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AN IMPULSIVE FRACTIONAL FUNCTIONAL BOUNDARY VALUE PROBLEM

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Abstract. In this article a mathematical model is presented for a non-instantaneous impulsive fractional functional boundary valued problem and concerned with the existence results of solution for considered model. Under the classical Caputo's derivative and more general conditions on model, existence results are obtained with the help of classical fixed point techniques on a arbitrary Banach space. At last, an application is provided to illustrate the existence results.

1. Introduction

The topic of differential equations with non-integer order has recently come out as a notable field of dynamical research due to it has extensive development area and found a lot of applications in several disciplines and various fields of science and engineering such as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electrodynamics of complex medium, visco-elasticity, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, etc. There are several type of qualitative properties such as existence, uniqueness, stability, etc. of solution for these models. To study these properties there are some remarkable monographs and the papers [1, 2, 3, 4, 5] which provide the main theoretical tools.

Differential equations with delay arise in the remote control, implicit functional differential equation like Wheeler-Feynman equations and in structured populations model which involve threshold phenomena etc. Delay differential equation has an important role in the modelling of scientific problems. Therefore, the existence and uniqueness results of solution of delay equations have been studied by several authors [6, 7, 8, 9, 10, 11, 12].

Integral boundary conditions have several apps in modelling of science and technology problems special in fluid mechanics like blood flow problems, underground water flow, unsteady biomedical computational fluid dynamics and other field of applied mathematics such as population dynamics, chemical engineering, thermo-elasticity, finite element method approaches with the minimization of constitutive

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error etc. A point of central importance in the study of nonlinear integral boundary value problems is to understand how the properties of nonlinearity in a problem influence the nature of the solutions. For a detail description of the integral boundary conditions, we refer to reader some papers [13, 14, 15] and the references therein.

Many practical dynamical systems generally represent in form of impulsive fractional differential equations which include the evolutionary processes that characterized by abrupt changes of the state at certain instants. In present, the theory about the impulsive fractional differential equations have received great attention and committed to many applications in medicine, mechanical, engineering, biology, ecology etc. However, it seems that the classical models with instantaneous impulses can not characterize the dynamics of evolution processes in pharmacotherapy. For example, consider the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. In fact, this situation is characterized by a new type of impulsive action, which starts at an arbitrary fixed point and stays active on a finite time interval. These impulse known as non-instantaneous impulse. There are few work available on this topic, we refer the papers [16, 17, 18, 19, 20, 21] for update theory of this topic. It is well known that the non-instantaneous impulsive effects are very important in control system.

In our previous paper [22, 23], we established the existence, uniqueness and continuous dependence of solution and mild solution for class of an abstract nonlocal fractional functional integro-differential equations with state dependent delay subject to non-instantaneous impulse with the help of fixed point theorems in a complex Banach spaces under the strong condition on nonlinear term. Author's [21] investigate periodic BVP for integer/fractional order nonlinear differential equations with non-instantaneous impulses and obtained the existence and uniqueness results under different conditions via fixed point technique as Banach contraction map, Krasnoselskii's theorem. Very recently, author's [24, 25] prove the existence of bounded solutions of a new class of retarded functional equation on an unbounded domain and Caputo fractional differential equations with non-instantaneous impulses.

We consider the following neutral fractional functional boundary value problem with non-instantaneous impulse

$$ {}_{t_0}^{C}D_t^{\alpha}[Q(y(t))] = J_t^{2-\alpha} f(t, y_t),\ t \in (s_i, t_{i+1}] \subset [t_0, T],\ i = 0, 1, \ldots, m, \tag{1.1} $$

$$ y(t) = \phi(t), t \in [-d, t_0]; ay'(t_0) + by'(T) = c \int_{t_0}^{T} \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)} y(s)ds, \tag{1.2} $$

$$ y(t) = g_i(t, y(t)); y'(t) = q_i(t, y(t)),\ t \in (t_i, s_i],\ i = 1, 2, \ldots, m, \tag{1.3} $$

where ${}_{t_0}^{C}D_t^{\alpha}$ denotes the classical Caputo's fractional derivative of order $\alpha \in (1, 2)$ with non zero lower bound and $J_t^{2-\alpha}$ denotes the Riemann-Liouville fractional integral operator of order $2 - \alpha > 0$. The neutral term $Q(y(t))$ is defined as

$$ Q(y(t)) = y(t) + \int_{t_0}^{t} (t-s)^{\beta} h(s, y_t) ds,\ \beta > 0, $$

and the state function $y(t)$ belong to a complex Banach space $(\mathbb{X}, \|\cdot\|)$. Functions f; $h : (s_i, t_{i+1}] \times PC_0 \to X$ are given for all $i = 0, 1, \ldots, m$, and g_i; $q_i : (t_i, s_i] \times X \to X$ are given for all $i = 1, 2, \ldots, m$. The history function $y_t : [-d, t_0] \to X$ is defined as $y_t(s) = y(t+s), s \in [-d, t_0]$ and $\phi(t)$ belong to the space PC_0. Numbers $a; b; c; \gamma \in \mathbb{R}$ are constants and $0 < \gamma < 2; a \neq 0$. y' denotes the ordinary derivative of y with respect to t. Here $[t_0, T]$ denotes the operational interval such that $t_0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_m \leq s_m \leq t_{m+1} = T$ are pre-fixed numbers.

