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SOME INCLUSION RELATIONS ASSOCIATED WITH GENERALIZED FRACTIONAL INTEGRAL OPERATOR

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Abstract. In this paper a known family of generalized fractional integral operator is used here to define some new subclasses of analytic function in the open unit disk U. For each of these new function classes, several inclusion relationships are established.

1. Introduction and definitions

Let $\mathbb{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$. If $f \in \mathbb{A}$ is given by (1) and $g \in \mathbb{A}$ is given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in $z \in U$, then the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let $P_k(\alpha)$ denotes the class of functions $h(z)$ analytic in the unit disk U satisfying the properties $h(0) = 1$ and

$$\int_0^{2\pi} \left| Re\left(\frac{h(z) - \alpha}{1 - \alpha}\right) \right| d\theta \leq k\pi \qquad (z = re^{i\theta}; \quad 0 \leq \alpha < 1; \quad k \geq 2). \tag{2}$$

This class $P_k(\alpha)$ has been introduced in [7]. Note that for $\alpha = 0$, we obtain the class P_k defined and studied in [8] and for $k = 2$, we have the class $P(\alpha)$ of functions with positive real part greater than α. In particular, $P(0)$ is the class P

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of functions with positive real part. From (2), we can easily deduce that $h \in P_k(\alpha)$ if and only if

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z), \quad h_1, h_2 \in P(\alpha). \tag{3}$$

Following the recent investigation [4] (see also [6], [9]), we have the following subclasses:

$$R_k(\alpha) = \{f \in A : \frac{zf'(z)}{f(z)} \in P_k(\alpha), z \in U\}, \tag{4}$$

$$V_k(\alpha) = \{f \in A : \frac{(zf'(z))'}{f'(z)} \in P_k(\alpha), z \in U\}, \tag{5}$$

$$P'_k(\alpha) = \{f \in A : f'(z) \in P_k(\alpha), z \in U\}, \tag{6}$$

$$T_k(\beta, \alpha) = \{f \in A : g \in R_2(\alpha) \text{ and } \frac{zf'(z)}{g(z)} \in P_k(\beta), z \in U\}, \tag{7}$$

$$T^*_k(\beta, \alpha) = \{f \in A : g \in V_2(\alpha) \text{ and } \frac{(zf'(z))'}{g'(z)} \in P_k(\beta), z \in U\}. \tag{8}$$

We note that the class $R_2(\alpha) = S^*(\alpha)$ and $V_2(\alpha) = k(\alpha)$ are respectively, the subclasses of $\mathbb{A}$ consisting of functions which are starlike of order α and convex of order α in U. The class $T^*_2(\beta, \alpha) = C^*(\beta, \alpha)$ was considered by Noor [2] and $T^*_2(0, 0) = C^*$ is the class of quasi-convex univalent functions which was first introduced and studied in [3]. It can be easily seen from the above definition that

$$f(z) \in V_k(\alpha) \Leftrightarrow zf'(z) \in R_k(\alpha), \tag{9}$$

and

$$f(z) \in T^*_k(\beta, \alpha) \Leftrightarrow zf'(z) \in T_k(\beta, \alpha). \tag{10}$$

For $\lambda > 0,\ \mu, \eta \in R$ and $min\{\lambda + \eta, -\mu + \eta, -\mu\} > -2$, Srivastava et al. [14] introduced a family of *fractional integral operators*

$$J^{\lambda,\mu,\eta}_{0,z} f(z) : \mathbb{A} \to \mathbb{A},$$

defined by

$$J^{\lambda,\mu,\eta}_{0,z} f(z) = \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\eta)}{\Gamma(2-\mu+\eta)} z^{\mu}\ I^{\lambda,\mu,\eta}_{0,z} f(z)\ , \tag{11}$$

where $I^{\lambda,\mu,\eta}_{0,z}$ is the *hypergeometric fractional integral operator* due to Saigo [13],:

$$I^{\lambda,\mu,\eta}_{0,z} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1}\ {}_2F_1\left(\lambda+\mu, -\eta; \lambda; (1-\frac{t}{z})\right) f(t)dt\ . \tag{12}$$

Here the ${}_2F_1$ - function in the kernel of (12) is the *Gauss Hypergeometric function*, the function $f(z)$ is analytic in a simply-connected region of the complex z-plane containing the origin, with the order

$$f(z) = 0(|z|^{\varepsilon})\ \ (z \to 0; \varepsilon > max\{0, \mu - \eta\} - 1)\,,$$

and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $log(z-t)$ to be real when $(z-t) > 0$.

