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NONSTANDARD FINITE DIFFERENCE SCHEME FOR THE FRACTIONAL ORDER SALMONELLA TRANSMISSION MODEL

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Abstract. In this paper, an asymptotically stable difference schema, that was built depending on the nonstandard finite difference technique, is proposed to study numerically the fractional order Salmonella transmission model. This model is generalized to the standard Salmonella model by using fractional Caputo operator. Analytically and numerically studies of the properties of the introduced system were run out. The proposed schema saves the properties of the analytic solutions of the proposed model like the boundedness and the positivity. Numerical examples to test the behavior of the this explicit and computationally inexpensive techniques and to show simplicity, applicability, and reliability of this scheme to obtain realistic numerical solutions, are run out with some comparison with the standard finite difference method.

1. Introduction

Fractional calculus has been popular and important due to its attractive applications as a new modeling act in a variety of scientific and engineering fields, like biology ([2], [3]), system control ([5], [9] and [10]), viscoelasticity [6], thermoelasticity ([7], [8]), hydrology [11], finance [19] and fractional dynamics [12]. These days fractional differential equations are the best way to describe these fractional models.

Epidemics are one of the most serious issues of health in the world which need to be transacted. Many studies, for a long time, of the dynamics of epidemic diseases have been introduced. Models that include the time derivatives and consist of systems of ordinary differential equations are the most effective approach to study the dynamics of epidemics. In these models, each equation represents the change in the number of bodies in different categories given by continuous variables.

The effect of epidemics is always delayed in time, thus amalgamating the memory into differential epidemics systems is necessary. So, the models on which the current state relate by all of its previous states not only upon its first or second prior one are more suitable to describe the flow of the epidemic. It is known, these days, that the fractional derivatives are defined by integrals overall history of the domain of the

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study [5], so the fractional derivatives enable to describe the memory and hereditary properties latent in several processes and materials. Hence the derivatives models which constructed using fractional derivatives are more attractive and suitable for the epidemics models. For many applications, that employ epidemic models, the models which have been constructed depending on the fractional order derivatives have been shown to produce a better appropriate real data conclusions than the models which structured depending on the integer order derivative ([2], [3] and [4]). Therefore in this work, we present a Salmonella fractional order model that includes the effect of previous values of the variables.

Unfortunately, often the analytic solution of such differential equations can not be obtained explicitly [13]. Hence, it is a very substantial issue to insert numerical methods to solve approximately these models. Some of the proposed numerical techniques: finite difference methods ([15]- [18]), finite element methods [14], homotopy perturbation methods [21], Adomian's decomposition method [20], Taylor collocation methods [22], spectral methods ([23], [24]), variational iteration methods [25] and higher order numerical techniques [26]. Being in mind that most of the biological systems are stiff, therefore, it is necessary to use efficient numerical methods for obtaining good results when approximate the solution of those systems.

Mickens ([27]-[31]) proposed the nonstandard finite difference method (NSFDM), which can be easy to contract [34], for improving the discretizations of many expressions in the differential equations, such that depending on the specific discretization and the denominator function the NSFDM will be more stable and more accurate than the standard finite difference method ([32], [33]). This technique is applied in many fields like engineering, chemistry, and physics ([17], [35], [36] and [37]). Essentially, the most fascinating applications of this method are in mathematical ecology and biology ([38], [39]). In these fields, the fitness of NSFDM has been seen conspicuously. In addition, the NSFDM are well used in approximating the solution of fractional order systems, like the fractional financial model [19], the fractionalorder neuron system [40], the fractional Hodgkin-Huxley model [41], system of fractional Schrödinger equations[17] and fractional TB biology model ([42], [43]).

The main contribution of this work is to present nonstandard finite difference scheme (NSFDS) to study numerically the following fractional order Salmonella epidemic model:

$$\begin{aligned}
{}_0^cD_t^\alpha S(t) &= \mu^\alpha N + r^\alpha R(t) - (\beta_c^\alpha I_c(t) + \beta_s^\alpha I_s(t) + \mu^\alpha)S(t),\\
{}_0^cD_t^\alpha I_c(t) &= f^\alpha(\beta_c^\alpha I_c(t) + \beta_s^\alpha I_s(t))S(t) - (e^\alpha + m^\alpha + \mu^\alpha)I_c(t),\\
{}_0^cD_t^\alpha I_s(t) &= (1-f^\alpha)(\beta_c^\alpha I_c(t) + \beta_s^\alpha I_s(t))S(t) + e^\alpha I_c(t) - (h_s^\alpha + \mu^\alpha)I_s(t),\\
{}_0^cD_t^\alpha R(t) &= h_s^\alpha I_s(t) - (r^\alpha + \mu^\alpha)R(t), \qquad (1)
\end{aligned}$$

with initial conditions $S(0) = s_0 \geq 0, I_c(0) = i_{c0} \geq 0, I_s(0) = i_{s0} \geq 0,$ and $R(0) = r_0 \geq 0$.

All parameters and variables in the proposed system (1) with their definitions are written in Table (1). In this system, all the parameters are related to the fractional order α. Here, for simplicity in the notations, in the sequence of paper we will delete the symbol α from above of the parameters.

System (1) describes the flows of individuals from and into the different groups. Such that, Once infected, a susceptible organism leaves the susceptible closet and

TABLE 1. The symbols in the system (1) and their definition

Symbol	Definition
t	Time, $t \geq 0$.
α	Order of the fractional derivative.
${}_0^c D_t^\alpha$	Caputo fractional derivative.
$S(t)$	Individuals that do not have the bacterial infection (susceptible).
$I_c(t)$	Individuals that have the bacterial infection (clinically infected).
$I_s(t)$	Individuals that recovered from the infection and have temporary immunity (subclinically infected).
$R(t)$	Number of recovered individuals.
N	Size population, $N = S(t) + I_c(t) + I_s(t) + R(t)$.
μ	Replacement and exit rate (denotes the mortality rate in every compartment).
β_c	Transmission coefficient for clinical animals.
β_s	Transmission coefficient for subclinical animals.
f	Proportion of infected animals that develop clinical case.
e	Rate of clinical cases that become subclinical.
h_s	Recovery rate for subclinical case.
m	Disease-induced induce mortality rate.
r	Immunity loss rate.

combines the infectious compartment, then it becomes infectious. The infected organisms pass into the recovered compartment. The organisms who have returned to a normal state of health have temporal immunity and classified into a compartment.