Stimulate from the spoken of earlier work and by survey, it is found that there is no work available on neutral fractional differential equations with impulse. So, we consider the problem (1.1) with a kind of integral boundary conditions and non-instantaneous impulsive effects, which is untouch topic yet in the literature. Main motto to study the problem (1.1)-(1.3) comes from physics which arise in modeling of underground water flow that is very useful model in fluid dynamics.

In the present paper, we first establish a standard framework to derive a suitable formula of solutions for fractional boundary problem with non-instantaneous impulses which inspire the researcher to study existence and others qualitative properties like periodicity, stability, oscillations of solution. In this context, we apply the fixed point theorems, like Banach, Schauder's on a generalized complete Banach space to study the existence results under the weak conditions on nonlinear term. Finally, an example is given to illustrate existence and uniqueness result.

2. Background and Preliminaries

Let $(X, \|\cdot\|_X)$ be a arbitrary complex Banach space equipped with the norm $\|y\|_X = \sup_{t\in J}\{|y(t)| : y \in X\}$ and $C([-d, t_0], X)$ (with $[-d, t_0] \subset \mathbb{R}$) is the space formed by all the continuous functions defined on $[-d, t_0]$ to X, endowed with the norm

$$\|y(t)\|_{C([-d,t_0],X)} = \sup_{t\in[-d,t_0]} \{\|y(t)\|_X : y \in C([-d, t_0], X)\}.$$

In case of impulse conditions, let $NPC([-d, T]; X)$ be a Banach space of all functions $y : [-d, T] \to X$, which are continuous on $[t_0, T]$ except for a finite number of points $s_i \in (t_0, T)$, at which $y(s_i^+) = \lim_{\epsilon\to 0} y(s_i + \epsilon)$ and $y(s_i^-) = y(s_i) = \lim_{\epsilon\to 0} y(s_i - \epsilon)$ exist for $i = 1, 2, \ldots, N$, and endowed with the norm

$$\|y\|_{NPC} = \sup_{t\in[-d,T]} \{\|y(t)\|_X : y \in NPC([-d, T]; X)\}.$$

Further, let $NPC^1([-d, T]; X)$ be a Banach space of all such functions $y : [-d, T] \to X$, which are continuously differentiable on $[t_0, T]$ except for a finite number of points $t_i \in (t_0, T)$ at which $y'(s_i^+) = \lim_{\epsilon\to 0} y'(s_i + \epsilon)$ and $y'(s_i^-) = y'(s_i) = \lim_{\epsilon\to 0} y'(s_i - \epsilon)$ exist for $i = 1, 2, \ldots, N$, and endowed with the norm

$$\|y\|_{NPC^1} = \sup_{t\in[-d,T]} \sum_{j=0}^{1} \left\{\|y^j(t)\|_X : y \in NPC^1([-d, T]; X)\right\}.$$

In in following lemma, we derive a formula of solution for non-instantaneous impulsive neutral fractional integral boundary problem for linear case.

Lemma 2.1. *A function $y(t)$ is a solution of the following fractional integral boundary value problem*

$${}^C_{t_0}D_t^\alpha [y(t) + h(t)] = J_t^{2-\alpha} f(t), \alpha \in (1, 2), t \in (s_i, t_{i+1}],\ i = 0, 1, \ldots, m, \quad (2.1)$$

$$y(t) = \phi(t), t \in [-d, t_0];\ ay'(t_0) + by'(T) = c \int_{t_0}^{T} q(s)ds, \quad (2.2)$$

$$y(t) = g_i(t);\ y'(t) = q_i(t),\ t \in (t_i, s_i],\ i = 1, 2, \ldots, m, \quad (2.3)$$

if $y(t)$ is a solution of following the fractional integral equation

$$y(t)=\begin{cases}\phi(t_0)+(t-t_0)c_1-h(t)+\int_{t_0}^{t}(t-s)f(s)ds, & t\in(t_0,t_1]\\ g_i(t), & t\in(t_i,s_i],\\ c_2+(t-s_i)c_3-h(t)+\int_{s_i}^{t}(t-s)f(s)ds, & t\in(s_i,t_{i+1}],\end{cases} \tag{2.4}$$

where constants $c_1; c_2$ and c_3 are as follow

$$\begin{aligned} c_1 &= \frac{c}{a}\int_{t_0}^{T}q(s)ds+\frac{b}{a}h'(T)+h(0)-\frac{b}{a}q_m(s_m)-\frac{b}{a}h'(s_m)-\frac{b}{a}\int_{s_m}^{T}f(s)ds,\\ c_2 &= g_i(s_i)+h(s_i);\;\; c_3=q_i(s_i)+h'(s_i),\;,\; i=1,2,\dots,m. \end{aligned}$$