If $f(z) \in \mathbb{A}$ is of the form (1), than the fractional integral operator $J_{0,z}^{\lambda,\mu,\eta}$ has the form

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = z + \frac{\Gamma(2-\mu)\Gamma(2+\lambda+\eta)}{\Gamma(2-\mu+\eta)} \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\mu+\eta+1)}{\Gamma(n-\mu+1)\Gamma(n+\lambda+\eta+1)} a_n z^n. \tag{13}$$

It is easily verified from (13) that

$$z\left(J_{0,z}^{\lambda+1,\mu,\eta} f(z)\right)' = (\lambda+\eta+2)\ J_{0,z}^{\lambda,\mu,\eta} f(z) - (\lambda+\eta+1)\ J_{0,z}^{\lambda+1,\mu,\eta} f(z). \tag{14}$$

Using the generalized fractional integral operator $J_{0,z}^{\lambda,\mu,\eta}$, we now define the following subclasses of $\mathbb{A}$:

- Let $f(z) \in \mathbb{A}$. Then $f(z) \in R^{\lambda,\mu,\eta}(k,\alpha)$ if and only if $J_{0,z}^{\lambda,\mu,\eta} f(z) \in R_k(\alpha)$, for $z \in U$.
- Let $f(z) \in \mathbb{A}$. Then $f(z) \in V^{\lambda,\mu,\eta}(k,\alpha)$ if and only if $J_{0,z}^{\lambda,\mu,\eta} f(z) \in V_k(\alpha)$, for $z \in U$.
- Let $f(z) \in \mathbb{A}$. Then $f(z) \in T^{\lambda,\mu,\eta}(k,\beta,\alpha)$ if and only if $J_{0,z}^{\lambda,\mu,\eta} f(z) \in T_k(\beta,\alpha)$, for $z \in U$.
- Let $f(z) \in \mathbb{A}$.Then $f(z) \in T_*^{\lambda,\mu,\eta}(k,\beta,\alpha,)$ if and only if $J_{0,z}^{\lambda,\mu,\eta} f(z) \in T_k^*(\beta,\alpha)$, for $z \in U$.
- Let $f(z) \in \mathbb{A}$. Then $f(z) \in P_k'(\lambda,\mu,\eta,\alpha)$ if and only if $J_{0,z}^{\lambda,\mu,\eta} f(z) \in P_k'(\alpha)$, for $z \in U$.

In this paper we establish some inclusion relationships and some other interesting properties for these subclasses.

2. Main inclusion relationships

We recall first the following necessary lemmas:

Lemma 1.([1]) Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let $\phi(u,v)$ be a complex-valued function satisfying the conditions:

(i) $\phi(u,v)$ is continuous in $D \subset C^2$

(ii) $(1,0) \in D$ and $Re\ \phi(1,0) > 0$,

(iii) $Re\ \phi(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1+u_2^2)$.

If $h(z) = 1 + \sum_{n=2}^{\infty} c_n z^n$, is a function analytic in U such that $(h(z), zh'(z)) \in D$ and

$Re(\phi(h(z), zh'(z)) > 0$, for $z \in U$, then $Re\ (h(z)) > 0$ for $z \in U$.

Lemma 2. ([11]) Let $p(z)$ be analytic in U with $p(0) = 1$ and $Re\ \{p(z)\} > 0$, $z \in U$. Then,for $s > 0$ and $\eta_1 \neq -1$ (complex),

$$\Re\left(p(z) + \frac{szp'(z)}{p(z)+\eta_1}\right) > 0, \qquad for\ \ |z| < r_0$$

where r_0 is given by $r_0 = \frac{|1+\eta_1|}{\sqrt{m+(m^2-|\eta_1^2-1|)^{\frac{1}{2}}}}$, $m = 2(s+1)^2 + |\eta_1|^2 - 1$ and this result is best possible.

Lemma 3. ([10]) Let ϕ be convex and g be starlike in U. Then for F analytic in U with $F(0) = 1$, $\frac{\Psi *\ Fg}{\Psi * g}$ is contained in the convex hull of $F(U)$. By convex hull of a set X, we mean the intersection of all convex sets that contain X.