These days Salmonella is considered a critical problem for the health everywhere in the world. It accounts for approximately 1.4 million clinical cases, 16,000 hospitalizations, and 600 expirations yearly in the USA. It is a major zoonotic disease which is inherited from animals to humans in beef, milk, eggs, and other dairy products, or during immediate contact with sick animals and their environment. Salmonella lives in the intestinal tracts of humans, animals, and birds. Usually, the illness holds on for 4 to 7 days, and most individuals recover without treatment However, in some persons, diarrhea may be so dangerous that the patient needs to visit a hospital.

Mathematical modeling of Salmonella infection played a significant role in understanding of the transmission of the disease. The above system was introduced by Lo [1] in case $\alpha = 1$ as the following:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \mu N + rR(t) - (\beta_c I_c(t) + \beta_s I_s(t) + \mu)S(t), \\
\frac{dI_c(t)}{dt} &= f(\beta_c I_c(t) + \beta_s I_s(t))S(t) - (e + m + \mu)I_c(t), \\
\frac{dI_s(t)}{dt} &= (1 - f)(\beta_c I_c(t) + \beta_s I_s(t))S(t) + eI_c(t) - (h + \mu)I_s(t), \\
\frac{dR(t)}{dt} &= hI_s(t) - (r + \mu)R(t). \qquad (2)
\end{aligned}$$

We attend in the fractional order case since the effect of epidemic is not instantaneous, so incorporating the memory is very important to explain and understand the flow of the Salmonella epidemic. For more details on the biological motivation and the associated assumptions of this system we refer to ([1]).

The article is organized as follows: In Section 2, we give the relevant definitions on fractional calculus and introduce the preliminaries of NSFDM. In Section 3, we will discuss the most important properties of the proposed model. We construct

NSFDS for the system (1) and prove that this schema preserve the boundedness and the positivity of the solutions of the studied model in section 4 then, in this section, we will give notice about the asymptotically stable of this scheme. In Section 5, some numerical simulations for the proposed model are introduced to show the applicability and the efficiency of NSFDM. Finally, the conclusion will be given in the final section (6).

2. Notations and Preliminaries

Here, we will recall some needful willingnesses for subsequent discussions. Firstly, we mention some useful definitions and mathematical preliminaries on the fractional calculus. Secondly, we define the NSFDM.

2.1. **Fractional calculus definitions.** In literatures, many of the fractional derivatives definitions were introduced (see e.g., [5], [45], [46]). Usually, the operators Riemann-Liouville, Grünwald-Letnikov and Caputo are used to define the time fractional derivatives. In these days the Caputo's definition of the fractional derivative is the one of the most common fractional derivative in applied scientific and engineering because this definition deals with any initial value problem in a appropriate manner.

Definition 2.1. *Let $\alpha \in \mathbb{R}^+$, the Caputo fractional derivative of order α are defined by (Caputo, 1967)*

$$({}_0^cD_t^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(x)}{(t-x)^{1-n+\alpha}} dx, \quad t > 0, \tag{3}$$

where $f(x) \in C^n[0, \infty[$, $n = [\alpha] + 1$.

From this definition, it is easy to see that the derivative of a constant function using Caputo operator is zero. When $\alpha \in \mathbb{N}$ the Caputo differential operator identifies with the usual differential operator of an integer order. Also, similar to the integer-order differentiation, Caputo's fractional differentiation is a linear operation; i.e.

$${}_0^cD_t^\alpha(\lambda f(t) + \gamma g(t)) = \lambda\ {}_0^cD_t^\alpha f(t) + \gamma\ {}_0^cD_t^\alpha g(t).$$

2.2. **The nonstandard finite difference method.** The NSFDM approach was firstly introduced by Mickens ([28], [31]). It depends on construction a numerical discrete scheme for partial differential equations (PDEs) or ordinary differential equations (ODEs). The NSFDM is able to saves the properties of the analytic exact solution of the studies ODEs or PDEs depending on the following steps:

1. Both the discrete approximation derivatives and the corresponding continous derivatives in the differential equations must be of the same order.
2. Terms which are Nonlinear must have nonlocal form approximation.
3. In general, the denominator function must be function of the step sizes of the discrtiztion.
4. The scheme should not have solutions that do not agree with solutions of the differential equations.
5. Special condition which holds in a solution of the differential equation must also be holds for the special discrete of the finite difference scheme.

When approximating $\frac{dy}{dt}$ using Euler method we use $\frac{y(t+h)-y(t)}{\phi(h)}$ instead of $\frac{y(t+h)-y(t)}{h}$, where the function $\phi(h)$ is a continuous of step size h, and

$$\phi(h) = h + O(h^2), \quad 0 < \phi(h) < 1, \quad h \longrightarrow 0.$$

With this replacement, when there is a nonlinear term in the studied differential model, we replace it by a non-local discretization, as we illustrate in this example:

$$yx \longrightarrow \begin{cases} y_n x_{n+1}, \\ y_{n+1} x_n. \end{cases}$$

3. Properties of the Solutions of the Proposed Model

3.1. **Positive solutions.** It is clear that the populations is non-negative. We will prove that the solution of the proposed model is positive. For this aim we use the following Lemma (generalized mean value theorem):

Lemma 3.1. [44] *If the function $g(t) \in C[a,b]$ and ${}_a^c D_t^\alpha g(t) \in C(a,b]$, for $0 < \alpha \leq 1$. Then we have:*

$$g(t) = g(a) + {}_0^c D_t^\alpha g(\xi) \frac{(t-a)^\alpha}{\Gamma(\alpha)},$$

with $0 \leq \xi \leq t$.

So, If $g(t) \in C[0,b]$ and ${}_0^c D_t^\alpha g(t) \in C(0,b]$ and if ${}_0^c D_t^\alpha g(t) \geq 0$ then the function g is nondecreasing.

