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# FRACTIONAL PARTIAL HYPERBOLIC DIFFERENTIAL INCLUSIONS WITH STATE-DEPENDENT DELAY

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Abstract. In this paper we investigate the existence of solutions of initial value problem for partial hyperbolic differential inclusions of fractional order involving Caputo fractional derivative with state-dependent Delay when the right hand side is convex valued by using a multi-valued version of nonlinear alternative of Leray-Schauder type.

## 1. Introduction

The first result of this paper deals with the existence of solutions to fractional order initial value problems ($IVP$ for short), for the system

$$({}^cD_0^r u)(t,x) \in F(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}),\ \text{if } (t,x)\in J, \tag{1}$$

$$u(t,x)=\phi(t,x),\ \text{if } (t,x)\in \tilde{J}, \tag{2}$$

$$u(t,0)=\varphi(t),\ u(0,x)=\psi(x),\ (t,x)\in J, \tag{3}$$

where $\varphi(0)=\psi(0),\ J:=[0,a]\times[0,b],\ a,b,\alpha,\beta>0,\ \tilde{J}:=[-\alpha,a]\times[-\beta,b]\backslash[0,a]\times[0,b],\ {}^cD_0^r$ is the standard Caputo's fractional derivative of order $r=(r_1,r_2)\in(0,1]\times(0,1], F: J\times C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n)\to\mathcal{P}(\mathbb{R}^n)$, is a compact valued multivalued maps, $\mathcal{P}$ is a family of all subsets of $\mathbb{R}^n,\ \rho_1: J\times C\to[-\alpha,a],\ \rho_2: J\times C\to[-\beta,b]$ are given functions, $\phi\in C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n)$ is a given continuous function with $\phi(t,0)=\varphi(t),\ \phi(0,x)=\psi(x)$ for each $(t,x)\in J, \varphi:[0,a]\to\mathbb{R}^n$, $\psi:[0,b]\to\mathbb{R}^n$ are given absolutely continuous functions.
We denote by $u_{(t,x)}$ the element of $C([-\alpha,0]\times[-\beta,0],\mathbb{R}^n)$ defined by

$$u_{(t,x)}(s,\tau)=u(t+s,x+\tau);\ (s,\tau)\in[-\alpha,0]\times[-\beta,0],$$

here $u_{(t,x)}(\cdot,\cdot)$ represents the history of the state $u$.

The second result deals with the existence of solutions to fractional order partial differential equations

$$({}^cD_0^r u)(t,x) \in F(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}),\ \text{if } (t,x)\in J, \tag{4}$$

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$$u(t,x) = \phi(t,x), \text{ if } (t,x) \in \tilde{J}', \tag{5}$$

$$u(t,0) = \varphi(t), \ u(0,x) = \psi(x), \ (t,x) \in J, \tag{6}$$

where $\varphi, \ \psi$ are as in problem (1)-(3), $\tilde{J}' := (-\infty, a] \times (-\infty, b] \backslash [0, a] \times [0, b]$, $F : J \times \mathcal{B} \to \mathcal{P}(\mathbb{R}^n)$, is a compact valued multivalued maps, $\rho_1 : J \times \mathcal{B} \to (-\infty, a], \rho_2 : J \times \mathcal{B} \to (-\infty, b]$ are given functions, $\phi : \tilde{J}' \to \mathbb{R}^n$ is a given continuous function with $\phi(t, 0) = \varphi(t), \ \phi(0, x) = \psi(x)$ for each $(t, x) \in J$ and $\mathcal{B}$ is called a phase space that will be specified in Section 4.

It is well known that differential equations and inclusions of fractional order play a very important role in describing some real world problems. For example some problems in physics, mechanics, viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [8, 30, 42, 45, 50]). The theory of differential equations and inclusions of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. Numerous research papers and monographs have appeared devoted to fractional differential equations and inclusions, for example see the monographs of Kilbas *et al.* [38], Lakshmikantham *et al.* [40], and the papers by Agarwal *et al* [3, 4], Belarbi *et al.* [7], Benchohra *et al.* [10] and the references therein.

Differential delay equations and inclusions, or functional differential equations and inclusions, have been used in modeling scientific phenomena for many years. Often, it has been assumed that the delay is either a fixed constant or is given as an integral in which case it is called a distributed delay; see for instance the books by Hale and Verduyn Lunel [27], Hino *et al.* [31], Kolmanovskii and Myshkis [37], Lakshmikantham *et al.* [41], Wu [54] and the papers [24].

In this paper, we present existence result for the problems (1)-(3) and (4)-(6). Our main result for this problem is based a multi-valued version of nonlinear alternative of Leray-Schauder type [21].

## 2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

By $L^1(J, \mathbb{R}^n)$ we denote the space of Lebesgue-integrable functions $u : J \to \mathbb{R}^n$ with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(t,x)\| dx dt,$$

where $\| \cdot \|$ denotes a suitable complete norm on $\mathbb{R}^n$.

**Definition 2.1**[52] Let $r = (r_1, r_2) \in (0, \infty) \times (0, \infty), \theta = (0, 0)$ and $u \in L^1(J, \mathbb{R}^n)$. The left-sided mixed Riemann-Liouville integral of order $r$ of $u$ is defined by

$$(I_\theta^r u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} u(s,\tau) d\tau ds.$$

In particular,

$$(I_\theta^\theta u)(t,x) = u(t,x), \ (I_\theta^\sigma u)(t,x) = \int_0^t \int_0^x u(s,\tau) d\tau ds; \text{ for almost all } (t,x) \in J,$$

where $\sigma = (1, 1)$.

For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0, \infty) \times (0, \infty)$, when $u \in L^1(J, \mathbb{R}^n)$. Note also that when $u \in C(J, \mathbb{R}^n)$, then $(I_\theta^r u) \in C(J, \mathbb{R}^n)$, moreover

$$(I_\theta^r u)(t, 0) = (I_\theta^r u)(0, x) = 0; \ (t, x) \in J.$$

**Example 2.2** Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0, \infty) \times (0, \infty)$, then

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2}, \text{ for almost all } (t,x) \in J.$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0,1) \times [0,1)$. Denote by $D_{tx}^2 := \frac{\partial^2}{\partial t \partial x}$, the mixed second order partial derivative.

**Definition 2.3**[52] Let $r \in (0,1] \times (0,1]$ and $u \in L^1(J, \mathbb{R}^n)$. The mixed fractional Riemann-Liouville derivative of order $r$ of $u$ is defined by the expression

$$D_\theta^r u(t,x) = (D_{tx}^2 I_\theta^{1-r} u)(t,x)$$

and the Caputo fractional-order derivative of order $r$ of $u$ is defined by the expression

$$({}^c D_0^r u)(t,x) = (I_\theta^{1-r} \frac{\partial^2}{\partial t \partial x} u)(t,x).$$

The case $\sigma = (1,1)$ is included and we have

$$(D_\theta^\sigma u)(t,x) = ({}^c D_\theta^\sigma u)(t,x) = (D_{tx}^2 u)(t,x), \text{ for almost all } (t,x) \in J.$$

**Example 2.4** Let $\lambda, \omega \in (-1, \infty)$ and $r = (r_1, r_2) \in (0,1] \times (0,1]$, then

$$D_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)} t^{\lambda-r_1} x^{\omega-r_2}, \text{ for almost all } (t,x) \in J.$$

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

**Lemma 2.5** [29] Let $\upsilon : J \to [0, \infty)$ be a real function and $\omega(.,.)$ be a nonnegative, locally integrable function on $J$. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that

$$\upsilon(t,x) \le \omega(t,x) + c \int_0^t \int_0^x \frac{\upsilon(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}} d\tau ds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\upsilon(t,x) \le \omega(t,x) + \delta c \int_0^t \int_0^x \frac{\omega(s,\tau)}{(t-s)^{r_1}(x-\tau)^{r_2}} d\tau ds,$$

for every $(t,x) \in J$.

