^{\alpha_{k+1}} D_a^{\alpha_{k+1}} f(x)) = \sum_{k=0}^{n} \frac{(x-a)^{\alpha_k}}{\Gamma(\alpha_k + 1)} (D_a^{\alpha_k} f)(a)$$

Then,

$$f(x) - I_a^{\alpha_{n+1}} D_a^{\alpha_{n+1}} f(x) = \sum_{k=0}^{n} \frac{(x-a)^{\alpha_k}}{\Gamma(\alpha_k + 1)} (D_a^{\alpha_k} f)(a) \tag{16}$$

Using the integral mean value theorem, we have

$$\begin{aligned} I_a^{\alpha_{n+1}} D_a^{\alpha_{n+1}} f(x) &= \tfrac{1}{\Gamma(\alpha_{n+1}+1)} \int_a^x (x-t)^{\alpha_{n+1}} (D_a^{\alpha_{n+1}} f)(t) dt \\ &= \tfrac{(D_a^{\alpha_{n+1}} f)(\xi)}{\Gamma(\alpha_{n+1}+1)} \int_a^x (x-t)^{\alpha_{n+1}} dt \\ &= \tfrac{(D_a^{\alpha_{n+1}} f)(\xi)}{\Gamma(\alpha_{n+1}+1)} (x-a)^{\alpha_{n+1}} \end{aligned} \tag{17}$$

From (16) and (17), we have (15). In case $\alpha_k = k$, the generalized fractional Taylor's formula (15) reduces to the classical Taylor's formula.

Theorem 4. Suppose that $D_a^{\alpha_k} f(x) \in C_\alpha(a,b]$for $k = 0, 1, 2, ..., n$; $k-1 < \alpha_k \leq k$, $\alpha_0 = 0$, then we have the generalized fractional Taylor's series

$$f(x) = \sum_{k=0}^{\infty} \frac{(D_a^{\alpha_k} f)(x_0)}{\Gamma(\alpha_k + 1)} (x - x_0)^{\alpha_k}, \tag{18}$$

with $a < x_0 < x < b$, $\forall x \in (a, b)$.

Proof. From (15), we take $n \to \infty$in the reminder

$$R_n = \frac{(D_a^{\alpha_{n+1}} f)(\xi)}{\Gamma(\alpha_{n+1} + 1)} (x - x_0)^{\alpha_{n+1}}. \tag{19}$$

Hence, we get

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(D_a^{\alpha_{n+1}} f)(\xi)}{\Gamma(\alpha_{n+1} + 1)} (x - x_0)^{\alpha_{n+1}} = 0$$

then we have the result.

Theorem 5. Suppose that $D_a^{\alpha_k} f(x) \in C_\alpha(a,b]$for $k = 0, 1, 2, ..., n$; $k-1 < \alpha_k \leq k$, $\alpha_0 = 0$, then we have the generalized fractional Maclaurin series

$$f(x) = \sum_{k=0}^{\infty} \frac{(D_a^{\alpha_k} f)(0)}{\Gamma(\alpha_k + 1)} x^{\alpha_k}, \tag{20}$$

with $a < 0 < x < b$, $\forall x \in (a, b)$.

Proof. In (18) taking $x_0 = 0$, gives the result.

3. Application

Since, many phenomena that appear in various applied sciences and engineering depend on the history of the previous time, thus the fractional calculus becomes very important subject to model these phenomena [2]. In this section we use our formula of the generalized Taylor's series to give a series for some functions.
Definition 3. A Mittag-Leffler like function can be defined as

$$\mathrm{E}_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^{1_k}}{\Gamma(\alpha_k + 1)}, \tag{21}$$

where$i_k = ik,\ i = 1,\, 2,\, ...$ and $\ k - 1 < \alpha_k \leq k,\ \alpha_0 = 0.$
Definition 4. A generalized Mittag-Leffler like function can be given in the form

$$\mathrm{E}_\alpha(x^{f(\alpha)}) = \sum_{k=0}^{\infty} \frac{x^{\alpha_k}}{\Gamma(\alpha_k + 1)}\ , \quad f(\alpha) \tag{22}$$

where $k - 1 < \alpha_k \leq k,\ \alpha_0 = 0.$
From Theorem 3, the generalized fractional Taylor's formula can be used to approximate functions at given points, and then we have the form

$$f(x) \cong \mathrm{P}_N^\alpha(x) = \sum_{k=0}^{N} \frac{(D_a^{\alpha_k} f)(a)}{\Gamma(\alpha_k + 1)} (x - a)^{\alpha_k}\ , \tag{23}$$

with an error term of the form

$$R_N^\alpha(x) = \frac{(D_a^{\alpha_{N+1}} f)(\xi)}{\Gamma(\alpha_{N+1} + 1)} (x - a)^{\alpha_{N+1}}, \tag{24}$$

with $\xi \in (a,\, x)$.
Application 1. The generalized Mittag-Leffler like function (22) can be approximated as

$$\mathrm{E}_\alpha(x^{f(\alpha)}) \cong \mathrm{P}_N^\alpha(x) = 1 + \frac{x^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + \frac{x^{\alpha_2}}{\Gamma(\alpha_2 + 1)} + ... + \frac{x^{\alpha_N}}{\Gamma(\alpha_N + 1)}\ , \tag{25}$$

and the error term take the form

$$R_N^\alpha(x) = \frac{\mathrm{E}_\alpha(\xi)}{\Gamma(\alpha_{N+1} + 1)} (x - a)^{\alpha_{N+1}}, \xi \in (a,\, x). \tag{26}$$

4. Conclusions

In this paper we successfully established a new generalization of the classic Taylor's formula. We discussed a new fractional Taylor's series with a new Caputo like-fractional derivative. The behavior of the new formula seems to be extremely interesting. Finally, we can say that the new fractional Taylor's formula is potentially very effective and accurate for solving fractional differential equations.

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