Theorem 3.2. *There is a unique solution for system (1) and the solution is positive.*

Proof. The existence and the uniqueness of the solutions of (1) follow from the results given in [48]. Depending on the above lemma and because we have :

$$\begin{aligned} &{}_0^c D_t^\alpha S(t) \quad |_{S=0,R=0} = \mu N \geq 0, \\ &{}_0^c D_t^\alpha I_c(t) \quad |_{I_c=0,I_s=0} \geq 0, \\ &{}_0^c D_t^\alpha I_s(t) \quad |_{I_s=0} = (1-f)(\beta I_c(t))S(t) + eI_c(t) \geq 0, \\ &{}_0^c D_t^\alpha R(t) \quad |_{R=0} = hI_s(t) \geq 0, \end{aligned} \tag{4}$$

So $S(t) \geq 0$, $I_c(t) \geq 0$, $I_s(t) \geq 0$, and $R(t) \geq 0$ for any t. □

3.2. **Stability analysis.**

Theorem 3.3. [16] *To find the equilibrium points for system (1), we solve the following equation: $g(t) = 0$, where $g(t)$ is the right hand side of system (1). If all the eigenvalues λ_i of the Jacobian matrix $J = \frac{\partial g}{\partial t}$ which calculated at the equilibrium point are satisfy $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$ then the equilibrium point is locally asymptotically stable.*

The disease free equilibrium point of system (1) are

$$\xi_1 = (N, 0, 0, 0).$$

The Jacobian matrix J for this system evaluated at the equilibrium point is:

$$\boldsymbol{J} = \begin{pmatrix} -(\beta_c I_c + \beta_s I_s + \mu) & -\beta_c S & -\beta_s S & r \\ f(\beta_c I_c + \beta_s I_s) & f\beta_c S - (e+m+\mu) & f\beta_s S & 0 \\ (1-f)(\beta_c I_c + \beta_s I_s) & (1-f)\beta_c S + e & (1-f)\beta_s S - (h+\mu) & 0 \\ 0 & 0 & h & -(r+\mu) \end{pmatrix},$$

such that the Jacobian matrix evaluated at the free equilibrium point is

$$\boldsymbol{J(\xi_1)} = \begin{pmatrix} -\mu & -\beta_c N & -\beta_s N & r \\ 0 & f\beta_c N - (e+m+\mu) & f\beta_s N & 0 \\ 0 & (1-f)\beta_c N + e & (1-f)\beta_s N - (h+\mu) & 0 \\ 0 & 0 & h & -(r+\mu) \end{pmatrix},$$

whose eigenvalues are given by,

$$\lambda_1 = -mu,$$

$$\lambda_2 = -mu - r,$$

$$\lambda_3 = \frac{(1-f)\beta_s N + f\beta_c N - h - m - e - 2\mu + \sqrt{(AN^2 + BN + C)}}{2},$$

$$\lambda_4 = \frac{(1-f)\beta_s N + f\beta_c N - h - m - e - 2\mu - \sqrt{(AN^2 + BN + C)}}{2},$$

where

$$A = (\beta_c^2 f^2 - 2\beta_c\beta_s f^2 + 2\beta_c\beta_s f + \beta_s^2 f^2 - 2\beta_s^2 f + \beta_s^2),$$

$$B = -2\beta_c ef + 2\beta_c fh - 2\beta_c fm + 2\beta_s ef + 2\beta_s e + 2\beta_s fh - 2\beta_s fm - 2\beta_s h + 2\beta_s m,$$

$$C = e^2 - 2eh + 2em + h^2 - 2hm + m^2,$$

So, the free equilibrium point for the model is asymptotically stable when $\lambda_3, \lambda_4 \leq 0$ or λ_3 and λ_4 has negative real part.

4. Construction of the NSFDS

In the current section, we will construct NSFDS to obtain an explicit discretization of system (1).

Let N_n be a natural number and the coordinate of the each mesh point is:

$$t_n = n\triangle t, \quad n = 0,\ 1,\ 2,\ ...,\ N_n,$$

where

$$h := \triangle t = \frac{t_{final}}{N_n}.$$

The numerical value of S, I_c, I_s and R at the grid point (t_n) is denoted by S_n, I_{cn}, I_{sn} and R_n. The nonstandard differences approximation of Caputo operator is given by the Grünwald-Letnikov approach:

$${}_0^c D_t^\alpha x(t)\,\big|_{t=t_n} = \frac{1}{(\phi(\triangle t))^\alpha}(x_{n+1} - \sum_{i=1}^{n+1} w_i x_{n+1-i} - q_{n+1}x_0), \tag{5}$$

where

$$w_i = (-1)^{i-1}\binom{\alpha}{i}, \quad w_1 = \alpha,$$

$$q_i = \frac{i^{-\alpha}}{\Gamma(1-\alpha)}, \quad i = 1,\ 2,\ \ldots,\ n+1.$$

Theorem 4.1. [47] *Assume that* $0 < \alpha < 1$, *then the coefficients* w_i *and* q_i *satisfy for* $i \geq 1$ *the properties*

$$0 < w_{i+1} < w_i < ... < w_1 = \alpha < 1, \tag{6}$$

$$0 < q_{i+1} < q_i < ... < q_1 = \frac{1}{\Gamma(1-\alpha)}. \tag{7}$$

Proof. see [47]. □

Using nonstandard technique and relation (5) we obtain the following nonstandard scheme for system (1)

$$\begin{aligned}
\frac{1}{(\phi(h))^\alpha}(S_{n+1} - \sum_{i=1}^{n+1} w_i S_{n+1-i} - q_{n+1}S_0) &= \mu N + rR_n - (\beta_c I_{cn} + \beta_s I_{sn} + \mu)S_{n+1},\\
\frac{1}{(\phi(h))^\alpha}(I_{c(n+1)} - \sum_{i=1}^{n+1} w_i I_{c(n+1-i)} - q_{n+1}I_{c0}) &= f(\beta_c I_{cn} + \beta_s I_{sn})S_{n+1} - (e+m+\mu)I_{c(n+1)},\\
\frac{1}{(\phi(h))^\alpha}(I_{s(n+1)} - \sum_{i=1}^{n+1} w_i I_{s(n+1-i)} - q_{n+1}I_{s0}) &= (1-f)(\beta I_{cn} + \beta_s I_{sn})S_{n+1} + eI_{c(n+1)} - (h_s+\mu)I_{s(n+1)},\\
\frac{1}{(\phi(h))^\alpha}(R_{n+1} - \sum_{i=1}^{n+1} w_i R_{n+1-i} - q_{n+1}R_0) &= h_s I_{s(n+1)} - (r+\mu)R_{n+1}.
\end{aligned} \tag{8}$$