## 3. Some Properties of Set-Valued Maps

Let $(X, \|\cdot\|)$ be a Banach space. Denote

- $\mathcal{P}(X) = \{Y \in X : Y \neq \emptyset\}$,
- $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$,
- $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$,
- $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$,
- $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact and convex}\}$.

For each $u \in C(J, \mathbb{R}^n)$, define the set of selections of $F$ by

$$S_{F,u} = \{f \in L^1(J, \mathbb{R}^n) : f(t,x) \in F(t,x,u(t,x)) \ a.e. \ (t,x) \in J\}.$$

Let $(X,d)$ be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b)=\inf\limits_{a\in A} d(a,b),\ d(a,B)=\inf\limits_{b\in B} d(a,b)$. Then $(\mathcal{P}_{b,cl}(X),H_d)$ is a metric space and $(\mathcal{P}_{cl}(X),H_d)$ is a generalized metric space (see [39]).

**Definition 3.1** A multivalued map $T: X\to\mathcal{P}(X)$ is convex(closed) valued if $T(x)$ is convex (closed) for all $x\in X$. $T$ is bounded on bounded sets if $T(B)=\bigcup\limits_{x\in B} T(x)$ is bounded in $X$ for all $B\in\mathcal{P}_b(X)$ (i.e. $\sup_{x\in B}\sup_{y\in T(x)}\|y\|<\infty$).

A multivalued map $T: X\to\mathcal{P}(X)$ is called upper semi-continuous (u.s.c.) on $X$ if for each $x_0\in X$, the set $T(x_0)$ is a nonempty closed subset of $X$, and if for each open set $N$ of $X$ containing $T(x_0)$, there exists an open neighborhood $N_0$ of $x_0$ such that $T(N_0)\subseteq N$. $T$ is lower semi-continuous (l.s.c.) if the set $\{x\in X: T(x)\cap A\neq\emptyset\}$ is open for any open subset $A\subseteq X$. $T$ is said to be completely continuous if $T(\mathcal{B})$ is relatively compact for every $\mathcal{B}\in\mathcal{P}_b(X)$. $T$ has a fixed point if there is $x\in X$ such that $x\in T(x)$. The fixed point set of the multivalued operator $T$ will be denoted by $FixT$.

A multivalued map $G: J\to\mathcal{P}_{cl}(\mathbb{R}^n)$ is said to be measurable if for every $v\in\mathbb{R}^n$, the function $(x)\longmapsto d(v,G(x))=\inf\{\|v-z\|: z\in G(x)\}$ is measurable.

**Lemma 3.2** [25] Let $G$ be a completely continuous multivalued map with nonempty compact values, then $G$ is u.s.c if and only if $G$ has a closed graph (i.e. $x_n\to x_*, y_n\to y_*, y_n\in G(x_n)$ imply $y_*\in G(x_*)$).

**Definition 3.3** A multivalued map $F: J\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ is said to be Carathéodory if

(i) $(t,x)\longmapsto F(t,x,u)$ is measurable for each $u\in\mathbb{R}^n$;
(ii) $u\longmapsto F(t,x,u)$ is upper semicontinuous for almost all $(t,x)\in J$.

$F$ is said to be $L^1$-Carathéodory if $(i),(ii)$ and the following condition holds;

(iii) for each $c>0$, there exists $\sigma_c\in L^1(J,\mathbb{R}_+)$ such that

$$\begin{aligned}\|F(t,x,u)\|_{\mathcal{P}} &= \sup\{\|f\|: f\in F(t,x,u)\}\\ &\leq \sigma_c(t,x)\ \text{ for all } \|u\|\leq c \text{ and for } a.e.\ (t,x)\in J.\end{aligned}$$

For more details on multivalued maps see the books of Aubin and Cellina [5], Aubin and Frankowska [6], Deimling [18], Gorniewicz [23], Hu and Papageorgiou [25] and Kisielewicz [39].

**Theorem 3.4** (Nonlinear alternative of Leray-Schauder type) [21] Let $X$ be a Banach space and $C$ a nonempty convex subset of $X$. Let $U$ a nonempty open subset of $C$ with $0\in U$ and $T:\overline{U}\to\mathcal{P}(C)$ an upper semicontinuous and compact multivalued operator. Then either

$(a)$ $T$ has a fixed points. Or
$(b)$ There exist $u\in\partial U$ and $\lambda\in[0,1]$ with $u\in\lambda T(u)$.

## 4. Existence Results for the Finite Delay Case

In this section, we give our main existence result for the problem (1)-(3).
For each $a,b>0$ we consider following set $C_{(a,b)}:=C([-\alpha,a]\times[-\beta,b],\mathbb{R}^n)$.
Let us start by defining what we mean by a solution of problem (1)-(3).

**Definition 4.1** A function $u\in C_{(a,b)}$ is said to be a solution of (1)-(3) if there exists a function $f\in L^1(J,\mathbb{R}^n)$ with $f(t,x)\in F(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))})$ such that $(^cD_0^r u)(t,x)=f(t,x)$ and $u$ satisfies equations (3) on $J$ and the condition (2) on $\tilde{J}$.

**Lemma 4.2** A function $u \in C_{(a,b)}$ is a solution of problem (1)-(3) if and only if $u$ satisfies the equation

$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)d\tau ds,$$

for all $(t,x) \in J$ and the condition (2) on $\tilde{J}$, where

$$z(t,x) = \varphi(t) + \psi(x) - \varphi(0).$$

Set $\mathcal{R} := \mathcal{R}_{(\rho_1^-,\rho_2^-)}$

$$= \{(\rho_1(s,\tau,u), \rho_2(s,\tau,u)) : (s,\tau,u) \in J \times C,\ \rho_i(s,\tau,u) \leq 0;\ i = 1,2\}.$$

We always assume that $\rho_1 : J \times C \to [-\alpha, a],\ \rho_2 : J \times C \to [-\beta, b]$ are continuous and the function $(s,\tau) \longmapsto u_{(s,\tau)}$ is continuous from $\mathcal{R}$ into $C$.

**Theorem 4.3** Assume the following hypotheses hold:

($H1$) $F : J \times \mathbb{R}^n \to \mathcal{P}_{cp,c}(\mathbb{R}^n)$ is a Carath?odory multi-valued map.

($H2$) There exist $p \in C(J, \mathbb{R}_+)$ and $\Psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$\|F(t,x,u)\|_{\mathcal{P}} \leq p(t,x)\Psi(||u||),\ \text{for } (t,x) \in J \text{ and each } u \in \mathbb{R}^n,$$

($H3$) There exists $\ell \in C(J, \mathbb{R}^+)$ such that

$$H_d(F(t,x,u).F(t,x,v)) \leq \ell(t,x)|u-v|, \text{ for any } u,v \in \mathbb{R}^n,$$

and

$$d(0,(F(t,x,0)) \leq \ell(t,x),\ \text{ a.e. } (t,x) \in J.$$

($H4$) There exists an numbre $M > 0$ such that

$$\frac{M}{||z||_\infty + \frac{\Psi(M)p^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}} > 1, \tag{7}$$

where $p^* = \sup_{(t,x)\in J} p(t,x)$.

