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A NEW STUDY OF A CLASS OF MULTI-FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this effort, we bring up a set of conditions to study the existence and uniqueness of solutions for a class of nonlocal initial value problems regarding pantograph equation with a generalized fractional derivative. The generalized derivative is taken for two fractional powers. This class of fractional differential operators extended the standard fractional calculus. Our tool is based on the Krasnoselskii's fixed point theorem (KFPT). An example is given in the sequel. Moreover, we discuss the stability of the fractional differential equation (FDE) in view of the fractional Ulam concept of stability (FUS). We suggest the power series formula depending on the initial condition of the FDE.

1. Introduction

A non-integer (arbitrary) order differential equations are able to realize the memory and hereditary residences of well known critical materials and methods. Fractional calculus (FC) has presently covered in plenty of exciting and crucial fields of study. The tons interest in the problem owes to its huge packages inside the mathematical modeling of several phenomena in almost all life sciences, computer sciences and social studies [3, 9, 12]. It's widely known that, in the deterministic scenario, there can be a very special class of delay differential equations known as the panto-graph equations. The recent development of pantograph equation with a fractional order can be seen in [1, 8]. The study depended on the classic fractional calculus of one fractional power (in terms of the Riemann-Liouville fractional operators and Caputo fractional differential operator). In this manifestation, we transact with a multi-fractional pantograph equation (two fractional powers).

Katugampola derived generalized fractional derivative and studied existence and uniqueness results involving this kind of derivative (see [10, 11]). Recently, Vivek et al. studied different classes of fractional differential equations involving generalized fractional derivative (see [14, 15]). Non-local initial value problems studies are very

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few in the field of fractional differential equations. This type of equations satisfies the non-local condition

$$z(0) = \sum_{i=1}^{m} c_i z(\tau_i).$$

This class has many applications in physical problems which yield better effect than the initial conditions [2] $z(0) = z_0$.

The Ulam-Hyers stability (UHS) of fractional differential equations has been studied in [4], [16]. While this form of stability has been formalized in a complex domain for the Cauchy problem in [5]-[7] .

Consider non-local initial value problem of generalized fractional derivative is as follows:

$$\begin{cases} {}^{\rho}D^{\nu} z(t) = g(t, z(t), z(\kappa t)), \quad t \in I := [0, T], T > 0, \\ z(0) = \sum_{i=1}^{m} c_i z(\tau_i), \quad \tau_i \in [0, T]. \end{cases} \tag{1}$$

where ${}^{\rho}D^{\nu}$ is generalized fractional derivative of order $\nu \in R$ and $\rho > 0,\ 0 < \kappa < 1$ where $g : I \times R \times R \to R$ is given continuous function, $\tau_i, i = 0, 1, ..., m$ are prefixed points satisfying $0 < \tau_1 \leq ... \leq \tau_m < T$ and c_i is real numbers.

In Section 2, we introduce some definitions and Lemmas that used throughout the paper. In Section 3, we study the theory of pantograph equation with the generalized derivative. We express an example to illustrate the theory. Section 4, we deal with the fractional Ulam stability based on polynomial functions.

2. Preliminaries

Some basic definitions and results imposed in this section. Throughout this paper, let $C(I)$ be the collection of the Banach space containing all continuous functions from I into R with the sup. norm

$$(\|\phi\| = \sup_{t \in I} \{|\phi(t)| : t \in I\}).$$

The Riemann-Liouville integral and derivative of order $\nu \in C$, $\Re(\nu) > 0$ are given by

$$I^{\nu}_{a^+} g(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t - s)^{\nu - 1} g(s) ds,$$

and

$$D^{\nu}_{a^+} g(t) = \left(\frac{d}{dt}^{n}\right) \left(I^{n-\nu}_{a^+} f\right)(t),\ t > a,$$

respectively, where $n = [\Re(\nu)]$ and $\Gamma(\nu)$ is the gamma function.

The Hadamard fractional integral and derivative are given by

$$I^{\nu}_{a^+} g(t) = \frac{1}{\Gamma(\nu)} \int_a^t \left(\log \frac{t}{s}\right)^{\nu - 1} g(s) \frac{ds}{s},$$

and

$$D^{\nu}_{a^+}g(t) = \frac{1}{\Gamma(n-\nu)}\left(t\frac{d}{dt}^n\right)\int_a^t \left(\log\frac{t}{s}\right)^{n-\nu+1} g(s)\frac{ds}{s},$$

respectively, for $t > a \geq 0$ and $\Re(\nu) > 0$.