Since each of these equations is linear in $S_{n+1},\ I_{c(n+1)},\ I_{s(n+1)}$ and R_{n+1} so, some calculations give us the following explicit expressions

$$\begin{aligned}
S_{n+1} &= \frac{1}{1+(\phi(h))^\alpha(\beta_c I_{cn} + \beta_s I_{sn} + \mu)}[\sum_{i=1}^{n+1} w_i S_{n+1-i} + q_{n+1}S_0 + (\phi(h))^\alpha \mu N + (\phi(h))^\alpha r R_n],\\
I_{c(n+1)} &= \frac{1}{1+(\phi(h))^\alpha(e+m+\mu)}[\sum_{i=1}^{n+1} w_i I_{c(n+1-i)} + q_{n+1}I_{c0} + (\phi(h))^\alpha f(\beta_c I_{cn} + \beta_s I_{sn})S_{n+1}],\\
I_{s(n+1)} &= \frac{1}{1+(\phi(h))^\alpha(h_s+\mu)}[\sum_{i=1}^{n+1} w_i I_{s(n+1-i)} + q_{n+1}I_{s0} + (\phi(h))^\alpha((1-f)(\beta_c I_{cn} + \beta_s I_{sn})S_{n+1}\\
&\qquad\qquad + eI_{c(n+1)})],\\
R_{n+1} &= \frac{1}{1+(\phi(h))^\alpha(r+\mu)}[\sum_{i=1}^{n+1} w_i R_{n+1-i} + q_{n+1}R_0 + (\phi(h))^\alpha h_s I_{s(n+1)}].
\end{aligned} \tag{9}$$

4.1. **Positivity and boundedness of scheme.** This subsection, analyzes some properties of the introduced scheme (9). Being in mind that system (1) have a unique non-negative solutions.

Theorem 4.2. *(Positivity). Suppose that* $S_0 \geq 0,\ I_{c0} \geq 0,\ I_{s0} \geq 0,$ *and* $R_0 \geq 0,$ *then* $S_n \geq 0,\ I_{cn} \geq 0,\ I_{sn} \geq 0,$ *and* $R_n \geq 0$ *is satisfied for all* $n = 1,\ 2,\ ...$

Proof. By induction. For $n = 0$, we have from system (9):

$$
\begin{aligned}
S_1 &= \frac{1}{1+(\phi(h))^{\alpha}(\beta_c I_{c0}+\beta_s I_{s0}+\mu)}[w_1 S_0 + q_1 S_0 + (\phi(h))^{\alpha}\mu N + (\phi(h))^{\alpha} r R_0] \geq 0,\\
I_{c(1)} &= \frac{1}{1+(\phi(h))^{\alpha}(e+m+\mu)}[w_1 I_{c0} + q_1 I_{c0} + (\phi(h))^{\alpha} f(\beta_c I_{c0}+\beta_s I_{s0}) S_1] \geq 0,\\
I_{s(1)} &= \frac{1}{1+(\phi(h))^{\alpha}(h_s+\mu)}[w_1 I_{s0} + q_1 I_{s0} + (\phi(h))^{\alpha}((1-f)(\beta_c I_{c0}+\beta_s I_{s0}) S_1 + e I_{c(1)})] \geq 0,\\
R_1 &= \frac{1}{1+(\phi(h))^{\alpha}(r+\mu)}[w_1 R_0 + q_1 R_0 + (\phi(h))^{\alpha} h_s I_{s(1)}] \geq 0. \qquad (10)
\end{aligned}
$$

Notice, all the parameter here are positive. We suppose that $S_n \geq 0,\ I_{cn} \geq 0,\ I_{sn} \geq 0$, and $R_n \geq 0$ for all $n < n+1$. Thus for $n+1$

$$
\begin{aligned}
S_{n+1} &= \frac{1}{1+(\phi(h))^{\alpha}(\beta_c I_{cn}+\beta_s I_{sn}+\mu)}[\sum_{i=1}^{n+1} w_i S_{n+1-i} + q_{n+1} S_0 + (\phi(h))^{\alpha}\mu N + (\phi(h))^{\alpha} r R_n] \geq 0,\\
I_{c(n+1)} &= \frac{1}{1+(\phi(h))^{\alpha}(e+m+\mu)}[\sum_{i=1}^{n+1} w_i I_{c(n+1-i)} + q_{n+1} I_{c0} + (\phi(h))^{\alpha} f(\beta_c I_{cn}+\beta_s I_{sn}) S_{n+1}] \geq 0,\\
I_{s(n+1)} &= \frac{1}{1+(\phi(h))^{\alpha}(h_s+\mu)}[\sum_{i=1}^{n+1} w_i I_{s(n+1-i)} + q_{n+1} I_{s0}\\
&\qquad +(\phi(h))^{\alpha}((1-f)(\beta_c I_{cn}+\beta_s I_{sn}) S_{n+1} + e I_{c(n+1)})] \geq 0,\\
R_{n+1} &= \frac{1}{1+(\phi(h))^{\alpha}(r+\mu)}[\sum_{i=1}^{n+1} w_i R_{n+1-i} + q_{n+1} R_0 + (\phi(h))^{\alpha} h_s I_{s(n+1)}] \geq 0. \qquad (11)
\end{aligned}
$$

□

Theorem 4.3. *(Boundedness). Suppose that the initial conditions are* $S_0 = N,\ I_{c0} = 0,\ I_{s0} = 0$, *and* $R_0 = 0$ *and so* $S_0 + I_{c0} + I_{s0} + R_0 = N$, *then* $S_n,\ I_{cn},\ I_{sn}$ *and* R_n *are bounded for all* $n = 1,\ 2,\ ...$