Lemma 4. ([12]) Let p is analytic in E with $p(0) = 1$, and λ is a complex number satisfying $Re(\lambda) \geq 0, (\lambda \neq 0)$, then $Re\ [p(z) + \lambda z p'(z)] > \beta$, $(0 \leq \beta < 1)$ implies $Re\ \{p(z)\} > \{\beta + (1-\beta)(2\gamma - 1)\}$ where γ is given by

$$\gamma = \int_0^1 (1 + t^{\ Re\lambda})^{-1} dt,$$

which is an increasing function of $Re\ (\lambda)$ and $\frac{1}{2} \leq \gamma \leq 1$. The estimate is sharp in the sense that bound cannot be improved.

Our first main inclusion relationship is given by the theorem below.

Theorem 1. *Let* $f \in \mathbb{A}, \lambda > 0,\ 0 \leq \alpha < 1,\ \lambda + \eta > -7/8$ *and min* $\{-\mu + \eta, -\mu\} > -2$. *Then*

$$R^{\lambda,\mu,\eta}(k,\alpha) \subset R^{\lambda+1,\mu,\eta}(k,\alpha_1) \ , \tag{15}$$

where

$$\alpha_1 = \frac{2(2\alpha\lambda + 2\alpha\eta + 2\alpha + 1)}{(2\lambda + 2\eta - 2\alpha + 3) + \sqrt{4(\lambda + \eta + \alpha + 1)^2 + 4(\lambda + \eta - \alpha + 1) + 9}} \tag{16}$$

and $\quad 0 \leq \alpha < \alpha_1 < 1$.

Proof. Let $f \in R^{\lambda,\mu,\eta}(k,\alpha)$. Then upon setting

$$\frac{z\left(J_{0,z}^{\lambda+1,\mu,\eta} f(z)\right)'}{J_{0,z}^{\lambda+1,\mu,\eta} f(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \quad (z \in U) \ , \tag{17}$$

we see that the function $p(z)$ is analytic in U, with $p(0) = 1$ in $z \in U$. Using identity (14) in (17) and differentiating with respect to z, we get

$$\frac{z\left(J_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{J_{0,z}^{\lambda,\mu,\eta} f(z)} = \left(p(z) + \frac{zp'(z)}{\lambda + \eta + 1 + p(z)}\right) \in P_k(\alpha) \quad (z \in U) \ .$$

Let

$$\phi(z) = \sum_{j=1}^{\infty} \frac{\lambda + \eta + 1 + j}{\lambda + \eta + 2} z^j$$

then, by convolution technique(see [5]), we have

$$p(z) * \frac{\phi(z)}{z} = p(z) + \frac{zp'(z)}{p(z) + \lambda + \eta + 1}$$

$$= \left(\frac{k}{4} + \frac{1}{2}\right)\left(p_1(z) * \frac{\phi(z)}{z}\right) - \left(\frac{k}{4} - \frac{1}{2}\right)\left(p_2(z) * \frac{\phi(z)}{z}\right)$$

and this implies that

$$\left(p_i(z) + \frac{zp_i'(z)}{p_i(z) + \lambda + \eta + 1}\right) \in P(\alpha) \qquad (z \in U,\ i = 1,2) \ . \tag{18}$$

We want to show that $p_i(z) \in P(\alpha_1)$ where α_1 is given by (16) and this will show that $p(z) \in P_k(\alpha)$ for $z \in U$. Let

$$p_i(z) = (1-\alpha_1)h_i(z) + \alpha_1 \qquad (z \in U, i = 1,2) \ , \tag{19}$$

then in view of (18) and (19), we obtain for $z \in U,\ \ i = 1,2$

$$Re\left((1-\alpha_1)h_i(z) + (\alpha_1 - \alpha) + \frac{(1-\alpha_1)zh_i'(z)}{(1-\alpha_1)zh_i(z) + \alpha_1 + \lambda + \eta + 1}\right) > 0 \ . \tag{20}$$

We now form a functional $\phi(u,v)$ by choosing $u = h_i(z)$ and $v = zh_i'(z)$ in (20). Thus

$$\phi(u,v) = (1-\alpha_1)u + \alpha_1 - \alpha + \frac{(1-\alpha_1)v}{(1-\alpha_1)u + \alpha_1 + \lambda + \eta + 1} \quad . \tag{21}$$