Then the IVP (1)-(3) has at least one solution on $[-\alpha, a] \times [-\beta, b]$.

**Proof:** Transform the problem (1)-(3) into a fixed point problem. Consider the operators $N : C_{(a,b)} \to \mathcal{P}(C_{(a,b)})$ defined by,

$$(Nu)(t,x) = h \in C_{(a,b)}$$

such that

$$h(t,x) = \begin{cases} \phi(t,x), & (t,x) \in \tilde{J}, \\ z(t,x) & \\ +\dfrac{1}{\Gamma(r_1)\Gamma(r_2)} \displaystyle\int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)d\tau ds, & (t,x) \in J, \end{cases}$$

where $f \in S_{F,u_{(\rho_1(t,x,u),\rho_2(t,x,u))}}$.

We shall show that $N$ satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof will be given in several steps.

**Step 1:** $N(u)$ is convex for each $u \in C_{(a,b)}$. Indeed, if $h_1, h_2$ belong to $N(u)$, then there exist $f_1, f_2 \in S_{F,u}$ such that for each $(t,x) \in J$ we have

$$h_i(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f_i(s,\tau)d\tau ds, \quad i = 1,2.$$

Let $0 \le d \le 1$. Then, for each $(t,x) \in J$ we have

$$\begin{aligned}[dh_1 + (1-d)h_2](t,x) &= z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad \times [df_1(s,\tau) + (1-d)f_2(s,\tau)] d\tau ds,\end{aligned}$$

and for each $(t,x) \in \tilde{J}$, we have

$$[dh_1 + (1-d)h_2](t,x) = \phi(t,x).$$

Since $S_{F,u}$ is convex (because $F$ has convex values), we have

$$[dh_1 + (1-d)h_2] \in N(u).$$

**Step 2:** $N$ maps bounded sets into bounded sets in $C_{(a,b)}$. Let $B_\eta = \{u \in C_{(a,b)} : \|u\|_\infty \le \eta\}$ be bounded set in $C_{(a,b)}$ and $u \in B_\eta$. Then for each $h \in N(u)$, there exists $f \in S_{F,u}$ such that

$$h(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau) d\tau ds.$$

By $(H2)$ we have for each $(t,x) \in J$,

$$\begin{aligned}||h(t,x)|| &\le ||z(t,x)|| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} ||f(s,\tau)|| d\tau ds \\ &\le ||z(t,x)|| + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad \times p(s,\tau)\Psi(||u_{(s,\tau)}||) d\tau ds.\end{aligned}$$

Then

$$||h||_\infty \le ||z||_\infty + \frac{p^*\Psi(\eta)a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} := \ell_1.$$

In other hand, for each $(t,x) \in \tilde{J}$,

$$||h||_\infty \le ||\phi||_\infty := \ell_2.$$

Thus, for each $(t,x) \in [-\alpha, a] \times [-\beta, b]$,

$$||h||_\infty \le \min\{\ell_1, \ell_2\} := \ell.$$

**Step 3:** $N$ maps bounded sets into equicontinuous sets in $C_{(a,b)}$. Let $(t_1,x_1)$, $(t_2,x_2) \in J, t_1 < t_2$ and $x_1 < x_2$, $B_\eta$ be a bounded set of $C_{(a,b)}$ as in Step 2, let $u \in B_\eta$ and $h \in N(u)$, then

$$||h(t_2,x_2) - h(t_1,x_1)|| \le ||z(t_1,x_1) - z(t_2,x_2)||$$

$$
\begin{aligned}
&+ \quad \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_0^{x_1}[(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}-(t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}] \\
&\quad\ \times||f(s,\tau)||d\tau ds \\
&+ \quad \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau)||d\tau ds \\
&+ \quad \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau)||d\tau ds \\
&+ \quad \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_0^{x_1}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau)||d\tau ds \\
&\leq \quad \|z(t_1,x_1)-z(t_2,x_2)\|+\frac{p^*\psi(\eta)}{\Gamma(r_1+1)\Gamma(r_2+1)}[2x_2^{r_2}(t_2-t_1)^{r_1} \\
&\quad\ +2t_2^{r_1}(x_2-x_1)^{r_2}+t_1^{r_1}x_1^{r_2}-t_2^{r_1}x_2^{r_2}-2(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}].
\end{aligned}
$$

As $t_1 \to t_2$ and $x_1 \to x_2$, the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 < 0,\ x_1 < x_2 < 0$ and $t_1 \leq 0 \leq t_2,\ x_1 \leq 0 \leq x_2$ is obvious. As a consequence of Steps 1 to 3 together with Arzela-Ascoli theorem, we can conclude that $N : C_{(a,b)} \to \mathcal{P}(C_{(a,b)})$ is a completely continuous.

**Step 4:** $N$ has a closed graph. Let $u_n \to u_*, h_n \in N(u_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(u_*)$.
$h_n \in N(u_n)$ means that there exists $f_n \in S_{F,u_n}$ such that for each $(t,x) \in J$,

$$h_n(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}f_n(s,\tau)d\tau ds$$

and for $(t,x) \in \tilde{J}, h_n(t,x) = \phi(t,x)$.
We must show that there exists $f_* \in S_{F,u_*}$ such that for each $(t,x) \in J$

$$h_*(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}f_*(s,\tau)d\tau ds$$

and for $(t,x) \in \tilde{J}, h_*(t,x) = \phi(t,x)$.
Since $F(t,x,.)$ is upper semicontinuous, then for every $\varepsilon > 0$, there exist $n_0(\varepsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$f_n(t,x) \in F(t,x,u_{n(t,x)}) \subset F(t,x,u_{*(t,x)}) + \varepsilon B(0,1), \quad a.e. \quad (t,x) \in J.$$

Since $F(.,.,.)$ has compact values, then there exists a subsequence $f_{n_m}$ such that

$$f_{n_m}(.,.) \to f_*(.,.) \quad as \quad m \to \infty$$

and

$$f_*(.,.) \in F(t,x,u_{*(t,x)}), \quad a.e. \quad (t,x) \in J.$$

For every $w \in F(t,x,u_{*(t,x)})$, we have

$$|f_{n_m}(.,.) - f_*(t,x)| \leq |f_{n_m}(.,.) - w| + |w - f_*(t,x)|.$$

Then

$$|f_{n_m}(.,.) - f_*(.,.)| \leq d(f_{n_m}(.,.), F(t,x,u_{*(t,x)})).$$

By an analogous relation, obtained by interchanging the roles of $f_{n_m}$ and $f_*$, it follows that

$$
\begin{aligned}
|f_{n_m}(.,.) - u_{*(t,x)}| &\leq H_d(F(t,x,u_{n(t,x)}), F(t,x,u_{*(t,x)})) \\
&\leq \ell(t,x)||u_n - u_*||_\infty.
\end{aligned}
$$