Proof. Multiplying each equation in system (9) by its denominator give:

$$
\begin{aligned}
S_{n+1}(1 &+ (\phi(h))^{\alpha}\mu) + I_{c(n+1)}(1+(\phi(h))^{\alpha}(m+\mu)) + I_{s(n+1)}(1+(\phi(h))^{\alpha}\mu) + R_{n+1}(1+(\phi(h))^{\alpha}(r+\mu))\\
&= \sum_{i=1}^{n+1} w_i(S_{n+1-i} + I_{c(n+1-i)} + I_{s(n+1-i)} + R_{n+1-i}) + q_{n+1} N + (\phi(h))^{\alpha}\mu N + (\phi(h))^{\alpha} r R_n, \qquad (12)
\end{aligned}
$$

using induction principle, for $n = 0$, we have:

$$
\begin{aligned}
S_1(1+(\phi(h))^{\alpha}\mu) &+ I_{c(1)}(1+(\phi(h))^{\alpha}(m+\mu)) + I_{s(1)}(1+(\phi(h))^{\alpha}\mu) + R_1(1+(\phi(h))^{\alpha}(r+\mu))\\
&= \sum_{i=1}^{1} w_i(S_{1-i} + I_{c(1-i)} + I_{s(1-i)} + R_{1-i}) + q_1 N + (\phi(h))^{\alpha}\mu N + (\phi(h))^{\alpha} r R_0,\\
&= w_1(S_0 + I_{c0} + I_{s0} + R_0) + q_1 N + (\phi(h))^{\alpha}\mu N,\\
&= w_1 N + q_1 N + (\phi(h))^{\alpha}\mu N,\\
&= (w_1 + q_1 + (\phi(h))^{\alpha}\mu) N,\\
&= (\alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha}\mu) N = M_1.
\end{aligned}
$$

(13)

So, we have

$$S_1 \leq \frac{M_1}{(1+(\phi(h))^{\alpha}\mu)}, \quad I_{c(1)} \leq \frac{M_1}{(1+(\phi(h))^{\alpha}(m+\mu))},$$

$$I_{s(1)} \leq \frac{M_1}{(1+(\phi(h))^{\alpha}\mu)}, \quad R_1 \leq \frac{M_1}{(1+(\phi(h))^{\alpha}(r+\mu))},$$

i.e.,

$$S_1 \leq M_1, \quad I_{c(1)} \leq M_1, \quad I_{s(1)} \leq M_1, \quad R_1 \leq M_1.$$

For $n = 1$, we have:

$$\begin{aligned}
S_2(1+(\phi(h))^{\alpha}\mu) &+ I_{c(2)}(1+(\phi(h))^{\alpha}(m+\mu)) + I_{s(2)}(1+(\phi(h))^{\alpha}\mu) + R_2(1+(\phi(h))^{\alpha}(r+\mu)) \\
&= \sum_{i=1}^{2} w_i(S_{2-i} + I_{c(2-i)} + I_{s(2-i)} + R_{2-i}) + q_2 N + (\phi(h))^{\alpha}\mu N + (\phi(h))^{\alpha} r R_1, \\
&\leq w_1(S_1 + I_{c(1)} + I_{s(1)} + R_1) + w_2(S_0 + I_{c0} + I_{s0} + R_0) + q_1 N + (\phi(h))^{\alpha}(\mu N + r M_1), \\
&\leq w_1(4M_1) + w_1 N + q_1 N + (\phi(h))^{\alpha}(\mu N + r M_1), \\
&= 4\alpha M_1 + \alpha N + \frac{1}{\Gamma(1-\alpha)} N + (\phi(h))^{\alpha}\mu N + (\phi(h))^{\alpha} r M_1, \\
&= (4\alpha + (\phi(h))^{\alpha} r) M_1 + (\alpha + \frac{1}{\Gamma(1-\alpha)} + (\phi(h))^{\alpha}\mu) N, \\
&= (4\alpha + (\phi(h))^{\alpha} r + 1) M_1 = M_2,
\end{aligned} \tag{14}$$

So,

$$S_2 \leq M_2, \quad I_{c(2)} \leq M_2, \quad I_{s(2)} \leq M_2, \quad R_2 \leq M_2.$$

For $n = 2$, we have:

$$\begin{aligned}
S_3(1+(\phi(h))^{\alpha}\mu) &+ I_{c(3)}(1+(\phi(h))^{\alpha}(m+\mu)) + I_{s(3)}(1+(\phi(h))^{\alpha}\mu) + R_3(1+(\phi(h))^{\alpha}(r+\mu)) \\
&= \sum_{i=1}^{3} w_i(S_{3-i} + I_{c(3-i)} + I_{s(3-i)} + R_{3-i}) + q_3 N + (\phi(h))^{\alpha}\mu N + (\phi(h))^{\alpha} r R_2, \\
&\leq w_1(S_2 + I_{c(2)} + I_{s(2)} + R_2) + w_2(S_1 + I_{c(1)} + I_{s(1)} + R_1) + w_3(S_0 + I_{c0} + I_{s0} + R_0) \\
&\qquad + q_1 N + (\phi(h))^{\alpha}(\mu N + r M_2), \\
&\leq w_1(4M_2) + w_1(4M_1) + w_1 N + q_1 N + (\phi(h))^{\alpha}(\mu N + r M_2), \\
&= 4\alpha M_2 + 4\alpha M_1 + \alpha N + \frac{1}{\Gamma(1-\alpha)} N + (\phi(h))^{\alpha}\mu N + (\phi(h))^{\alpha} r M_2, \\
&= (4\alpha + (\phi(h))^{\alpha} r) M_2 + (4\alpha + 1) M_1 = M_3,
\end{aligned} \tag{15}$$

So,

$$S_3 \leq M_3, \quad I_{c(3)} \leq M_3, \quad I_{s(3)} \leq M_3, \quad R_3 \leq M_3.$$