The generalized left-sided fractional integral ${}^{\rho}I^{\nu}_{a^+}f$ of order $\nu \in C(\Re(\nu))$ is defined by

$$\left({}^{\rho}I^{\nu}_{a^+}\right)f(t) = \frac{\rho^{1-\nu}}{\Gamma(\nu)}\int_a^t (t^{\rho}-s^{\rho})^{\nu-1}s^{\rho-1}g(s)ds,\ t > a, \tag{2}$$

if the integral exists. The generalized fractional derivative, corresponding to the generalised fractional integral (2), is defined for $0 \leq a < t$, by

$$\left({}^{\rho}D^{\nu}_{a^+}f\right)(t) = \frac{\rho^{\nu-n-1}}{\Gamma(n-\nu)}\left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t (t^{\rho}-s^{\rho})^{n-\nu+1}s^{\rho-1}g(s)ds, \tag{3}$$

if the integral exists.

Let $z \in C^1(J)$, then

$${}^{\rho}I^{\nu\,\rho}D^{\nu}z(t) = z(t) - z(0)$$

for some $c_i \in R,\ i = 0, 1, ..., n-1,\ n = [\nu]+1$. (KFPT) Let Z be a Banach space, let Θ be a bounded closed convex subset of Z and let Π_1, Π_2 be mapping from Θ into Z such that $\Pi_1 z + \Pi_2 y, \in \Theta$ for every pair $z, y \in \Omega$. If Π_1 is contraction and Π_2 is completely continuous, then the equation $\Pi_1 z + \Pi_2 z = z$ has a solution on Θ.

3. Concurrence of outcomes

The following Lemma shows the equivalent integral equation of problem (1): A function z is a solution of the mixed type integral equation

$$z(t) = \begin{cases} \frac{1}{1-\sum_{i=1}^m c_i}\sum_{i=1}^m c_i \frac{\rho^{1-\nu}}{\Gamma(\nu)}\int_0^{\tau_i}(\tau_i^{\rho}-s^{\rho})^{\nu-1}s^{\rho-1}g(s,z(s),z(\kappa s))ds \\ \\ +\frac{\rho^{1-\nu}}{\Gamma(\nu)}\int_0^t (t^{\rho}-s^{\rho})^{\nu-1}s^{\rho-1}g(s,z(s),z(\kappa s))ds \end{cases} \tag{4}$$

if and only if z is a solution of the fractional initial value problem

$$\begin{aligned} {}^{\rho}D^{\nu}z(t) &= g(t, z(t), z(\kappa t)), \quad t \in I, \\ z(0) &= \sum_{i=1}^m c_i z(\tau_i), \quad \tau_i \in [0, T]. \end{aligned}$$

Proof. According to Lemma 2, a solution of Eq (1) can be expressed as

$$z(t) = z(0) + \frac{1}{\Gamma(\nu)}\int_0^t \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\nu-1} s^{\rho-1}g(s, z(s), z(\kappa s))ds. \tag{5}$$

Next, we substitute $t = \tau_i$ and multiply by c_i, we can write

$$c_i z(\tau_i) = c_i z(0) + \frac{1}{\Gamma(\nu)} c_i \int_0^{\tau_i} \left(\frac{\tau_i^\rho - s^\rho}{\rho} \right)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds.$$

Thus, we have

$$\begin{aligned} z(0) &= \sum_{i=1}^m c_i z(\tau_i) \\ &= \sum_{i=1}^m c_i z(0) + \frac{1}{\Gamma(\nu)} \sum_{i=1}^m c_i \int_0^{\tau_i} \left(\frac{\tau_i^\rho - s^\rho}{\rho} \right)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds, \end{aligned}$$

which implies

$$z(0) = \frac{1}{1 - \sum_{i=1}^m c_i} \sum_{i=1}^m c_i \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{\tau_i} (\tau_i^\rho - s^\rho)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds. \tag{6}$$