Then

$$\begin{aligned}|h_n(t,x)-h_*(t,x)| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad \times|f_m(s,\tau)-f_*(s,\tau)|d\tau ds \\ &\leq \frac{\ell^*||u_{n_m}-u_*||_\infty}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds,\end{aligned}$$

where $\ell^* = \sup_{(t,x)\in J} \ell(t,x)$. Hence

$$||h_{n_m}-h_*||_\infty \leq \frac{a^{r_1}b^{r_2}\ell^*}{\Gamma(r_1+1)\Gamma(r_2+1)}||u_{n_m}-u_*||_\infty \to 0 \quad as \quad m\to\infty.$$

**Step 5: (A priori bounds)** Let $u$ be a possible solution of the problem (1)-(3). Then, there exists $f \in S_{F,u}$ such that, for each $(t,x)\in J$,

$$\begin{aligned}|u(t,x)| &\leq |z(t,x)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}|f(s,\tau)|d\tau ds \\ &\leq |z(t,x)| + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad \times p(s,\tau)\Psi(||u_{(s,\tau)}||)d\tau ds \\ &\leq |z(t,x)| + \frac{\Psi(||u_{(s,\tau)}||)}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}p(s,\tau)d\tau ds \\ &\leq ||z||_\infty + \frac{\Psi(||u||_\infty)p^*a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)},\end{aligned}$$

and for each $(t,x)\in\tilde{J}, |u(t,x)| = |\phi(t,x)|$. This implies by $(H2)$ that, for each $(t,x)\in J$, we have

$$\frac{||u||_\infty}{||z||_\infty + \frac{\Psi(||u||_\infty)p^*a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}} < 1.$$

Then by condition (7), there exists $M$ such that $||u||_\infty \neq M$.
Let

$$U = \{u\in C_{(a,b)} : ||u||_\infty < M^*\},$$

where $M^* = \min\{M, ||\phi||_C\}$. The operator $N : \overline{U} \to \mathcal{P}(C_{(a,b)})$ is upper semicontinuous and completely continuous. From the choice of $U$, there is no $u\in\partial U$ such that $u\in\lambda N(u)$ for some $\lambda\in(0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that $N$ has a fixed point $u\in\overline{U}$ which is a solution of the problem (1)-(3).

## 5. Existence Results for the Infinite Delay Case

5.1. **The phase space $\mathcal{B}$.** The notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato (see [24]). For further applications see for instance the books [27, 31, 41] and their references.

For any $(t,x)\in J$ denote $E_{(t,x)} := [0,t]\times\{0\}\cup\{0\}\times[0,x]$, furthermore in case $t=a,\ x=b$ we write simply $E$. Consider the space $(\mathcal{B}, \|(.,.)\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty,0]\times(-\infty,0]$ into $\mathbb{R}^n$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

($A_1$) If $y : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n$ continuous on $J$ and $y_{(t,x)} \in \mathcal{B}$, for all $(t,x) \in E$, then there are constants $H, K, M > 0$ such that for any $(t,x) \in J$ the following conditions hold:

($i$) $y_{(t,x)}$ is in $\mathcal{B}$;

($ii$) $\|y(t,x)\| \le H\|y_{(t,x)}\|_{\mathcal{B}}$,

($iii$) $\|y_{(t,x)}\|_{\mathcal{B}} \le K \sup_{(s,\tau)\in[0,t]\times[0,x]} \|y(s,\tau)\| + M \sup_{(s,\tau)\in E_{(t,x)}} \|y_{(s,\tau)}\|_{\mathcal{B}}$,

($A_2$) For the function $y(.,.)$ in ($A_1$), $y_{(t,x)}$ is a $\mathcal{B}$-valued continuous function on $J$.

($A_3$) The space $\mathcal{B}$ is complete.

Now, we present some examples of phase spaces [15, 16].

**Example 5.1.1** Let $\mathcal{B}$ be the set of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0]$, $\alpha, \beta \ge 0$, with the seminorm

$$\|\phi\|_{\mathcal{B}} = \sup_{(s,\tau)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,\tau)\|.$$

Then we have $H = K = M = 1$. The quotient space $\widehat{\mathcal{B}} = \mathcal{B}/\|.\|_{\mathcal{B}}$ is isometric to the space $C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n)$ of all continuous functions from $[-\alpha, 0] \times [-\beta, 0]$ into $\mathbb{R}^n$ with the supremum norm, this means that partial differential functional equations with finite delay are included in our axiomatic model.

**Example 5.1.2** Let $\gamma \in \mathbb{R}$ and let $C_\gamma$ be the set of all continuous functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ for which a limit $\lim_{\|(s,\tau)\|\to\infty} e^{\gamma(s+\tau)}\phi(s,\tau)$ exists, with the norm

$$\|\phi\|_{C_\gamma} = \sup_{(s,\tau)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(s+\tau)}\|\phi(s,\tau)\|.$$

Then we have $H = 1$ and $K = M = \max\{e^{-(a+b)}, 1\}$.

**Example 5.1.3** Let $\alpha, \beta, \gamma \ge 0$ and let

$$\|\phi\|_{CL_\gamma} = \sup_{(s,\tau)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,\tau)\| + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\gamma(s+\tau)}\|\phi(s,\tau)\|d\tau ds,$$

be the seminorm for the space $CL_\gamma$ of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0]$ measurable on $(-\infty, -\alpha] \times (-\infty, 0] \cup (-\infty, 0] \times (-\infty, -\beta]$, and such that $\|\phi\|_{CL_\gamma} < \infty$. Then

$$H = 1,\ K = \int_{-\alpha}^{0}\int_{-\beta}^{0} e^{\gamma(s+\tau)}d\tau ds,\ M = 2.$$

5.2. **Main Results.** Let us start in this section by defining what we mean by a solution of the problem (4)-(6). Let the space
$\Omega := \{u : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n : u_{(t,x)} \in \mathcal{B}$ for $(t,x) \in E$ and $u|_J$ is continuous $\}$.

**Definition 5.2.1** A function $u \in \Omega$ is said to be a solution of (4)-(6) if there exists a function $f \in L^1(J, \mathbb{R}^n)$ with $f(t,x) \in F(t, x, u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))})$ such that $({}^cD_0^r u)(t,x) = f(t,x)$ and $u$ satisfies equations (6) on $J$ and the condition (5) on $\tilde{J}$.

Set $\mathcal{R}' := \mathcal{R}'_{(\rho_1^-,\rho_2^-)}$

$$= \{(\rho_1(s,\tau,u), \rho_2(s,\tau,u)) : (s,\tau,u) \in J\times \mathcal{B},\ \ \rho_i(s,\tau,u) \le 0;\ i = 1,2\}.$$

We always assume that $\rho_1 : J\times\mathcal{B}\to(-\infty, a]$, $\rho_2 : J\times\mathcal{B}\to(-\infty, b]$ are continuous and the function $(s,\tau)\longmapsto u_{(s,\tau)}$ is continuous from $\mathcal{R}'$ into $\mathcal{B}$.

We will need to introduce the following hypothesis:

$(H_\phi)$ There exists a continuous bounded function $L : \mathcal{R}'_{(\rho_1^-,\rho_2^-)} \to (0,\infty)$ such that

$$\|\phi_{(s,\tau)}\|_{\mathcal{B}} \leq L(s,\tau)\|\phi\|_{\mathcal{B}}, \text{ for any } (s,\tau)\in\mathcal{R}'.$$

In the sequel we will make use of the following generalization of a consequence of the phase space axioms ([[26], Lemma 2.1]).