For $n = 3$, we have:

$$\begin{aligned}
S_4(1+(\phi(h))^{\alpha}\mu) &+ I_{c(4)}(1+(\phi(h))^{\alpha}(m+\mu)) + I_{s(4)}(1+(\phi(h))^{\alpha}\mu) + R_4(1+(\phi(h))^{\alpha}(r+\mu)) \\
&= \sum_{i=1}^{4} w_i(S_{4-i}+I_{c(4-i)}+I_{s(4-i)}+R_{4-i}) + q_4N + (\phi(h))^{\alpha}\mu N + (\phi(h))^{\alpha} r R_3, \\
&\leq w_1(S_3+I_{c(3)}+I_{s(3)}+R_3) + w_2(S_2+I_{c(2)}+I_{s(2)}+R_2) \\
&\quad + w_3(S_1+I_{c(1)}+I_{s(1)}+R_1) + w_4(S_0+I_{c0}+I_{s0}+R_0) \\
&\quad + q_1N + (\phi(h))^{\alpha}(\mu N + rM_3), \\
&\leq w_1(4M_3) + w_1(4M_2) + w_1(4M_1) + w_1N + q_1N + (\phi(h))^{\alpha}(\mu N + rM_3), \\
&= 4\alpha M_3 + 4\alpha M_2 + 4\alpha M_1 + \alpha N + \frac{1}{\Gamma(1-\alpha)}N + (\phi(h))^{\alpha}\mu N + (\phi(h))^{\alpha} r M_3, \\
&= (4\alpha + (\phi(h))^{\alpha} r)M_3 + 4\alpha M_2 + (4\alpha+1)M_1 = M_4,
\end{aligned} \tag{16}$$

So,

$$S_4 \leq M_4, \quad I_{c(4)} \leq M_4, \quad I_{s(4)} \leq M_4, \quad R_4 \leq M_4.$$

Now we suppose that

$$\begin{aligned}
S_n(1+(\phi(h))^{\alpha}\mu) &+ I_{c(n)}(1+(\phi(h))^{\alpha}(m+\mu)) + I_{s(n)}(1+(\phi(h))^{\alpha}\mu) + R_n(1+(\phi(h))^{\alpha}(r+\mu)) \\
&\leq (4\alpha + (\phi(h))^{\alpha} r)M_{n-1} + 4\alpha(M_{n-2} + ... + M_3 + M_2) + (4\alpha+1)M_1, \\
&\leq M_n,
\end{aligned} \tag{17}$$

i.e.,

$$S_n \leq M_n, \quad I_{c(n)} \leq M_n, \quad I_{s(n)} \leq M_n, \quad R_n \leq M_n.$$

Now, we will proof

$$S_{n+1} \leq M_{n+1}, \quad I_{c(n+1)} \leq M_{n+1}, \quad I_{s(n+1)} \leq M_{n+1}, \quad R_{n+1} \leq M_{n+1},$$

where

$$M_{n+1} = (4\alpha + (\phi(h))^{\alpha} r)M_n + 4\alpha(M_{n-1} + ... + M_3 + M_2) + (4\alpha+1)M_1.$$

From Eq. (12) we have

$$\begin{aligned}
S_{n+1}(1+(\phi(h))^{\alpha}\mu) &+ I_{c(n+1)}(1+(\phi(h))^{\alpha}(m+\mu)) + I_{s(n+1)}(1+(\phi(h))^{\alpha}\mu) + R_{n+1}(1+(\phi(h))^{\alpha}(r+\mu)) \\
&\leq w_1(S_n+I_{c(n)}+I_{s(n)}+R_n) + w_2(S_{n-1}+I_{c(n-1)}+I_{s(n-1)}+R_{n-1}) + ... \\
&\quad + w_n(S_1+I_{c(1)}+I_{s(1)}+R_1) + w_{n+1}(S_0+I_{c0}+I_{s0}+R_0) \\
&\quad + q_1N + (\phi(h))^{\alpha}(\mu N + rM_n), \\
&\leq w_1(4M_n) + w_1(4M_{n-1}) + ... + w_1(4M_2) + w_1(4M_1) + w_1N + q_1N + (\phi(h))^{\alpha}(\mu N + rM_n), \\
&\leq (4\alpha + (\phi(h))^{\alpha} r)M_n + 4\alpha(M_{n-1} + ... + M_3 + M_2) + (4\alpha+1)M_1, \\
&\leq M_{n+1},
\end{aligned} \tag{18}$$

so,

$$S_{n+1} \leq M_{n+1}, \quad I_{c(n+1)} \leq M_{n+1}, \quad I_{s(n+1)} \leq M_{n+1}, \quad R_{n+1} \leq M_{n+1},$$

□

4.2. **Stability.**

Definition 4.4. *Scheme (9) is called asymptotically stable, if there are constants* L_1, L_2, L_3 *and* L_4 *as* $\alpha \longrightarrow 1$, *such that*

$$S_{n+1} \leq L_1, \quad I_{c(n+1)} \leq L_2, \quad I_{s(n+1)} \leq L_3, \quad R_{n+1} \leq L_4,$$

hold for any arbitrary initial values $0 < S_0 + I_{c0} + I_{s0} + R_0 = N$.

From boundness theorem we conclude that the proposed NSFDS (9) is asympotically stable.

5. Numerical Simulations

Here, we will use the proposed NSFD scheme (9) to simulate the solution of (1). NSFDM can decrease the calculation's time since it is able to use larger time steps and it provides good approximations for (1). Through the numerical simulation section, we used $\phi(h) = 1 - e^{-h}$, even though, we can use other choice for the function $\phi(h)$ in our technique and we obtain similar results.

Let $N = 100$, $\mu = 0.0011$, $\beta_c = 0.0016$, $\beta_s = 0.00006$, $f = 0.5$, $e = 0.25$, $h_s = 0.041$ $m = 0.011$, and $r = 0.01$. The initial conditions are $S_0 = N - 1$, $I_{c0} = 1$, $I_{s0} = 0$, $R_0 = 0$. So the eigenvalues now are $\lambda_1 = -0.0011 \leq 0$, $\lambda_2 = -0.0111 \leq 0$, $\lambda_3 = -0.0325 \leq 0$, and $\lambda_3 = -0.1887 \leq 0$. All these eigenvalues satisfy the condition $|arg(\lambda_i)| = |arg(\pi)| > \frac{\alpha\pi}{2}$, therefore the free equilibrium point for the model is asymptotically stable.