Substituting (6) in (5) we obtain (4).
Next we prove the sufficiency: Substitute $t = 0$ in (4), we get

$$z(0) = \frac{1}{1 - \sum_{i=1}^m c_i} \sum_{i=1}^m c_i \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{\tau_i} (\tau_i^\rho - s^\rho)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds. \tag{7}$$

Next, substitiing $t = \tau_i$ and multiply by c_i in (4). Then we derive

$$\begin{aligned} \sum_{i=1}^m c_i z(t) =& \sum_{i=1}^m c_i \left(\frac{1}{1 - \sum_{i=1}^m c_i} \sum_{i=1}^m c_i \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{\tau_i} (\tau_i^\rho - s^\rho)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds \right. \\ & \left. + \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{\tau_i} (\tau_i^\rho - s^\rho)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds \right) \\ =& \sum_{i=1}^m c_i \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{\tau_i} (\tau_i^\rho - s^\rho)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds \left(1 + \frac{\sum_{i=1}^m c_i}{1 - \sum_{i=1}^m c_i} \right) \\ =& \frac{1}{1 - \sum_{i=1}^m c_i} \sum_{i=1}^m c_i \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{\tau_i} (\tau_i^\rho - s^\rho)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds. \end{aligned} \tag{8}$$

It follows from (7) and (8),

$$z(0) = \sum_{i=1}^m c_i z(\tau_i).$$

Now we apply ${}^\rho D_{a+}^\nu$ on both sides of (4), hence it reduced to

$${}^\rho D_{a+}^\nu z(t) = g(t, z(t), z(\kappa t)).$$

The results are proved completely. □

Here, we introduce the following hypotheses:

(H1) Let $g : I \times R \times R \to R$ be a continuous function and there exists a positive constant $\ell > 0$, such that

$$|g(t, z_1, z_2) - g(t, y_1, y_2)| \leq \ell \left(|z_1 - y_1| + |z_2 - y_2|\right),$$

$$\Big(z_1, z_2, y_1, y_2 \in R, t \in I\Big).$$

(H2) Assume that

$$\sigma = \frac{2\ell}{1 - \sum_{i=1}^{m} c_i} \sum_{i=1}^{m} c_i \frac{\tau_i^{\rho\nu}}{\rho^\nu \Gamma(\nu + 1)} + \frac{\ell T^{\rho\nu}}{\rho^\nu \Gamma(\nu + 1)} < 1.$$

The result is based upon Theorem 2.

(Existence) Suppose that [H1] and [H2] are achieved. Then, Eq.(1) admits at least one outcome on I.

Proof. Define the operator $N : C(I) \to C(I)$, it is well defined and given by

$$(Nx)(t) = \begin{cases} \frac{1}{1-\sum_{i=1}^{m} c_i} \sum_{i=1}^{m} c_i \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{\tau_i} (\tau_i^\rho - s^\rho)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds \\ + \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^t (t^\rho - s^\rho)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds. \end{cases} \tag{9}$$

Set $\widetilde{g}(s) = g(s, 0, 0)$ and

$$\omega = \left(\frac{1}{1 - \sum_{i=1}^{m} c_i} \sum_{i=1}^{m} c_i \frac{\tau_i^{\rho\nu}}{\rho^\nu \Gamma(\nu + 1)} + \frac{T^{\rho\nu}}{\rho^\nu \Gamma(\nu + 1)} \right) \|\widetilde{g}\|$$

Consider the ball

$$B_r = \{z \in C(I) : \|x\| \leq r, \quad r > 0\}.$$

Now we subdivide the operator N into two operators A and B on B_r as follows

$$(Az)(t) = \frac{1}{1 - \sum_{i=1}^{m} c_i} \sum_{i=1}^{m} c_i \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{\tau_i} (\tau_i^\rho - s^\rho)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds$$

and

$$(Bx)(t) = \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^t (t^\rho - s^\rho)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds.$$

Now we verify the conditions of Theorem 2.