We can easily see that the first two conditions of Lemma 1, are easily satisfied as $\phi(u,v)$ is continuous in $D = C - \left(-\frac{\alpha_1+\lambda+\eta+1}{1-\alpha_1}\right) \times C$, $(1,0) \in D$, and $Re\,\{\phi(1,0)\} > 0$. Now for $v_1 \le -\frac{1}{2}(1+u_2^2)$ we obtain

$$Re\{\phi(iu_2,v_1)\} = Re\left((1-\alpha_1)iu_2 + \alpha_1 - \alpha + \frac{(1-\alpha_1)v_1}{(1-\alpha_1)iu_2 + \alpha_1 + \lambda + \eta + 1}\right)$$

$$= \alpha_1 - \alpha + \frac{(1-\alpha_1)v_1\{\alpha_1 + \lambda + \eta + 1\}}{(\alpha_1+\lambda+\eta+1)^2 + (1-\alpha_1)^2 u_2^2}$$

$$\le \alpha_1 - \alpha - \frac{1}{2}\frac{(1-\alpha_1)(\alpha_1+\lambda+\eta+1)(1+u_2^2)}{(\alpha_1+\lambda+\eta+1)^2 + (1-\alpha_1)^2 u_2^2} = \frac{A + \; Bu_2^2}{2\; C} \quad ,$$

where
$A = (\alpha_1+\lambda+\eta+1)\{2(\alpha_1-\alpha)(\alpha_1+\lambda+\eta+1) - (1-\alpha_1)\}$
$B = (1-\alpha_1)\{2(\alpha_1-\alpha)(1-\alpha_1) - (\alpha_1+\lambda+\eta+1)\}$
$C = (\alpha_1+\lambda+\eta+1)^2 + (1-\alpha_1)^2 u_2^2 > 0$.
We note that $Re\,\{\phi(iu_2, v_1\} \le 0$ if and only if $A \le 0$ and $B \le 0$. From $A \le 0$, we obtain α_1 as given by (16) and $B \le 0$ gives us $0 \le \alpha_1 < 1$. Therefore, Lemma 1 is applied to conclude that $Re\,\{h_i(z)\} > 0$ in U and this implies $Re\,\{p_i(z)\} > \alpha_1$. This completes the proof of Theorem 1.

Theorem 2. *Let* $f \in \mathbb{A}$, $\lambda > 0$, $0 \le \alpha < 1$, $\lambda + \eta > -7/8$ *and min* $\{-\mu+\eta, -\mu\} > -2$. *Then*

$$V^{\lambda,\mu,\eta}(k,\alpha) \subset \; V^{\lambda+1,\mu,\eta}(k,\alpha_1) \quad , \tag{22}$$

where α_1 *is given by (16).*

Proof. To prove the inclusion relationship, we observe from Theorem 1, that

$$f(z) \in \; V^{\lambda,\mu,\eta}(k,\alpha) \Leftrightarrow zf'(z) \in \; R^{\lambda,\mu,\eta}(k,\alpha) \Rightarrow zf'(z) \in \; R^{\lambda+1,\mu,\eta}(k,\alpha_1) \Leftrightarrow$$
$$f(z) \in \; V^{\lambda+1,\mu,\eta}(k,\alpha_1),$$

which establishes Theorem 2.

Theorem 3. *Let* $f \in \mathbb{A}, \lambda > 0$, $0 \le \alpha, \beta < 1$, $\lambda + \eta > -7/8$ *and min* $\{-\mu+\eta, -\mu\} > -2$. *Then*

$$T^{\lambda,\mu,\eta}(k,\beta,\alpha) \subset \; T^{\lambda+1,\mu,\eta}(k,\beta_1,\alpha_1) \quad , \tag{23}$$

where α_1 *is given by (16) and* $\beta < \beta_1 < 1$ *is defined in the proof.*

Proof. Let $f(z) \in T^{\lambda,\mu,\eta}(k,\beta,\alpha)$. Then there exists $g(z) \in R^{\lambda,\mu,\eta}(2,\alpha)$ such that

$$\frac{z\left(\; J_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{J_{0,z}^{\lambda,\mu,\eta} g(z)} \in P_k(\beta) \qquad (z \in U, \;\; 0 \le \beta < 1) \quad . \tag{24}$$