Proof. If $t\in(t_0,t_1]$ then by using Riemann-Liouvill fractional integral operator on Eq. (2.1) we get

$$y(t)+h(t)=b_0+c_1t+\int_{t_0}^{t}(t-s)f(s)ds, \tag{2.5}$$

using the initial condition $y(t_0)=\phi(t_0)$, then Eq. (2.5) we have

$$y(t)+h(t)=\phi(t_0)+h(t_0)+c_1t+\int_{t_0}^{t}(t-s)f(s)ds. \tag{2.6}$$

If $t\in(t_1,s_1]$, then the solution of Eq. (2.3) will be

$$y(t)=g_1(t).$$

If $t\in(s_1,t_2]$, then again by using Riemann- Liouvill fractional integral operator on Eq. (2.1) we get

$$y(t)+h(t)=e_2+e_3(t-s_1)+\int_{s_1}^{t}(t-s)f(s)ds. \tag{2.7}$$

By the impulsive conditions $y(s_1)=g_1(s_1);\;\; y'(s_1)=q_1(s_1)$, Eq. (2.7) becomes

$$y(t)+h(t)=g_1(s_1)+h(s_1)+(t-s_1)q_1(s_1)+(t-s_1)h'(s_1)+\int_{s_1}^{t}(t-s)f(s)ds.$$

Now, for the general subinterval if $t\in(t_i,s_i],\; i=1,2,\dots,m$, the solution of Eq. (2.3) will be

$$y(t)=g_i(t),\quad i=1,2,\dots,m, \tag{2.8}$$

and again for the subinterval $t\in(s_i,t_{i+1}],\; i=1,2,\dots,m$, and using Riemann-Liouvill fractional integral operator on Eq. (2.1) we get

$$y(t)+h(t)=c_2+c_3(t-s_i)+\int_{s_i}^{t}(t-s)f(s)ds, \tag{2.9}$$

by the impulsive conditions $y(s_i)=g_i(s_i);\;\; y'(s_i)=q_i(s_i)$ the Eq. (2.9) becomes

$$y(t)+h(t)=g_i(s_i)+h(s_i)+(t-s_i)q_i(s_i)+(t-s_i)h'(s_i)+\int_{s_i}^{t}(t-s)f(s)ds. \tag{2.10}$$

Now, differentiation of Eq. (2.6) and Eq. (2.10) with respect to t, we get

$$y'(t)+h'(t)=c_1+\int_{0}^{t}f(s)ds, \tag{2.11}$$

$$y'(t)+h'(t)=q_i(s_i)+h'(s_i)+\int_{s_i}^{t}f(s)ds. \tag{2.12}$$

Using the boundary condition $ay'(t_0)+by'(T)=c\int_{t_0}^{T}q(s)ds$ and by Eq. (2.11) and Eq. (2.12)

$$c_1=\frac{c}{a}\int_{t_0}^{T}q(s)ds+\frac{b}{a}h'(T)+h(t_0)-\frac{b}{a}q_m(s_m)-\frac{b}{a}h'(s_m)-\frac{b}{a}\int_{s_m}^{T}f(s)ds. \quad (2.13)$$

It is obvious that Eq. (2.6), Eq. (2.8) and Eq. (2.13) gives the result of Eq. (2.4). □

Remark 2.2. *We assume that the functions f, h, g_i, q_i are smooth enough, such that the boundary value problem (2.1)-(2.3) has a solution under some sufficient condition.*

Now, we are going to present the definition of solution for the problem (1.1)-(1.3) based on cited research paper [16].

Definition 2.3. *A function $y(t)$ is a solution of the problem (1.1)-(1.3) if $y(t)$ is a solution of following the fractional integral equation*

$$y(t)=\begin{cases}\phi(t_0)+(t-t_0)C_1-\int_{t_0}^{t}(t-s)^{\beta}h(s,y_t)ds+\int_{t_0}^{t}(t-s)f(s,y_t)ds, & t\in(t_0,t_1],\\ g_i(t,y(t)), & t\in(t_i,s_i],\\ C_2+(t-s_i)C_3-\int_{0}^{t}(t-s)^{\beta}h(s,y_t)ds+\int_{s_i}^{t}(t-s)f(s,y_t)ds, & t\in(s_i,t_{i+1}],\end{cases}$$

where constants $C_1; C_2$ and C_3 are as follow

$$\begin{aligned}C_1 &= \frac{c}{a}\int_{t_0}^{T}\frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}y(s)ds+\frac{b}{a}\int_{t_0}^{T}\beta(T-s)^{\beta-1}h(s,y_t)ds-\frac{b}{a}q_m(s_m)\\ &\quad -\frac{b}{a}\int_{t_0}^{s_m}\beta(s_m-s)^{\beta-1}h(s,y_t)ds-\frac{b}{a}\int_{s_m}^{T}f(s,y_t)ds,\\ C_2 &= g_i(s_i,y(s_i))+\int_{t_0}^{s_i}(s_i-s)^{\beta}h(s,y_t)ds,\\ C_3 &= q_i(s_i,y(s_i))+\int_{t_0}^{s_i}\beta(s_i-s)^{\beta-1}h(s,y_t)ds.\end{aligned}$$

Theorem 2.4. *(Banach fixed point theorem) Let $\mathcal{C}$ be a closed subset of a Banach space X and let $\mathcal{J}$ be a contraction mapping from $\mathcal{C}$ in to $\mathcal{C}$. i.e.*

$$\|\mathcal{J}(y)-\mathcal{J}(z)\|\le\delta\|y-z\| \ \ \forall\, y,z\in\mathcal{C};\ 0<\delta<1.$$

Then there exists a unique $z\in\mathcal{J}$ such that $\mathcal{J}(z)=z$.