Step.1 Boundedness. We aim to show that $Az + By \in B_r$ for every $z, y \in B_r$.

$$
\begin{aligned}
&|(Az)(t)+(By)(t)| \\
&\leq \frac{1}{1-\sum_{i=1}^{m} c_i} \sum_{i=1}^{m} c_i \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{\tau_i} (\tau_i^{\rho}-s^{\rho})^{\nu-1} s^{\rho-1} |g(s,z(s),z(\kappa s))|\, ds \\
&+ \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{t} (t^{\rho}-s^{\rho})^{\nu-1} s^{\rho-1} |g(s,y(s),y(\kappa s))|\, ds \\
&\leq \frac{1}{1-\sum_{i=1}^{m} c_i} \sum_{i=1}^{m} c_i \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{\tau_i} (\tau_i^{\rho}-s^{\rho})^{\nu-1} s^{\rho-1} \left(|g(s,z(s),z(\kappa s))-g(s,0,0)|+|g(s,0,0)|\right) ds \\
&+ \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{t} (t^{\rho}-s^{\rho})^{\nu-1} s^{\rho-1} |g(s,y(s),y(\kappa s))-g(s,0,0)|+|g(s,0,0)|\, ds \\
&\leq \frac{1}{1-\sum_{i=1}^{m} c_i} \sum_{i=1}^{m} c_i \frac{\tau_i^{\rho\nu}}{\rho^{\nu}\Gamma(\nu+1)} \left(2\ell r+\|\widetilde{g}\|\right) + \frac{t^{\rho\nu}}{\rho^{\nu}\Gamma(\nu+1)} \left(2\ell r+\|\widetilde{g}\|\right) \\
&\leq \sigma r+\omega \\
&\leq r.
\end{aligned}
$$

Step.2 Contracting. Our aim is to prove that A is a contraction mapping, for any $z, y \in B_r$.

$$
\begin{aligned}
&|(Az)(t)-(Ay)(t)| \\
&\leq \frac{1}{1-\sum_{i=1}^{m} c_i} \sum_{i=1}^{m} c_i \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{\tau_i} (\tau_i^{\rho}-s^{\rho})^{\nu-1} s^{\rho-1} |g(s,z(s),z(\kappa s))-g(s,y(s),y(\kappa s))|\, ds \\
&\leq \frac{1}{1-\sum_{i=1}^{m} c_i} \sum_{i=1}^{m} c_i \frac{2\ell\tau_i^{\rho\nu}}{\rho^{\nu}\Gamma(\nu+1)} \|x-y\|.
\end{aligned}
$$

The operator A is contraction mapping due to hypothesis [H2].

Step.3 Compactness. We have to impose that the operator B is compact and continuous.

$$
\begin{aligned}
|(Bz)(t)| &\leq \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{t} (t^{\rho}-s^{\rho})^{\nu-1} s^{\rho-1} |g(s,z(s))|\, ds \\
&\leq \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{t} (t^{\rho}-s^{\rho})^{\nu-1} s^{\rho-1} |g(s,z(s),z(\kappa s))-g(s,0,0)|+|g(s,0,0)|\, ds \\
&\leq \frac{T^{\rho\nu}}{\rho^{\nu}\Gamma(\nu+1)} \left(2\ell\|x\|+\|\widetilde{g}\|\right).
\end{aligned}
$$

So operator B is uniformly bounded.

Now we verify the compactness of operator B.
For $0<t_2<t_1<T$, we have

$$\begin{aligned}
&|(Bz)(t_1) - (Bz)(t_2)| \\
&= \left| \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{t_1} (t_1^\rho - s^\rho)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds - \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{t_2} (t_2^\rho - s^\rho)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds \right| \\
&\leq \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{t_1} (t_1^\rho - s^\rho)^{\nu-1} s^{\rho-1} |g(s, z(s), z(\kappa s))| \, ds - \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{t_2} (t_2^\rho - s^\rho)^{\nu-1} s^{\rho-1} |g(s, z(s), z(\kappa s))| \, ds \\
&\leq \frac{\|g\|}{\rho^\nu \Gamma(\nu+1)} \left| t_1^{\rho\nu} - t_2^{\rho\nu} \right|
\end{aligned}$$

tending to zero as $t_1 \to t_2$. Thus B is equi-continuous. Hence, the operator B is compact on B_r by the Arzela-Ascoli Theorem. Thus, all the hypotheses of Theorem 3 are fulfilled. Consequently, the conclusion of Theorem 3 applies and the problem (1) admits at least one outcome. □

Putting the generalized pantograph equation as follows:

$$\begin{aligned}
{}^\rho D^\nu z(t) &= g(t, z(t), z(\kappa t)), \quad t \in [0,1], \\
z(0) &= 0.5 z\,(0.2)\,, \quad \tau_i \in [0,1].
\end{aligned}$$