In the following simulation, to show that the introduced schema is efficient, we will take different values for the final time with different values of the time step h

First, when $t_{final} = 100$ and $\triangle t = 0.01$, in figure (1) we show how the solutions of system (1) obtiend by the proposed NSFDS change when α takes different values. We see that NSFDM is stable.

Second, when $t_{final} = 700$ and $\triangle t = 0.1$, in figure (2) we show how the solutions of system (1) obtiend by the proposed NSFDM change when α takes different values. We see that NSFDM is stable.

Third, when $t_{final} = 1000$ and $\triangle t = 10$, in figure (3) we show how the solutions of system (1) obtiend by the proposed NSFDM change when α takes different values. We see that NSFDM is still stable.

Fourth, Let $t_{final} = 1000$ and $\triangle t = 10$, figure (4) shows the unstable solutions of system (1) using SFDM when α takes different values.

The above comparisons show that NSFDM is more stable than SFDM.

Table (2), in case $\alpha = 1$, reports the convergence behavior of following numerical methods: NSFDM, SFDM, ode45 and od23s. We can conclude from this table that NSFDM and ode23s is convergent for big t_{final} while SFDM and ode45 converge only when t_{final} is small. That the studied system has stiffness properties.

6. Conclusion

In this article, we proposed a fractional order Salmonella model. This dynamical model is more compatible to describe the biological phenomena with memory than the integer order model. We studied some proprieties of the proposed model. We used NSFDM to simulate the solutions of the fractional Salmonella model. NSFDS is structured such that the numerical solutions obtained by mean of it

TABLE 2. Comparing between NSFDM, SFDM ($N_n = 500$), ode45 and ode23s when t_{final} takes different values.

t_{final}	NSFDM	SFDM	ode45	ode23s
100	Convergent	Convergent	Convergent	Convergent
200	Convergent	Convergent	Convergent	Convergent
500	Convergent	Convergent	Convergent	Convergent
1000	Convergent	Divergent	Divergent	Convergent
1500	Convergent	Divergent	Divergent	Convergent
2000	Convergent	Divergent	Divergent	Convergent

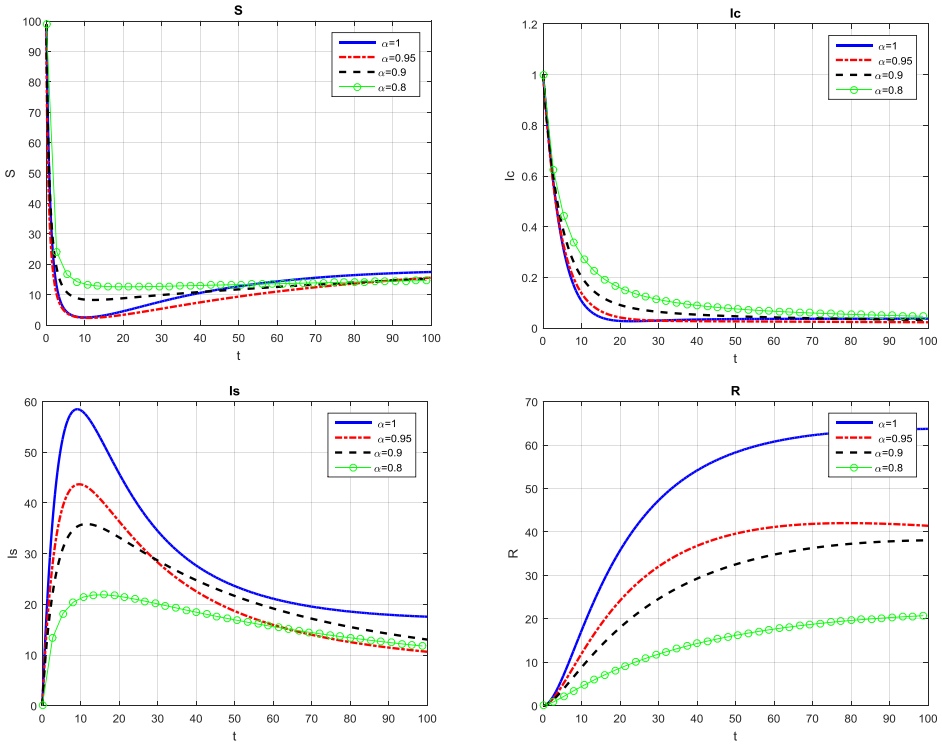

FIGURE 1. Profiles obtained by NSFDM for solving fractional Salmonella model with different value of α, $\triangle t = 0.01$

having properties like the properties of the analytical solutions. It has good properties in stability more than SFDM for solving like this fractional model. NSFDM saves the computational time when the final time is very big and provides valid approximations.

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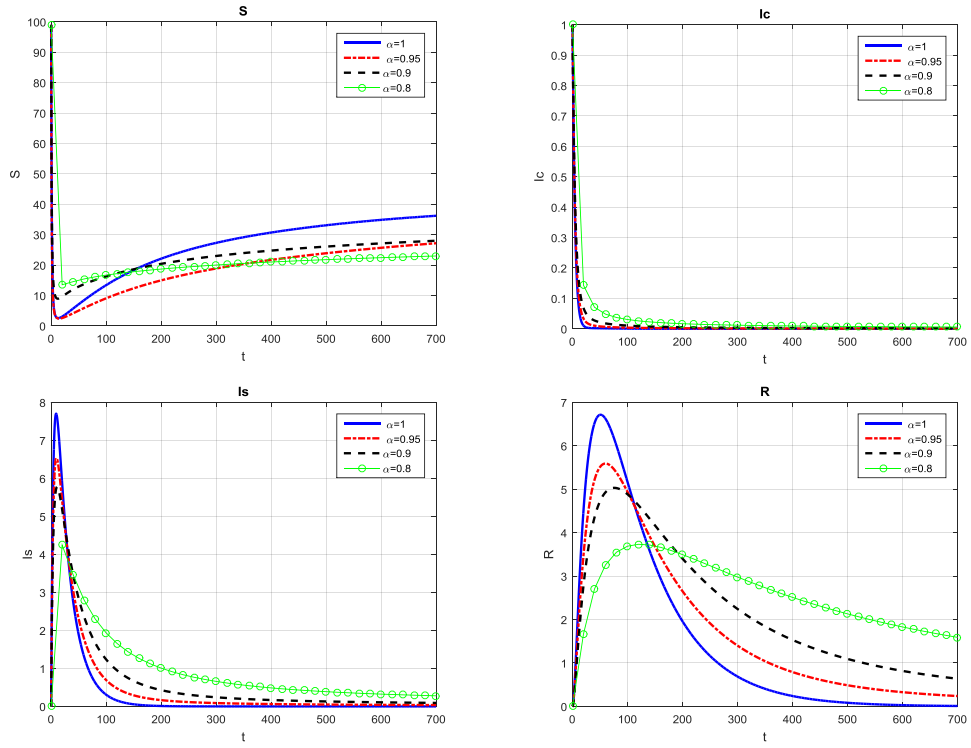