Theorem 2.5. *(Schauder fixed point theorem) Let* B *be a nonempty closed convex subset of a Banach space X, and let $\mathcal{K}$ be a continuous map with a compact image from* B *to* B, *then $\mathcal{K}$ has a fixed point.*

Theorem 2.6. *(Ascoli-Arzela Theorem) Let $\mathcal{L}$ be a class of continuous functions defined over some interval J. Then $\mathcal{L}$ is relatively compact iff $\mathcal{L}$ is equi-continuous and uniformly bounded.*

3. Existence Results

To established the existence and uniqueness results of the model problem (1.1)-(1.3), we assume that functions $g_i\,; q_i$ are constants at the impulse moments t_i and further, we have the following basic and weak assumptions:

(A_1) $f;h:(s_i,t_{i+1}]\times C([-d,t_0],X)\to X;\; g:(t_i,s_i]\times C([-d,t_0],X)\to X$ are jointly continuous functions and there exist $p_1,u_1,r_1\in(1,\alpha)$ and real functions $l_1(t)\in L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+), v_1(t)\in L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+), w_1(t)\in L^{\frac{1}{r_1}}((t_i,s_i],\mathbb{R}^+)$ such that

$$\|f((s_i,t_{i+1}],\varphi)\|_X\le l_1(t);\; \|h((s_i,t_{i+1}],\varphi)\|_X\le v_1(t),\;\; t\in(s_i,t_{i+1}],\;\; \varphi\in C([-d,t_0],X),$$
$$\|g((t_i,s_i],\varphi)\|_X\le w_1(t),\;\; t\in(t_i,s_i],\;\; \varphi\in C([-d,t_0],X).$$

(A_2) There exist $p_2,u_2,r_2\in(1,\alpha)$ and real functions $l_2(t)\in L^{\frac{1}{p_2}}((s_i,t_{i+1}],\mathbb{R}^+), v_2(t)\in L^{\frac{1}{u_2}}((s_i,t_{i+1}],\mathbb{R}^+), w_2(t)\in L^{\frac{1}{r_2}}((t_i,s_i],\mathbb{R}^+)$ such that

$$\|f((s_i,t_{i+1}],\varphi)-f((s_i,t_{i+1}],\psi)\|_X\le l_2(t)\|\varphi-\psi\|_{C([-d,t_0],X)},\;\; t\in(s_i,t_{i+1}],\;\; \varphi\in C([-d,t_0],X),$$
$$\|h((s_i,t_{i+1}],\varphi)-h((s_i,t_{i+1}],\psi)\|_X\le v_2(t)\|\varphi-\psi\|_{C([-d,t_0],X)},\;\; t\in(s_i,t_{i+1}],\;\; \varphi\in C([-d,t_0],X),$$
$$\|g((t_i,s_i],\varphi)-g((t_i,s_i],\psi)\|_X\le w_2(t)\|\varphi-\psi\|_{C([-d,t_0],X)},\;\; t\in(t_i,s_i],\;\; \varphi\in C([-d,t_0],X).$$

Theorem 3.1. *Let the assumption A_1 hold and let $\mathcal{B}(r)$ be a nonempty closed convex subset of a Banach space X. Let $\mathcal{T}:\mathcal{B}(r)\to\mathcal{B}(r)$ be a mapping such that $\mathcal{T}(y)=y$ and defines as*

$$\mathcal{T}(y)=\begin{cases}\phi(t_0)+(t-t_0)C_1-\int_{t_0}^t(t-s)^\beta h(s,y_t)ds+\int_{t_0}^t(t-s)f(s,y_t)ds, & t\in(t_0,t_1],\\ g_i(t,y(t)), & t\in(t_i,s_i],\\ C_2+(t-s_i)C_3-\int_0^t(t-s)^\beta h(s,y_t)ds+\int_{s_i}^t(t-s)f(s,y_t)ds, & t\in(s_i,t_{i+1}].\end{cases}\tag{3.1}$$

Then $\mathcal{T}$ is well defined and $\mathcal{T}(y)\subseteq\mathcal{B}$.

Proof. Let us consider the polynomial p that satisfies the conditions of Eq. (1.2)-(1.3) and is defined as

$$p(t)=\begin{cases}\phi(t_0)+(t-t_0)C_1, & t\in(t_0,t_1],\\ 0, & t\in(t_i,s_i],\\ C_2+(t-s_i)C_3, & t\in(s_i,t_{i+1}],\end{cases}$$

and the set $\mathcal{B}(r)=\{y\in NPC_T^1:\|y-p\|_X\le r\}$. It is evident that $\mathcal{B}(r)$ is a closed and convex subset of the Banach space of NPC_T^1. Since the polynomial p is an element of $\mathcal{B}(r)$, so it is nonempty set. It is obvious by assumption A_1 that f,h are continuous functions. Therefore, this implies that $\mathcal{T}$ is well defined map on $\mathcal{B}(r)$. Next, we show that $\mathcal{T}(y)\subseteq\mathcal{B}(r)$.