**Lemma 5.2.2** If $u\in\Omega$, then

$$\|u_{(s,\tau)}\|_{\mathcal{B}} = (M+L')\|\phi\|_{\mathcal{B}} + K \sup_{(\theta,\eta)\in[0,\max\{0,s\}]\times[0,\max\{0,\tau\}]} \|u(\theta,\eta)\|,$$

where

$$L' = \sup_{(s,\tau)\in\mathcal{R}'} L(s,\tau).$$

**Theorem 5.2.3** Assume $(H_\phi)$ and that the following hypotheses hold:

$(H1)$ $F : J\times\mathcal{B}\to\mathcal{P}_{cp}(\mathbb{R})$ is a Carath?odory multi-valued map.

$(H2)$ There exists $\ell\in L^\infty(J,\mathbb{R}^+)$ such that

$$H_d(F(t,x,u).F(t,x,v)) \leq \ell(t,x)||u-v||_{\mathcal{B}}, \text{ for every } u,v\in\mathcal{B},$$

and

$$d(0,(F(t,x,0)) \leq \ell(t,x), \text{ a.e. } (t,x)\in J.$$

Then the $IVP$ (4)-(6) has at least one solution on $(-\infty,a]\times(-\infty,b]$.

**Proof:** Transform the problem (4)-(6) into a fixed point problem. Consider the operator $A : \Omega\to\mathcal{P}(\Omega)$ defined by,

$$(Au)(t,x) = h\in\Omega$$

such that

$$h(t,x) = \begin{cases} \phi(t,x), & (t,x)\in\tilde{J}, \\ z(t,x) & \\ +\dfrac{1}{\Gamma(r_1)\Gamma(r_2)}\displaystyle\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}f(s,\tau)d\tau ds, \ f\in S_{F,u} & (t,x)\in J. \end{cases}$$

Let $v(.,.) : (-\infty,a]\times(-\infty,b]\to\mathbb{R}^n$ be a function defined by,

$$v(t,x) = \begin{cases} z(t,x), & (t,x)\in J. \\ \phi(t,x), & (t,x)\in\tilde{J}, \end{cases}$$

Then $v_{(t,x)} = \phi$ for all $(t,x)\in E$.

For each $w\in C(J,\mathbb{R}^n)$ with $w(t,x) = 0$ for each $(t,x)\in E$ we denote by $\overline{w}$ the function defined by

$$\overline{w}(t,x) = \begin{cases} w(t,x) & (t,x)\in J. \\ 0, & (t,x)\in\tilde{J}, \end{cases}$$

If $u(.,.)$ satisfies the integral equation,

$$u(t,x) = z(t,x) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}f(s,\tau)d\tau ds,$$

we can decompose $u(.,.)$ as $u(t,x) = \overline{w}(t,x) + v(t,x);\ (t,x) \in J$, which implies $u_{(t,x)} = \overline{w}_{(t,x)} + v_{(t,x)}$, for every $(t,x) \in J$, and the function $w(.,.)$ satisfies

$$w(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)d\tau ds,$$

where $f \in S_{F,\overline{w}+v}$. Set

$$C_0 = \{w \in C(J,\mathbb{R}^n):\ w(t,x) = 0 \ \text{ for } (t,x) \in E\},$$

and let $\|.\|_{(a,b)}$ be the seminorm in $C_0$ defined by

$$\|w\|_{(a,b)} = \sup_{(t,x)\in E} \|w_{(t,x)}\|_{\mathcal{B}} + \sup_{(t,x)\in J} \|w(t,x)\| = \sup_{(t,x)\in J} \|w(t,x)\|,\ w \in C_0.$$

$C_0$ is a Banach space with norm $\|.\|_{(a,b)}$. Let the operators $P:\ C_0 \to \mathcal{P}(C_0)$ defined by

$$(Pw)(t,x) = h \in C_0,$$

such that

$$h(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)d\tau ds,$$

where $f \in S_{F,\overline{w}_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}+v_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}}$. Obviously, that the operator $A$ has a fixed point is equivalent to $P$ has a fixed point.

**Step 1:** $P(w)$ is convex for each $w \in C_0$. Indeed, if $h_1, h_2$ belong to $P(w)$, then there exist $f_1, f_2 \in S_{F,\overline{w}+v}$ such that for each $(t,x) \in J$ we have

$$h_i(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f_i(s,\tau)d\tau ds, \quad i = 1,2.$$

Let $0 \le \xi \le 1$. Then, for each $(t,x) \in J$ we have

$$\begin{aligned}[\xi h_1 + (1-\xi)h_2](t,x) \quad &= \quad \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} \\ &\quad \times [\xi f_1(s,\tau) + (1-\xi)f_2(s,\tau)]d\tau ds,\end{aligned}$$

Since $S_{F,\overline{w}+v}$ is convex (because $F$ has convex values), we have

$$[\xi h_1 + (1-\xi)h_2] \in P(w).$$

**Step 2:** $P$ maps bounded sets into bounded sets in $C_0$. Indeed, it is enough to show that there exists a positive constant $\ell$ such that, for each $w \in B_\eta = \{w \in C_0 : ||w||_{(a,b)} \le \eta\}$, one has $||P(w)|| \le \tilde{\ell}$. Let $w \in B_\eta$ and $h \in P(w)$, then there exists $f \in S_{F,\overline{w}+v}$ such that, for each $(t,x) \in J$, we have

$$h(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} f(s,\tau)d\tau ds.$$

Then, for each $(t,x)\in J$,

$$\begin{aligned}||h(t,x)|| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}||f(s,\tau)||d\tau ds\\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell(s,\tau)\\ &\quad\times(1+||\overline{w}_{(s,\tau)}+v_{(s,\tau)}||_{\mathcal{B}})d\tau ds\\ &\leq \frac{\ell^*(1+\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1}d\tau ds\\ &\leq \frac{a^{r_1}b^{r_2}\ell^*(1+\eta^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}:=\tilde{\ell},\end{aligned}$$

where $\ell^*=\sup_{(t,x)\in J}\ell(t,x)$ and

$$\begin{aligned}\|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}} &\leq \|\overline{w}_{(s,\tau)}\|_{\mathcal{B}}+\|v_{(s,\tau)}\|_{\mathcal{B}}\\ &\leq K\eta+K\|\phi(0,0)\|+(M+L')\|\phi\|_{\mathcal{B}}=\eta^*.\end{aligned}$$

Hence $||P(w)||\leq\tilde{\ell}$.