Let

$$\frac{z\left(\; J_{0,z}^{\lambda+1,\mu,\eta} f(z)\right)'}{J_{0,z}^{\lambda+1,\mu,\eta} g(z)} = (1-\beta_1)h(z) + \beta_1 = \; H(z)$$

$$= \left(\frac{k}{4}+\frac{1}{2}\right)[(1-\beta_1)h_1(z)+\beta_1] - \left(\frac{k}{4}-\frac{1}{2}\right)[(1-\beta_1)h_2(z)+\beta_1] \quad , \tag{25}$$

where $h(z)$ is analytic in U with $h(0)=1$. Since $g(z) \in R^{\lambda,\mu,\eta}(2,\alpha)$, by Theorem 1 we know that $g(z) \in R^{\lambda+1,\mu,\eta}(2,\alpha_1)$. Hence there exist an analytic function $q(z) \in P$ such that

$$\frac{z\left(J_{0,z}^{\lambda+1,\mu,\eta} g(z)\right)'}{J_{0,z}^{\lambda+1,\mu,\eta} g(z)} = (1-\alpha_1)q(z)+\alpha_1 = H_0(z) \quad . \tag{26}$$

Now using identity (14) , we obtain

$$\frac{z\left(J_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{J_{0,z}^{\lambda,\mu,\eta} g(z)} = H(z) + \frac{z\,H'(z)}{H_0(z)+\lambda+\eta+1}$$

$$= \left(\frac{k}{4}+\frac{1}{2}\right)\left((1-\beta_1)h_1(z)+\beta_1+\frac{(1-\beta_1)zh_1'(z)}{H_0(z)+\lambda+\eta+1}\right) -$$

$$\left(\frac{k}{4}-\frac{1}{2}\right)\left((1-\beta_1)h_2(z)+\beta_1+\frac{(1-\beta_1)zh_2'(z)}{H_0(z)+\lambda+\eta+1}\right) \in P_k(\beta) \quad (z \in U)$$

and this implies that

$$Re\left((1-\beta_1)h_i(z)+\beta_1-\beta+\frac{(1-\beta_1)zh_i'(z)}{H_0(z)+\lambda+\eta+1}\right) > 0 \quad (z \in U, i=1,2) \quad .$$

We form a functional $\phi(u,v)$ by taking $u = h_i(z), v = zh_i'(z)$. Thus

$$\phi(u,v) = (1-\beta_1)u + (\beta_1-\beta) + \frac{(1-\beta_1)v}{H_0(z)+\lambda+\eta+1} \quad .$$

It can be easily seen that $\phi(u,v)$ satisfies the conditions (i) and (ii) of Lemma 1 and to verify the condition (iii) we proceed with $H_0(z) = (1-\alpha_1)(q_1+iq_2)+\alpha_1$ and $v_1 \le -\frac{1}{2}(1+u_2^2)$ as follows:

$$Re\,(\phi(iu_2,v_1) \le (\beta_1-\beta) - \frac{1}{2}\frac{(1-\beta_1)\{(1-\alpha_1)q_1+\alpha_1+\lambda+\eta+1\}(1+u_2^2)}{\{(1-\alpha_1)q_1+\alpha_1+\lambda+\eta+1\}^2+\{(1-\alpha_1)^2q_2^2\}}$$

$= \frac{A+Bu_2^2}{2C}$,
where
$A = 2(\beta_1-\beta)\{((1-\alpha_1)q_1+\alpha_1+\lambda+\eta+1)^2+(1-\alpha_1)^2q_2^2\} - (1-\beta_1)\{(1-\alpha_1)q_1+\alpha_1+\lambda+\eta+1\}$ $B = -(1-\beta_1)\{(1-\alpha_1)q_1+\alpha_1+\lambda+\eta+1)\} \le 0$
$C = ((1-\alpha_1)q_1+\alpha_1+\lambda+\eta+1)^2 + (1-\alpha_1)^2q_2^2 > 0$.
Thus $Re\{\phi(iu_2,v_1)\} \le 0$ if $A \le 0$ and therefore

$$\beta_1 = \frac{2\beta\left[((1-\alpha_1)q_1+\alpha_1+\lambda+\eta+1)^2+(1-\alpha_1)^2q_2^2\right]+[(1-\alpha_1)q_1+\alpha_1+\lambda+\eta+1]}{2\left[((1-\alpha_1)q_1+\alpha_1+\lambda+\eta+1)^2+(1-\alpha_1)^2q_2^2\right]+[(1-\alpha_1)q_1+\alpha_1+\lambda+\eta+1]}.$$