Let $y(t),p(t)\in\mathcal{B}(r)$ and for $t\in(t_0,t_1]$, we have

$$\begin{aligned}\|\mathcal{T}y(t)-p(t)\|_X &\le \int_{t_0}^t(t-s)^\beta\|h(s,y_t)\|_X ds+\int_{t_0}^t(t-s)\|f(s,y_t)\|_X ds\\ &\le \frac{T^{\beta+1}}{\beta+1}\|v_1\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)}+\frac{T^2}{2}\|l_1\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)}.\end{aligned}$$

For the $t\in(t_i,s_i]$, we obtain

$$\|\mathcal{T}y(t)-p(t)\|_X \le \|w_1\|_{L^{\frac{1}{p_1}}((t_i,s_i],\mathbb{R}^+)}.$$

For $t\in(s_i,t_{i+1}]$, we have

$$\begin{aligned}\|\mathcal{T}y(t)-p(t)\|_X &\le \int_{t_0}^t(t-s)^\beta h(s,y_t)ds+\int_{s_i}^t(t-s)f(s,y_t)ds\\ &\le \frac{T^{\beta+1}}{\beta+1}\|v_1\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)}+\frac{T^2}{2}\|l_1\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)}.\end{aligned}$$

Gathering the results for operational intervals, we get

$$\|\mathcal{T}y(t)-p(t)\|_X \leq$$
$$\max\left\{\frac{T^{\beta+1}}{\beta+1}\|v_1\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)}+\frac{T^2}{2}\|l_1\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)},\|w_1\|_{L^{\frac{1}{p_1}}((t_i,s_i],\mathbb{R}^+)}\right\}\leq r.$$

This implies that $\mathcal{T}(y)\subseteq\mathcal{B}(r)$. The proof is now completed. □

Theorem 3.2. *Let the assumptions A_1 A_2 hold and there exists a constant $\Pi<1$. Then problem (1.1)-(1.3) has unique solution.*

Proof. Consider the operator $\mathcal{T}:\mathcal{B}(r)\to\mathcal{B}(r)$ defined in Theorem 3.1 by the Eq. (3.1) as

$$\mathcal{T}(y)=\begin{cases}\phi(t_0)+(t-t_0)C_1-\int_{t_0}^t(t-s)^\beta h(s,y_t)ds+\int_{t_0}^t(t-s)f(s,y_t)ds, & t\in(t_0,t_1],\\ g_i(t,y(t)), & t\in(t_i,s_i],\\ C_2+(t-s_i)C_3-\int_{t_0}^t(t-s)^\beta h(s,y_t)ds+\int_{s_i}^t(t-s)f(s,y_t)ds, & t\in(s_i,t_{i+1}].\end{cases}\quad(3.2)$$

We prove that $\mathcal{T}$ is a contraction. For this, let $y,y^*\in\mathcal{B}(r)$.

First, for $t\in(t_0,t_1]$, we have

$$\begin{aligned}\|\mathcal{T}y-\mathcal{T}y^*\|_X &\leq \int_{t_0}^t(t-s)^\beta\|h(s,y_t)-h(s,y_t^*)\|_X ds+\int_{t_0}^t(t-s)\|f(s,y_t)-f(s,y_t^*)\|_X ds\\ &\quad \|v_1\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)}\|y-y^*\|_X\int_{t_0}^t(t-s)^\beta ds\\ &\quad +\|l_1\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)}\|y-y^*\|_X\int_{t_0}^t(t-s)ds\\ &\quad +\frac{T^{\beta+1}}{\beta+1}\|v_1\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)}\|y-y^*\|_X+\frac{T^2}{2}\|l_1\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)}\|y-y^*\|_X\\ &\leq \left\{\frac{T^{\beta+1}}{\beta+1}\|v_1\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)}+\frac{T^2}{2}\|l_1\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)}\right\}\|y-y^*\|_X.\end{aligned}$$

Now for the $t\in(t_i,s_i]$, we have

$$\|\mathcal{T}y-\mathcal{T}y^*\|_X \leq \|w_1\|_{L^{\frac{1}{p_1}}((t_i,s_i],\mathbb{R}^+)}\|y-y^*\|_X.$$

Finally for $t\in(s_i,t_{i+1}]$, we have

$$\begin{aligned}\|\mathcal{T}y-\mathcal{T}y^*\|_X &\leq \int_{t_0}^t(t-s)^\beta\|h(s,y_t)-h(s,y_t^*)\|_X ds+\int_{s_i}^t(t-s)\|f(s,y_t)-f(s,y_t^*)\|_X ds\\ &\quad \|v_1\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)}\|y-y^*\|_X\int_{t_0}^t(t-s)^\beta ds\\ &\quad +\|l_1\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)}\|y-y^*\|_X\int_{s_i}^t(t-s)ds\\ &\quad +\frac{T^{\beta+1}}{\beta+1}\|v_1\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)}\|y-y^*\|_X+\frac{T^2}{2}\|l_1\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)}\|y-y^*\|_X\\ &\leq \left\{\frac{T^{\beta+1}}{\beta+1}\|v_1\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)}+\frac{T^2}{2}\|l_1\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)}\right\}\|y-y^*\|_X.\end{aligned}$$

For the operational intervals, let

$$\Pi = \left\{ \frac{T^{\beta+1}}{\beta+1} \|v_1\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)} + \frac{T^2}{2} \|l_1\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)}, \|w_1\|_{L^{\frac{1}{p_1}}((t_i,s_i],\mathbb{R}^+)} \right\}.$$

Gathering above results, we get

$$\|\mathcal{T}y - \mathcal{T}y^*\|_X \leq \Pi \|y - y^*\|_X.$$

Since $\Pi < 1$, therefore the operator $\mathcal{T}$ is a contraction and hence by the Theorem 2.4 there exists a unique fixed point which is the unique solution of problem (1.1)-(1.3). The proof is now completed. □

Now, if relax the assumption A_2, then problem (1.1)-(1.3) lost the uniqueness property of the solution. In this case to show the existence result of the problem (1.1)-(1.3) we apply the Theorem 2.5.