**Step 3:** $P(B_\eta)$ is equicontinuous. Let $B_\eta$ as in Step 2 and let $(t_1,x_1),(t_2,x_2)\in J, t_1<t_2$ and $x_1<x_2$, let $w\in B_\eta$ and $h\in P(w)$, then there exists $f\in S_{F,\overline{w}+v}$ such that for each $(t,x)\in J$, we have

$$||h(t_2,x_2)-h(t_1,x_1)||=$$

$$\begin{aligned}&= \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_0^{x_1}[(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}-(t_1-s)^{r_1-1}(x_1-\tau)^{r_2-1}]\\ &\quad\times||f(s,\tau)||d\tau ds+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau)||d\tau ds\\ &\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{t_1}\int_{x_1}^{x_2}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau)||d\tau ds\\ &\quad+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{t_1}^{t_2}\int_0^{x_1}(t_2-s)^{r_1-1}(x_2-\tau)^{r_2-1}||f(s,\tau)||d\tau ds\\ &\leq \frac{\ell^*(1+\eta^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}[2x_2^{r_2}(t_2-t_1)^{r_1}\\ &\quad+2t_2^{r_1}(x_2-x_1)^{r_2}+t_1^{r_1}x_1^{r_2}-t_2^{r_1}x_2^{r_2}-2(t_2-t_1)^{r_1}(x_2-x_1)^{r_2}].\end{aligned}$$

As $t_1\to t_2$ and $x_1\to x_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with Arzela-Ascoli theorem, we can conclude that $P:C_0\to\mathcal{P}_{cp}(C_0)$ is a completely continuous.

**Step 4:** $P$ has a closed graph. Let $w_n\to w_*, h_n\in P(w_n)$ and $h_n\to h_*$. We need to show that $h_*\in P(w_*)$.
$h_n\in P(w_n)$ means that there exists $f_n\in S_{F,\overline{w}+v}$ such that for each $(t,x)\in J$,

$$h_n(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}f_n(s,\tau)d\tau ds.$$

We must show that there exists $f_*\in S_{F,\overline{w}+v}$ such that for each $(t,x)\in J$

$$h_*(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}f_*(s,\tau)d\tau ds.$$

Since $F(t,x,.)$ is upper semicontinuous, then for every $\varepsilon > 0$, there exist $n_0(\varepsilon) \geq 0$ such that for every $n \geq n_0$, we have

$$f_n(t,x) \in F(t,x,\overline{w}_{n(t,x)}+v_{(t,x)}) \subset F(t,x,\overline{w}_{*(t,x)}+v_{(t,x)})+\varepsilon B(0,1), \ \ a.e. \ (t,x) \in J.$$

Since $F(.,.,.)$ has compact values, then there exists a subsequence $f_{n_m}$ such that

$$f_{n_m}(.,.) \to f_*(.,.) \ \ as \ \ m \to \infty$$

and

$$f_*(t,x) \in F(t,x,\overline{w}_{*(t,x)}+v_{(t,x)}), \ \ a.e. \ \ (t,x) \in J.$$

Then for every $w \in F(t,x,\overline{w}_{(t,x)}+v_{(t,x)})$, we have

$$||f_{n_m}(t,x)-f_*(t,x)|| \leq ||f_{n_m}(t,x)-w||+||w-f_*(t,x)||.$$

Then

$$||f_{n_m}(t,x)-f_*(t,x)|| \leq d(f_{n_m}(t,x),F(t,x,\overline{w}_{*(t,x)}+v_{(t,x)})).$$

By an analogous relation, obtained by interchanging the roles of $f_{n_m}$ and $f_*$, it follows that

$$\begin{aligned} ||f_{n_m}(t,x)-f_*(t,x)|| &\leq H_d(F(t,x,\overline{w}_{n(t,x)}+v_{(t,x)}),F(t,x,\overline{w}_{*(t,x)}+v_{(t,x)})) \\ &\leq \ell(t,x)||\overline{w}_n-\overline{w}_*||_{\mathcal{B}}. \end{aligned}$$

Then

$$||h_{n_m}(t,x)-h_*(t,x)|| \leq$$

$$\begin{aligned} &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell(t,x)||\overline{w}_{n_m(s,\tau)}-\overline{w}_{*(s,\tau)}||d\tau ds \\ &\leq \frac{Ka^{r_1}b^{r_2}\ell^*}{\Gamma(r_1+1)\Gamma(r_2+1)}||\overline{w}_{n_m}-\overline{w}_*||_{(a,b)} \end{aligned}$$

Hence

$$||h_{n_m}-h_*||_{(a,b)} \leq \frac{Ka^{r_1}b^{r_2}\ell^*}{\Gamma(r_1+1)\Gamma(r_2+1)}||\overline{w}_{n_m}-\overline{w}_*||_{(a,b)} \to 0 \ \ as \ \ m \to \infty.$$

**Step 5: (A priori bounds)**. We now show there exists an open $U \subseteq C_0$ with $w \in \lambda P(w)$, for $\lambda \in (0,1)$ and $w \in \partial U$. Let $w \in \lambda P(w)$ for some $0 < \lambda < 1$. Thus there exists $f \in S_{F,\overline{w}_{(t,x)}+v_{(t,x)}}$ such that, for each $(t,x) \in J$,

$$w(t,x) = \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}f(s,\tau)d\tau ds.$$

This implies by $(H2)$ that, for each $(t,x) \in J$, we have

$$\begin{aligned} ||w(t,x)|| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}\ell(s,\tau) \\ &\quad \times(1+||\overline{w}_{(s,\tau)}+v_{(s,\tau)}||_{\mathcal{B}})d\tau ds \\ &\leq \frac{\ell^*a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &+ \frac{\ell^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-s)^{r_1-1}(x-\tau)^{r_2-1}||\overline{w}_{(s,\tau)}+v_{(s,\tau)}||_{\mathcal{B}}d\tau ds. \end{aligned}$$

But

$$\begin{aligned} \|\overline{w}_{(s,\tau)}+v_{(s,\tau)}\|_{\mathcal{B}} &\leq \|\overline{w}_{(s,\tau)}\|_{\mathcal{B}}+\|v_{(s,\tau)}\|_{\mathcal{B}} \\ &\leq K\sup\{w(\tilde{s},\tilde{\tau}) : (\tilde{s},\tilde{\tau}) \in [0,s]\times[0,\tau]\} \end{aligned}$$

$$+ (M + L')\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\|. \tag{8}$$

If we name $y(s,\tau)$ the right hand side of (8), then we have

$$\|\overline{w}_{(s,\tau)} + v_{(s,\tau)}\|_{\mathcal{B}} \leq y(t,x),$$

and therefore, for each $(t,x) \in J$ we obtain

$$\|w(t,x)\| \leq \frac{\ell^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}$$

$$+ \frac{\ell^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} y(s,\tau) d\tau ds. \tag{9}$$

Using the above inequality and the definition of $y$ for each $(t,x) \in J$, we have

$$\begin{aligned} y(t,x) \quad &\leq \quad (M+L')\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\| + \frac{K\ell^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &\quad + \frac{K\ell^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} y(s,t) d\tau ds. \end{aligned}$$

Then by Lemma 2, there exists $\delta = \delta(r_1, r_2)$ such that we have

$$\|y(t,x)\| \quad \leq \quad R + \delta \frac{K\ell^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-s)^{r_1-1}(x-\tau)^{r_2-1} R d\tau ds,$$

where

$$R = (M+L')\|\phi\|_{\mathcal{B}} + K\|\phi(0,0)\| + \frac{K\ell^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

Hence

$$\|y\|_\infty \leq R + \frac{R\delta K\ell^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} := \widetilde{R}.$$

Then, (9) implies that

$$\|w\|_\infty \leq \frac{\ell^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}(1+\widetilde{R}) := R^*.$$

Set

$$U = \{w \in C_0 : ||w||_{(a,b)} < R^* + 1\}.$$

The operator $P : \overline{U} \to C_0$ is continuous and completely continuous. From the choice of $U$, there is no $w \in \partial U$ such that $w \in \lambda P(w)$ for some $\lambda \in (0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type, we deduce that $P$ has a fixed point $w \in \overline{U}$ which is a solution of the problem (4)-(6).