Denote

$$\nu = 0.6, \quad \rho = 0.3, \quad \kappa = 0.5.$$

Set the function g as follows:

$$g(t, z(t), z(\kappa t)) = 0.1 z(t) + 0.1 z(0.5t).$$

Moreover, it satisfies

$$|g(t, x_1, y_1) - g(t, x_2, y_2)| \leq 0.1 \left(|x_1 - x_2| + |y_1 - y_2| \right).$$

On the other hand

$$\begin{aligned}
\sigma &= \frac{2\ell}{1 - \sum_{i=1}^m c_i} \sum_{i=1}^m c_i \frac{\tau_i^{\rho\nu}}{\rho^\nu \Gamma(\nu+1)} + \frac{\ell T^{\rho\nu}}{\rho^\nu \Gamma(\nu+1)} \\
&= 0.4536 < 1.
\end{aligned}$$

Here, hypothesis [H1] and [H2] are satisfied then the problem (1) has at least one solution.

4. Fractional Ulam stability (FUS)

In this section, we present a new formal of FUS. Let

$$g(t, z) = \int_0^t (t^\rho - s^\rho)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds,$$

$$G_i(t, z) = \int_0^{\tau_i} (\tau_i^\rho - s^\rho)^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds$$

and

$$C := \sum_{i=1}^m c_i < 1.$$

We say that the solution z of Eq.(1) is a FUS if there exists a constant $\ell > 0$ with the following property: for every $\epsilon > 0$ if

$$|{}^{\rho}D^{\nu}z(t) - g(t, z(t), z(\kappa t))| < \epsilon,$$

then there exists $y \in B_r$ achieving

$$^{\rho}D^{\nu}y(t) - g(t, y(t), y(\kappa t)) = 0, \quad y(0) = z(0) \tag{10}$$

such that

$$|z(t) - y(t)| \le \epsilon(\ell + 1), \quad z \in B_r.$$

We have the following result:

Let the assumptions of Theorem 3 hold. If there occurs $y \in B_r$ satisfying (10), then Eq.(1) has a FUS.

Proof. Let x be a solution of Eq.(1). In view of Lemma 3, we have

$$\begin{aligned}
|z(t)| &\le \left| \frac{1}{1 - \sum_{i=1}^{m} c_i} \sum_{i=1}^{m} c_i \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{\tau_i} (\tau_i^{\rho} - s^{\rho})^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds \right| \\
&\quad + \left| \frac{\rho^{1-\nu}}{\Gamma(\nu)} \int_0^{t} (t^{\rho} - s^{\rho})^{\nu-1} s^{\rho-1} g(s, z(s), z(\kappa s)) ds \right| \\
&= \left| \frac{1}{1-C} \sum_{i=1}^{m} c_i \frac{\rho^{1-\nu}}{\Gamma(\nu)} G_i(t, z) \right| + \left| \frac{\rho^{1-\nu}}{\Gamma(\nu)} g(t, z) \right| \\
&\le | \sum_{i=0}^{m} \frac{\rho^{1-\nu}|c_{i+1}|}{\Gamma(\nu)(1-C)} \|G\| + \frac{\rho^{1-\nu}}{\Gamma(\nu)} \|G\| \\
&\le \frac{\epsilon}{1-C} \sum_{i=0}^{\infty} \frac{|c_i|^p}{2^i} + \epsilon, \quad p \in (0, \infty) \\
&= \epsilon \frac{2 \max_i |c_i|^p}{1-C} + \epsilon := \epsilon(\ell + 1).
\end{aligned}$$

Consequently, we obtain

$$|z(t) - y(t)| \le \epsilon(\ell + 1), \quad z \in B_r.$$

Hence, Eq. (1) is FUS. □

It can be seen that for a suitable $\epsilon > 0.09$, the equation in Example 3.3 is FUS.

5. Conclusion

We established some requests for the existence and uniqueness of solutions for a special class of the fractional Cauchy equations with a non-local initial condition. Our tool is based on the modified fractional operators of two fractional powers. Moreover, we illustrated a new idea of FUS, which suggested by using power series. The main tool of our work was by employing the KFPT. For the future work, one may establish the occurrence of the outcome of sequential fractional differential equation.

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