Theorem 3.3. *Let the assumption A_1 hold. Then problem (1.1)-(1.3) has at-least one solution.*

Proof. Consider the operator $\mathcal{T} : \mathcal{B}(r) \to \mathcal{B}(r)$ defined in Theorem 3.1 by the Eq. (3.1) as

$$\mathcal{T}(y) = \begin{cases} \phi(t_0) + (t-t_0)C_1 - \int_{t_0}^t (t-s)^\beta h(s,y_t)ds + \int_{t_0}^t (t-s) f(s,y_t)ds, & t \in (t_0,t_1], \\ g_i(t,y(t)), & t \in (t_i,s_i], \\ C_2 + (t-s_i)C_3 - \int_{t_0}^t (t-s)^\beta h(s,y_t)ds + \int_{s_i}^t (t-s) f(s,y_t)ds, & t \in (s_i,t_{i+1}]. \end{cases} \quad (3.3)$$

Now, to prove our desire existence result, we have to show that $\mathcal{T}(y) = y$ has a fixed point. Our first target in this step is to show that $\mathcal{T}$ is continuous. To this end, consider a convergent sequence y^n which converge to y in $\mathcal{B}(r)$.

Primarily for $t \in (t_0,t_1]$, we have

$$\begin{aligned} \|\mathcal{T}(y^n) - \mathcal{T}(y)\|_X \leq & \int_{t_0}^t (t-s)^\beta \|h(s,y_t^n) - h(s,y_t)\|_X ds \\ & + \int_{t_0}^t (t-s) \|f(s,y_t^n) - f(s,y_t)\|_X ds. \end{aligned} \quad (3.4)$$

Secondly for the $t \in (t_i,s_i]$ we obtain

$$\|\mathcal{T}(y^n) - \mathcal{T}(y)\|_X \leq \|g_i(t,y^n(t)) - g_i(t,y(t))\|_X. \quad (3.5)$$

Now for the interval $t \in (s_i,t_{i+1}]$,

$$\begin{aligned} \|\mathcal{T}(y^n) - \mathcal{T}(y)\|_X = & \int_{t_0}^t (t-s)^\beta \|h(s,y_t^n) - h(s,y_t)\|_X ds \\ & + \int_{s_i}^t (t-s) \|f(s,y_t^n) - f(s,y_t)\|_X ds. \end{aligned} \quad (3.6)$$

It is clear that, the function f, h, g are continuous, and by the dominant convergent theorem, the expressions on the right-hand side of (3.4),(3.5) and (3.6) converges to 0 as y^n converge to y. Which proves that $\mathcal{T}$ is a continuous operator. Our next target is to show that the space $\mathcal{T}(\mathcal{B}) = \{\mathcal{T}(y) : y \in \mathcal{B}\}$ is a relatively compact. For this, first we show that $\mathcal{T}(\mathcal{B})$ is uniformly bounded. Let $\bar{y} \in \mathcal{T}(\mathcal{B})$.

For $t \in (t_0,t_1]$, we have

$$\|\bar{y}\| \leq \|\mathcal{T}y(t)\|_X \leq \|\phi(t_0)\|_X + T\|C_1\|_X$$

$$\begin{aligned} &+\int_{t_0}^{t}(t-s)^{\beta}\|h(s,y_t)\|_X ds+\int_{t_0}^{t}(t-s)\|f(s,y_t)\|_X ds \\ \leq\ & \|\phi(t_0)\|_X+T\|C_1\|_X+\frac{T^{\beta+1}}{\beta+1}\|v_1\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)} \\ &+\frac{T^2}{2}\|l_1\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)}, \end{aligned}$$

where

$$\begin{aligned} \|C_1\|_X &= \frac{c}{a}\int_{t_0}^{T}\frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)}y(s)ds+\frac{b}{a}\int_{t_0}^{T}\beta(T-s)^{\beta-1}\|h(s,y_t)\|_X ds-\frac{b}{a}q_m(s_m) \\ &\quad -\frac{b}{a}\int_{t_0}^{s_m}\beta(s_m-s)^{\beta-1}\|h(s,y_t)\|_X ds-\frac{b}{a}\int_{s_m}^{T}\|f(s,y_t)\|_X ds, \\ &= \frac{rcT^{\gamma}}{a\Gamma(\gamma+1)}+\frac{2b\beta T^{\beta}}{a\beta}\|v_1(t)\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)}+\frac{b}{a}\|\omega_1(t)\|_{L^{\frac{1}{\kappa_1}}((s_i,t_{i+1}],\mathbb{R}^+)} \\ &\quad +\frac{bT}{a}\|l_1(t)\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)}. \end{aligned}$$