FIGURE 2. Profiles obtained by NSFDM for solving fractional Salmonella model with different value of α, $\triangle t = 0.1$

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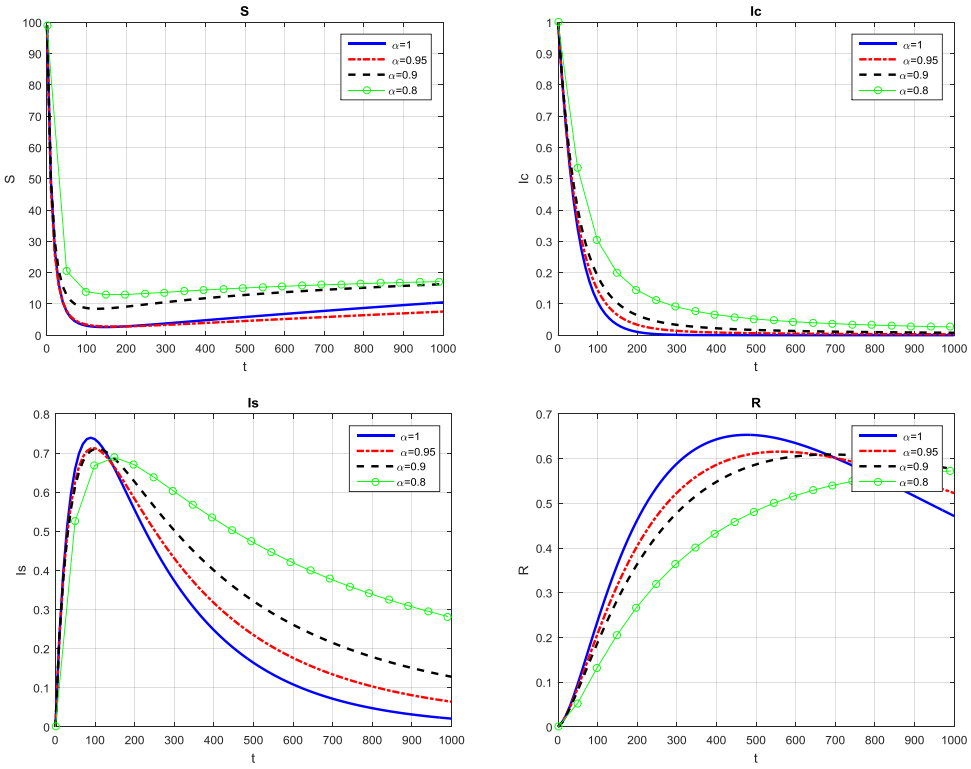

FIGURE 3. Profiles obtained by NSFDM for solving fractional Salmonella model with different value of α, $\triangle t = 10$

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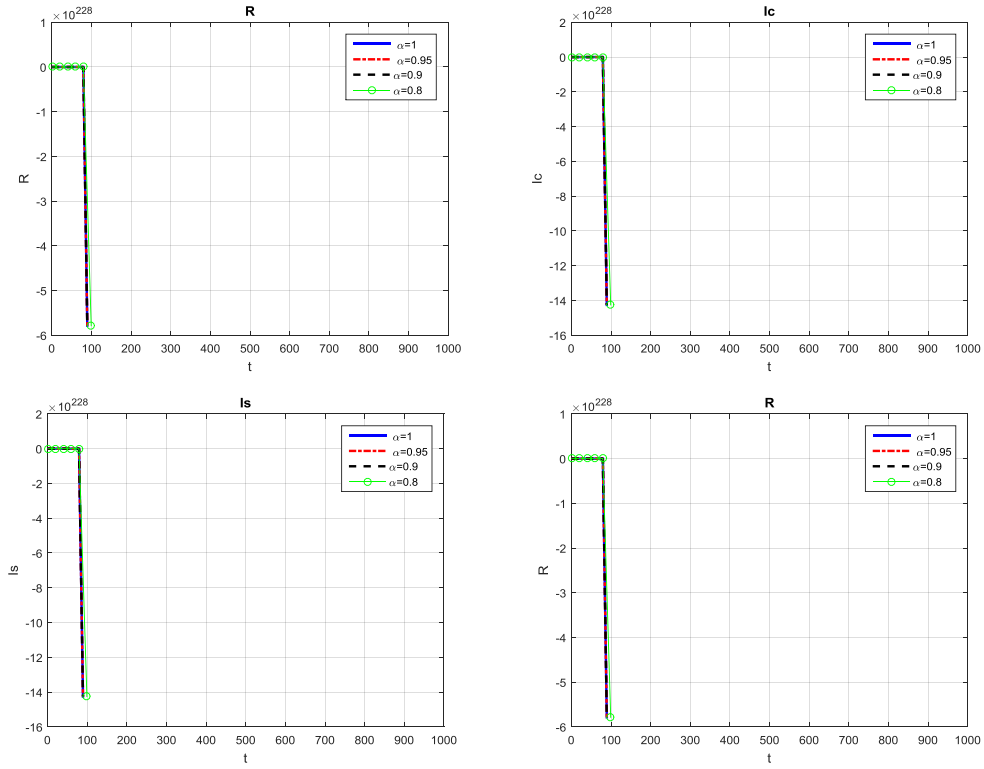

FIGURE 4. Unstable profiles obtained by SFDM for solving fractional Salmonella model with different value of α, $\triangle t = 10$.

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