## 6. Examples

**Example 6.1** As an application of our results we consider the following fractional differential inclusions with finite delay of the form

$$({}^cD_0^r u)(t,x) \in F(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}), \quad \text{a.e. } (t,x) \in J := [0,1]\times[0,1], \tag{10}$$

$$u(t,x) = t + x, \text{a.e. } (t,x) \in \tilde{J} := [-1,1]\times[-2,1]\backslash(0,1]\times(0,1], \tag{11}$$

$$u(t,0) = t, \ u(0,x) = x^2, \ (t,x) \in J, \tag{12}$$

where $r = (r_1, r_2)$ and $0 < r_1, r_2 \leq 1$.
Set

$$F(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}) =$$

$\{u \in \mathbb{R} : f_1(t, x, u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}) \leq u \leq f_2(t, x, u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))})\}$,
where $f_1, f_2 : J \times C(\tilde{J}, \mathbb{R}) \to \mathbb{R}$. We assume that for each $(t, x) \in J, f_1(t, x, \cdot)$ is lower semi-continuous (i.e, the set $\{u \in C(\tilde{J}, \mathbb{R}) : f_1(t, x, u) > \mu\}$ is open for each $\mu \in \mathbb{R}$) and assume that for each $(t, x) \in J, f_2(t, x, \cdot)$ is upper semi-continuous (i.e, the set $\{u \in C(\tilde{J}, \mathbb{R}) : f_2(t, x, u) < \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume that there are $p \in C(J, \mathbb{R}^+)$ and $\psi^* : [0, \infty) \to [0, \infty)$ continuous and nondecreasing such that

$$\max(|f_1(t, x, u)|, |f_2(t, x, u)|) \leq p(t, x)\psi^*(||u||),$$

for each $(t, x) \in J$ and all $u \in C(\tilde{J}, \mathbb{R})$.
It is clear that $F$ is compact and convex valued, and it is upper semi-continuous. Since all the conditions of Theorem 4.3 are satisfied, problem (10)-(12) has at least one solution $u$ on $[-1, 1] \times [-2, 1]$.

**Example 6.2** We consider now the following fractional differential inclusions with infinite delay of the form

$$({}^cD_0^r u)(t, x) \in F(t, x, u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}), \text{ if } (t, x) \in J := [0, 1]\times[0, 1], \quad (13)$$

$$u(t, x) = t + x, \text{if } (t, x) \in \tilde{J} := (-\infty, 1] \times (-\infty, 1]\backslash(0, 1] \times (0, 1], \quad (14)$$

$$u(t, 0) = t, \ u(0, x) = x^2, \ (t, x) \in J, \quad (15)$$

where $r = (r_1, r_2)$ and $0 < r_1, r_2 \leq 1$.Let $\gamma \geq 0$

$\mathcal{B}_\gamma = \{u \in C((-\infty, 0] \times (-\infty, 0], \mathbb{R}) : \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u(\theta, \eta) \text{ exists} \in \mathbb{R}\}$.
The norm of $\mathcal{B}_\gamma$ is given by

$$\|u\|_\gamma = \sup_{(\theta,\eta)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(\theta+\eta)}|u(\theta, \eta)|.$$

Let

$$E := [0, 1] \times \{0\} \cup \{0\} \times [0, 1],$$

and $u : (-\infty, 1] \times (-\infty, 1] \to \mathbb{R}$ such that $u_{(t,x)} \in \mathcal{B}_\gamma$ for $(t, x) \in E$, then

$$\begin{aligned} \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta+\eta)}u_{(t,x)}(\theta, \eta) &= \lim_{\|(\theta,\eta)\|\to\infty} e^{\gamma(\theta-t+\eta-x)}u(\theta, \eta) \\ &= e^{\gamma(t+x)} \lim_{\|(\theta,\eta)\|\to\infty} u(\theta, \eta) < \infty. \end{aligned}$$

Hence $u_{(t,x)} \in \mathcal{B}_\gamma$. Finally we prove that

$\|u_{(t,x)}\|_\gamma = K \sup\{|u(s, \tau)| : (s, \tau) \in [0, t]\times[0, x]\}+M \sup\{\|u_{(s,\tau)}\|_\gamma : (s, \tau) \in E_{(t,x)}\}$,
where $K = M = 1$ and $H = 1$,
If $t + \theta \leq 0, \ x + \eta \leq 0$ we get

$$\|u_{(t,x)}\|_\gamma = \sup\{|u(s, \tau)| : (s, \tau) \in (-\infty, 0] \times (-\infty, 0]\},$$

and if $t + \theta \geq 0, \ x + \eta \geq 0$ then we have

$$\|u_{(t,x)}\|_\gamma = \sup\{|u(s, \tau)| : (s, \tau) \in [0, t] \times [0, x]\}.$$

Thus for all $(t + \theta, x + \eta) \in [0, 1] \times [0, 1]$, we get

$$\begin{aligned} \|u_{(t,x)}\|_\gamma &= \sup\{|u(s, \tau)| : (s, \tau) \in (-\infty, 0] \times (-\infty, 0]\} \\ &\quad + \sup\{|u(s, \tau)| : (s, \tau) \in [0, t] \times [0, x]\}. \end{aligned}$$

Then

$$\|u_{(t,x)}\|_\gamma = \sup\{\|u_{(s,\tau)}\|_\gamma : (s, \tau) \in E\} + \sup\{|u(s, \tau) : (s, \tau) \in [0, t] \times [0, x]|\}.$$

$(\mathcal{B}_\gamma, \|.\|_\gamma)$ is a Banach space. We conclude that $\mathcal{B}_\gamma$ is a phase space.
Set

$$F(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}) =$$

$$\{u \in \mathbb{R} : f_1(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))}) \leq u \leq f_2(t,x,u_{(\rho_1(t,x,u_{(t,x)}),\rho_2(t,x,u_{(t,x)}))})\},$$

where $f_1, f_2 : J \times \mathcal{B}_\gamma \to \mathbb{R}$. We assume that for each $(t,x) \in J, f_1(t,x,\cdot)$ is lower semi-continuous (i.e, the set $\{u \in \mathcal{B}_\gamma : f_1(t,x,u) > \nu\}$ is open for each $\nu \in \mathbb{R}$) and assume that for each $(t,x) \in J, f_2(t,x,\cdot)$ is upper semi-continuous (i.e, the set $\{u \in \mathcal{B}_\gamma : f_2(t,x,u) < \nu\}$ is open for each $\mu \in \mathbb{R}$). Assume that there are $\ell \in L^\infty(J,\mathbb{R}_+)$ and $\psi : [0,\infty) \to [0,\infty)$ continuous and nondecreasing such that

$$\max(|f_1(t,x,u)|, |f_2(t,x,u)|) \leq \ell(t,x)\psi(||u||),$$

for each $(t,x) \in J$ and all $u \in \mathcal{B}_\gamma$.
It is clear that $F$ is compact and convex valued, and it is upper semi-continuous. Since all the conditions of Theorem 5.2.3 are satisfied, problem (13)-(15) has at least one solution defined on $(-\infty,1] \times (-\infty,1]$.