For the $t\in(t_i,s_i]$ we have

$$\|\bar{y}\|\leq\|\mathcal{T}y(t)\|_X \leq \|w_1\|_{L^{\frac{1}{r_1}}((t_i,s_i],\mathbb{R}^+)}.$$

At last for the $t\in(s_i,t_{i+1}]$, we have

$$\begin{aligned} \|\bar{y}\|\leq\|\mathcal{T}y(t)\|_X &\leq \|C_2\|_X+|(t-s_i)|\|C_3\|_X+\int_{t_0}^{t}(t-s)^{\beta}h(s,y_t)ds+\int_{s_i}^{t}(t-s)f(s,y_t)ds \\ &\leq \|C_2\|_X+T\|C_3\|_X+\frac{T^{\beta+1}}{\beta+1}\|v_1\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)}+\frac{T^2}{2}\|l_1\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)}, \end{aligned}$$

where

$$\begin{aligned} \|C_2\|_X &= \|g_i(s_i,y(s_i))\|_X+\int_{t_0}^{s_i}(s_i-s)^{\beta}\|h(s,y_t)\|_X ds, \\ &= \|w_1(t)\|_{L^{\frac{1}{r_1}}((t_i,s_i],\mathbb{R}^+)}+\frac{T^{\beta+1}}{\beta+1}\|v_1(t)\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)}, \\ \|C_3\|_X &= \|q_i(s_i,y(s_i))\|_X+\int_{t_0}^{s_i}\beta(s_i-s)^{\beta-1}\|h(s,y_t)\|_X ds \\ &= \|\omega_1(t)\|_{L^{\frac{1}{\kappa_1}}((t_i,s_i],\mathbb{R}^+)}+\frac{\beta T^{\beta}}{\beta}\|v_1(t)\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)}. \end{aligned}$$

Gathering results for the operational intervals, we get

$$\begin{aligned} \|\bar{y}\|\leq\|\mathcal{T}y(t)\|_X &\leq \max\{\|\phi(t_0)\|_X+T\|C_1\|_X,\|C_2\|_X+T\|C_3\|_X,\|w_1\|_{L^{\frac{1}{r_1}}((t_i,s_i],\mathbb{R}^+)}\} \\ &\quad +\frac{T^{\beta+1}}{\beta+1}\|v_1\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)}+\frac{T^2}{2}\|l_1\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)}, \\ \|\bar{y}\|\leq\|\mathcal{T}y(t)\|_X &\leq \mathcal{C}^*. \end{aligned}$$

This implies that $\mathcal{T}(\mathcal{B})$ has the uniformly bounded-ness property. Next, we show $\mathcal{T}(\mathcal{B})$ is a family of equi-continuous functions. This property can be derived as fellow.

Let $x_1, x_2 \in (t_0, t_1]$ such that $x_1 < x_2$, then we have

$$\begin{aligned}
\|\mathcal{T}(y)(x_2) - \mathcal{T}(y)(x_1)\|_X &\leq C_1|x_2 - x_1| + \int_{x_1}^{x_2} (x_2 - s)\|f(s, y_t)\|_X ds \\
&\quad + \int_{x_1}^{x_2} (x_2 - s)^\beta \|h(s, y_t)\|_X ds \\
&\quad + \int_{t_0}^{x_1} [(x_2 - s)^\beta - (x_1 - s)^\beta]\|h(s, y_t)\|_X ds \\
&\quad + \int_{t_0}^{x_1} [(x_2 - s) - (x_1 - s)]\|f(s, y_t)\|_X ds \\
&\leq C_1|x_2 - x_1| + \|l_1\|_{L^{\frac{1}{p_1}}((s_i, t_{i+1}], \mathbb{R}^+)} \left(\frac{x_2^2}{2} - \frac{x_1^2}{2}\right) \\
&\quad + \|v_1\|_{L^{\frac{1}{u_1}}((s_i, t_{i+1}], \mathbb{R}^+)} \left(\frac{x_2^{\beta+1}}{\beta+1} - \frac{x_1^{\beta+1}}{\beta+1}\right). \qquad (3.7)
\end{aligned}$$

For the $x_1, x_2 \in (t_i, s_i]$, we obtain

$$\|\mathcal{T}(y)(x_2) - \mathcal{T}(y)(x_1)\|_X \leq \|g_i(x_2, y(x_2)) - g_i(x_1, y(x_1))\|_X. \qquad (3.8)$$

Similarly, for $x_1, x_2 \in (s_i, t_{i+1}]$ we have

$$\begin{aligned}
\|\mathcal{T}(y)(x_2) - \mathcal{T}(y)(x_1)\|_X &\leq C_3|x_2 - x_1| + \int_{x_1}^{x_2} (x_2 - s)\|f(s, y_t)\|_X ds \\
&\quad + \int_{x_1}^{x_2} (x_2 - s)^\beta \|h(s, y_t)\|_X ds \\
&\quad + \int_{t_0}^{x_1} [(x_2 - s)^\beta - (x_1 - s)^\beta]\|h(s, y_t)\|_X ds \\
&\quad + \int_{s_i}^{x_1} [(x_2 - s) - (x_1 - s)]\|f(s, y_t)\|_X ds \\
&\leq C_3|x_2 - x_1| + \|l_1\|_{L^{\frac{1}{p_1}}((s_i, t_{i+1}], \mathbb{R}^+)} \left(\frac{(x_2 - s_i)^2}{2} - \frac{(x_1 - s_i)^2}{2}\right) \\
&\quad + \|v_1\|_{L^{\frac{1}{u_1}}((s_i, t_{i+1}], \mathbb{R}^+)} \left(\frac{(x_2)^{\beta+1}}{\beta+1} - \frac{(x_1)^{\beta+1}}{\beta+1}\right). \qquad (3.9)
\end{aligned}$$