Now, applying Lemma 1, we get $h_i(z) \in$ P, $i=1,2$ and consequently $h(z) \in P_k(\beta_1)$ and therefore $f \in T^{\lambda+1,\mu,\eta}(k,\beta_1,\alpha_1)$. Using the same techniques and relation (10) with Theorem 3, we have the following result:

Theorem 4. *Let $f \in \mathbb{A}, \lambda > 0, \quad 0 \leq \alpha, \beta < 1, \quad \lambda + \eta > -7/8$ and min $\{-\mu + \eta, -\mu\} > -2$. Then*

$$T_*^{\lambda,\mu,\eta}(k, \beta, \alpha) \subset T_*^{\lambda+1,\mu,\eta}(k, \beta_1, \alpha_1) \ ,$$

where β_1 and α_1 are as in Theorem 3.

Theorem 5. *Let $z \in U$ and $f \in R^{\lambda+1,\mu,\eta}(k, 0)$. Then $f \in R^{\lambda,\mu,\eta}(k, 0)$ for $|z| < r_0$, where*

$$r_0 = \frac{\lambda + \eta + 2}{\sqrt{m + \sqrt{m^2 - |(\lambda + \eta + 1)^2 - 1|}}}, \qquad m = 7 + (\lambda + \eta + 1)^2 \ . \tag{27}$$

This radius is exact.

Proof.

Let

$$\frac{z\left(J_{0,z}^{\lambda+1,\mu,\eta} f(z)\right)'}{J_{0,z}^{\lambda+1,\mu,\eta} f(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \ ,$$

where $p \in P_k$ and $p_1, p_2 \in P$ in U. Using similar argument as in Theorem 1, we obtain

$$\frac{z\left(J_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{J_{0,z}^{\lambda,\mu,\eta} f(z)} = p(z) + \frac{zp'(z)}{\lambda + \eta + 1 + p(z)}$$

$$= \left(\frac{k}{4} + \frac{1}{2}\right)\left(p_1(z) + \frac{zp_1'(z)}{p_1(z) + \lambda + \eta + 1}\right) - \left(\frac{k}{4} - \frac{1}{2}\right)\left(p_2(z) + \frac{zp_2'(z)}{p_2(z) + \lambda + \eta + 1}\right) .$$

Applying Lemma 2, we get

$$Re\left(p_i(z) + \frac{zp_i'(z)}{p_i(z) + \lambda + \eta + 1}\right) > 0 \text{ for } |\, z \,| < r_0,$$

where r_0 is given by (27), This completes our proof.

Theorem 6. *Let $\lambda > 0$ and min $\{\lambda + \eta, -\mu + \eta, -\mu\} > -2$. Then*

$$P_k'(\lambda, \mu, \eta, \alpha) \subset P_k'(\lambda + 1, \mu, \eta, \ \alpha + (1 - \alpha)(2\gamma - 1)) \ ,$$

where

$$\gamma = \int_0^1 \left(1 + t^{\frac{1}{\lambda+\eta+2}}\right)^{-1} dt,$$

which is an increasing function of $\frac{1}{\lambda+\eta+2}$ and $\frac{1}{2} \leq \gamma < 1$.

Proof. Let $f(z) \in P_k'(\lambda, \mu, \eta, \alpha)$. Then upon setting

$$\left(J_{0,z}^{\lambda+1,\mu,\eta} f(z)\right)' = H(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z), \tag{28}$$

where $H(z)$ is analytic and $H(0) = 1$ in U. Identity (14), gives us

$$\left(J_{0,z}^{\lambda,\mu,\eta} f(z)\right)' = H(z) + \frac{zH'(z)}{\lambda + \eta + 2}$$

$$= \left(\frac{k}{4} + \frac{1}{2}\right)\left(h_1(z) + \frac{zh_1'(z)}{\lambda + \eta + 2}\right) - \left(\frac{k}{4} - \frac{1}{2}\right)\left(h_2(z) + \frac{zh_2'(z)}{\lambda + \eta + 2}\right).$$

This implies that

$$Re\left(h_i(z)+\frac{zh_i^{'}(z)}{\lambda+\eta+2}\right)>\alpha, \qquad i=1,2.$$

Now using Lemma 4. we get desired result.