## References


[1] S. Abbas and M. Benchohra, Partial hyperbolic differential equations with finite delay involving the Caputo fractional derivative, *Commun. Math. Anal.* **7** (2) (2009), 62-72.

[2] S. Abbas and M. Benchohra, Darboux problem for perturbed partial differential equations of fractional order with finite delay, *Nonlinear Anal.: Hybrid Systems* **3** (2009), 597-604.

[3] R. P. Agarwal, M. Belmekki and M. Benchohra, A survey on semilinear differential equations and inclusions involving Riemann-Liouville fractional derivative. *Adv Differ. Equat.* **2009**(2009) Article ID 981728, 1-47.

[4] R.P Agarwal, M. Benchohra and S. Hamani, A survey on existence result for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta. Appl. Math.* **109** (3) (2010), 973-1033.

[5] J. P. Aubin and A. Cellina, *Differential Inclusions,* Springer-Verlag, Berlin-Heidelberg, New-York, 1984.

[6] J. P. Aubin and H. Frankowskas, *Set-Valued Analysis,* Birkhauser, Boston, 1990.

[7] A. Belarbi, M. Benchohra and A. Ouahab, Uniqueness results for fractional functional differential equations with infinite delay in Fréchet spaces, *Appl. Anal.* **85** (2006), 1459-1470.

[8] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific Publishing, New York, 2012.

[9] M. Benchohra, S. Hamani and S.K. Ntouyas, Boundary value problems for differential equations with fractional order, *Surv. Math. Appl.* **3** (2008), 1-12.

[10] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, Existence results for functional differential equations of fractional order, *J. Math. Anal. Appl.* **338** (2008),1340-1350.

[11] M. Benchohra and M. Hellal, Perturbed partial fractional order functional differential equations with Infinite delay in Fréchet spaces, *Nonlinear Dynamics and System Theory.* **14. 3**, (2014), 244-257.

[12] M. Benchohra and M. Hellal, A global uniqueness result for fractional partial hyperbolic differential equations with state-dependent delay, *Annales Polonici Mathemaici.* **110.3** (2014), 259-281.

[13] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics **580**, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[14] H. Covitz and S. B. Nadler Jr., Multivalued contraction mappings in generalized metric spaces, *Israel J. Math.* **8** (1970), 5-11.

[15] T. Czlapinski, On the Darboux problem for partial differential-functional equations with infinite delay at derivatives. *Nonlinear Anal.* **44** (2001), 389–398.

[16] T. Czlapinski, Existence of solutions of the Darboux problem for partial differential-functional equations with infinite delay in a Banach space. *Comment. Math. Prace Mat.* **35** (1995), 111–122.

[17] M. Dawidowski and I. Kubiaczyk, An existence theorem for the generalized hyperbolic equation $z''_{xy} \in F(x, y, z)$ in Banach space, *Ann. Soc. Math. Pol. Ser. I, Comment. Math.*, **30** (1) (1990), 41-49.

[18] K. Deimling *Multivalued Differential Equations,* Walter De Gruyter, Berlin-New York, 1992.

[19] K. Diethelm and A.D. Freed, On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity, in "Scientifice Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties" (F. Keil, W. Mackens, H. Voss, and J. Werther, Eds), pp 217-224, Springer- Verlag, Heidelberg, 1999.

[20] K. Diethelm and N. J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265** (2002), 229-248.

[21] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York 2003.

[22] L. Gaul, P. Klein and S. Kempfle, Damping description involving fractional operators, *Mech. Systems Signal Processing* **5** (1991), 81-88.

[23] L. Gorniewicz, Topological Fixed Point Theory of Multivalued Mappings, Mathematics and its Applications, 495, Kluwer Academic Publishers, Dordrecht, 1999.

[24] J. Hale and J. Kato, Phase space for retarded equationswith infinite delay, *Funkcial. Ekvac.* **21**, (1978),11-41.

[25] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Theory* **I**, Kluwer, Dordrecht, 1997.

[26] E. Hernández, A. Prokopczyk and L. Ladeira, A note on partial functional differential equations with state-dependent delay. *Nonlinear Anal. Real World Applications* **7** (2006), 510-519.

[27] J. K. Hale and S. Verduyn Lunel, *Introduction to Functional -Differential Equations*, Applied Mathematical Sciences, 99, Springer-Verlag, New York, 1993.

[28] J. K. Hale and S. Verduyn Lunel, *Introduction to Functional -Differential Equations*, Applied Mathematical Sciences, 99, Springer-Verlag, New York, 1993.

[29] D. Henry, *Geometric Theory of Semilinear Parabolic Partial Differential Equations*, Springer-Verlag, Berlin-New York, 1989.

[30] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[31] Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Infinite Delay,* in: Lecture Notes in Mathematics, 1473, Springer-Verlag, Berlin, 1991.

[32] Z. Kamont, *Hyperbolic Functional Differential Inequalities and Applications*. Kluwer Academic Publishers, Dordrecht, 1999.

[33] Z. Kamont, and K. Kropielnicka, Differential difference inequalities related to hyperbolic functional differential systems and applications. *Math. Inequal. Appl.* **8** (4) (2005), 655–674.

[34] A. A. Kilbas, B. Bonilla and J. Trujillo, Nonlinear differential equations of fractional order in a space of integrable functions, *Dokl. Ross. Akad. Nauk,* **374** (4) (2000), 445-449.

[35] A. A. Kilbas and S. A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, *Differential Equations* **41** (2005), 84-89.

[36] A. A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[37] V. Kolmanovskii, and A. Myshkis, *Introduction to the Theory and Applications of Functional-Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1999.

[38] A.A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[39] M. Kisielewicz, *Differential Inclusions and Optimal Control,* Kluwer, Dordrecht, The Netherlands, 1991.

[40] V. Lakshmikantham, S. Leela and J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[41] V. Lakshmikantham, L. Wen and B. Zhang, *Theory of Differential Equations with Unbounded Delay*, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 1994.

[42] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity. An introduction to mathematical models.* Imperial College Press, London, 2010.

[43] F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, *J. Chem. Phys.* **103** (1995), 7180-7186.

[44] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.
[45] K.B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, London, 1974.
[46] S. G. Pandit, Monotone methods for systems of nonlinear hyperbolic problems in two independent variables, *Nonlinear Anal.*, **30** (1997), 235-272.
[47] I. Podlubny, I. Petraš, B. M. Vinagre, P. O'Leary and L. Dorčak, Analogue realizations of fractional-order controllers. fractional order calculus and its applications, *Nonlinear Dynam.* **29** (2002), 281-296.
[48] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.
[49] N. P. Semenchuk, On one class of differential equations of noninteger order, *Differents. Uravn.*, **10** (1982), 1831-1833.
[50] V. E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Heidelberg; Higher Education Press, Beijing, 2010.
[51] A. N. Vityuk, Existence of Solutions of partial differential inclusions of fractional order, *Izv. Vyssh. Uchebn. , Ser. Mat.* , **8** (1997), 13-19.
[52] A. N. Vityuk and A. V. Golushkov, Existence of solutions of systems of partial differential equations of fractional order, *Nonlinear Oscil.* **7** (3) (2004), 318-325.
[53] A.N. Vityuk, and A.V. Mikhailenko, On a class of fractional-order differential equations. *Nonlinear Oscil.* **11** (2008), 307–319.
[54] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer-Verlag, New York, 1996.

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