Noting that the expressions on the right-hand side of (3.7), (3.8) and (3.9) are independent of y and x_1 and x_2, proving the equi-continuity of $\mathcal{T}(\mathcal{B})$. In either case the Theorem 2.6 yields that $\mathcal{T}(\mathcal{B})$ is relatively compact, and hence Theorem 2.5 asserts that $\mathcal{T}$ has a fixed point. By construction, a fixed point of $\mathcal{T}$ is a solution of our boundary value problem (1.1)-(1.3). The proof is now completed. □

Remark 3.4. *If we take $l_1(t), v_1(t), w_1(t), l_2(t), v_2(t), w_2(t)$ as a constants, then conditions A_1, A_2 reduces to the Osgood condition and become simple and easy.*

4. Example

Here a numerical example is presented to verify the existence and uniqueness results:

$$ {}_0^C D_t^\alpha [Q(y(t))] = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \left[\frac{y(s-d)\sin s^2}{16(1+e^{s^2})(1+|y(s-d)|)}\right] ds, $$

$$t \in (0, \frac{1}{2}] \cup (\frac{1}{2}, T], \quad (4.1)$$

$$y(t) = \phi(t), t \in [-d, 0], ay'(0) + by'(T) = c\int_0^T \frac{(T-s)^{\gamma-1}}{\Gamma(\gamma)} y(s)ds, \quad (4.2)$$

$$y(t) = \frac{1}{3}\sin y(t) + e^t,\ t \in (0, \frac{1}{2}];\ y'(t) = \frac{1}{3}\cos y(t) + e^t,\ t \in (\frac{1}{2}, T], (4.3)$$

where ${}^C D_t^\alpha$ denotes the classical Caputo's derivative, $0 < t_i = \frac{1}{2} < T, i = 1$ and $Q(y(t))$ is defined as

$$Q(y(t)) = y(t) + \int_0^t (t-s)\left[\frac{y(s-d)}{25(1+e^{s^2})(1+|y(s-d)|)}\right] ds.$$

Let $y(t) \in NPC_t^1$ and set the following function as

$$f(t,\varphi) = \frac{\varphi \sin t^2}{16(1+e^{t^2})(1+|\varphi|)};\ h(t,\varphi) = \frac{\varphi}{25(1+e^{t^2})(1+|\varphi|)};$$

$$g_i(t,y) = \frac{1}{3}\sin y(t) + e^t, q_i(t,y) = \frac{1}{3}\cos y(t) + e^t.$$

By simple computations, we can show that

$$\begin{aligned} \|f(t,\varphi) - f(t,\phi)\| &= \|\frac{\varphi \sin t^2}{16(1+e^{t^2})(1+|\varphi|)} - \frac{\phi \sin t^2}{16(1+e^{t^2})(1+|\phi|)}\| \\ &\le \|\frac{\sin t^2}{16(1+e^{t^2})}\| \times \|\frac{\varphi}{(1+|\varphi|)} - \frac{\phi}{(1+|\phi|)}\| \\ &\le \frac{1}{32}\|\varphi - \phi\|. \end{aligned}$$

Similarly, by same computations, we have

$$\|h(t,\varphi) - h(t,\phi)\| \le \frac{1}{50}\|\varphi - \phi\|;\ \|g_i(t,\varphi) - g_i(t,\phi)\| \le \frac{1}{3}\|\varphi - \phi\|.$$

It is obvious that the functions $f; h; g$ are followed the conditions of A_2 with $l_1 = \frac{1}{32}, v_1 = \frac{1}{50}, w_1 = \frac{1}{3}$. If we take $T = 1\ \beta = 1$, then we can calculate

$$\begin{aligned} \Pi &= \max\left\{\frac{T^{\beta+1}}{\beta+1}\|v_1\|_{L^{\frac{1}{u_1}}((s_i,t_{i+1}],\mathbb{R}^+)} + \frac{T^2}{2}\|l_1\|_{L^{\frac{1}{p_1}}((s_i,t_{i+1}],\mathbb{R}^+)}, \|w_1\|_{L^{\frac{1}{r_1}}((t_i,s_i],\mathbb{R}^+)}\right\} \\ &\approx 0.33 < 1. \end{aligned}$$

Thus, our first result can be applied to the problem (4.1)-(4.3), i.e., problem (4.1)-(4.1) has a unique solution.

Further, it is clear from the problem that $f; h; g$ are continuous functions and

$$\|f(t,\varphi)\| \le \frac{1}{32}; \|h(t,\varphi)\| \le \frac{1}{50}; \|g_i(t,\varphi)\| \le \frac{1}{3}.$$

Thus the conditions of Theorem 2.5 are satisfied which implies that problem (4.1)-(4.3) has at least one solution.

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