Theorem 7. *Let ϕ be a convex function and $f\in R^{\lambda,\mu,\eta}(2,\alpha)$. Then $\phi * f\in R^{\lambda,\mu,\eta}(2,\alpha)$*

Proof. Let $G=\phi * f$ and let

$$\phi(z)=z+\sum_{n=2}^{\infty}b_n z^n$$

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \quad .$$

Then

$$J_{0,z}^{\lambda,\mu,\eta}\ G(z)=\ J_{0,z}^{\lambda,\mu,\eta}\left[z+\sum_{n=2}^{\infty}a_n b_n z^n\right]$$

$$=z+\frac{\Gamma(2-\mu)\Gamma(2+\lambda+\eta)}{\Gamma(2-\mu+\eta)}\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(n-\mu+\eta+1)}{\Gamma(n-\mu+1)\Gamma(n+\lambda+\eta+1)}a_n b_n z^n$$

$$=\left(\phi *\ J_{0,z}^{\lambda,\mu,\eta}f\right)(z) \quad . \tag{29}$$

Also, $f(z)\in R^{\lambda,\mu,\eta}(2,\alpha)$. Therefore, $J_{0,z}^{\lambda,\mu,\eta}f(z)\in R_2(\alpha)=\ S^*(\alpha)$. By logarithmic differentiation of (29), we have

$$\frac{z\left(\ J_{0,z}^{\lambda,\mu,\eta}\ G(z)\right)^{'}}{J_{0,z}^{\lambda,\mu,\eta}\ G(z)}=\frac{\phi(z)*\ F\ J_{0,z}^{\lambda,\mu,\eta}f(z)}{\phi(z)*\ J_{0,z}^{\lambda,\mu,\eta}f(z)} \quad ,$$

where

$$F(z)=\frac{z(\ J_{0,z}^{\lambda,\mu,\eta}f(z))^{'}}{J_{0,z}^{\lambda,\mu,\eta}f(z)}$$

is analytic in U and $F(0)=1$. From Lemma 3 we see that $\frac{z(J_{0,z}^{\lambda,\mu,\eta}\ G(z))^{'}}{J_{0,z}^{\lambda,\mu,\eta}G(z)}$ is contained in the convex hull of $F(U)$. Since $\frac{z(J_{0,z}^{\lambda,\mu,\eta}\ G(z))^{'}}{J_{0,z}^{\lambda,\mu,\eta}\ G(z)}$ is analytic in U and $F(U)\subset\Omega=\{W:\frac{z(J_{0,z}^{\lambda,\mu,\eta}\ W(z))^{'}}{J_{0,z}^{\lambda,\mu,\eta}\ W(z)}\in\ P_2(\alpha)\}$, then $\frac{z(J_{0,z}^{\lambda,\mu,\eta}\ G(z))^{'}}{J_{0,z}^{\lambda,\mu,\eta}\ G(z)}$ lies in Ω. This implies that $G=\phi * f\in R^{\lambda,\mu,\eta}(2,\alpha)$.

Application of Theorem 7.

Corollary 1. *The class $R^{\lambda,\mu,\eta}(2,\alpha)$ is invariant under the following integral operators. That is if $f\in R^{\lambda,\mu,\eta}(2,\alpha)$ then so does f_i where f_i are given as:*

(i) $f_1(z)=\int\limits_0^z\frac{f(t)}{t}dt$

(ii) $f_2(z)=\frac{2}{z}\int\limits_0^z f(t)dt$

(iii) $f_3(z) = \int\limits_0^z \frac{f(t)-f(xt)}{t-xt} dt \quad | x | \leq 1, x \neq 1,$

(iv) $f_4(z) = \frac{1+c}{z^c} \int\limits_0^z t^{c-1} f(t) dt, \quad \mathbb{R}e\ (c) > 0.$

The proof immediately follows from Theorem 7. Since we can write $f_i = f * \phi_i$ with
$\phi_1(z) = -log(1-z)$
$\phi_2(z) = -2\left[\frac{z+log(1-z)}{z}\right]$
$\phi_3(z) = \frac{1}{1-x} log\left(\frac{1-xz}{1-z}\right)$
$\phi_4(z) = \sum\limits_{m=1}^{\infty} \frac{1+c}{m+c} z^m \qquad Re\ c > 0$
and each ϕ_i is convex for i = 1, 2, 3, 4.

Remarks

(i) In Theorems 1, 2, 3 and 4 taking $\alpha = 0$ we get the results obtained by Prajapat [9].
(ii) Taking $\mu = 0$ in the operator $J_{0,z}^{\lambda,\mu,\eta} f(z)$ and making some suitable changes in parameters in Theorems 1 to 8, we obtain the results derived by Noor et al.[6] and Noor[4].

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