ANALYTIC SOLUTION FOR SECOND-ORDER FRACTIONAL DIFFERENTIAL EQUATIONS VIA OHAM

M. ILIE, J. BIAZAR, Z. AYATI

ABSTRACT. Fractional differential equations are often seeming perplexing to solve. Therefore, finding comprehensive methods for solving them sounds of high importance. In this article, optimal homotopy asymptotic method is presented to solve specific second-order conformable fractional differential equations that is named conformable fractional optimal homotopy asymptotic method. The results obtained demonstrate the efficiency of the declared method for fractional differential equations. Some numerical examples are presented to illustrate the proposed approach.

1. INTRODUCTION

Many phenomena in our real world are described by fractional differential equations [3, 14]. Although having the exact solution of fractional equations in analyzing the phenomena is essential, there are many fractional differential equations, which cannot be solved exactly [31]. Due to this fact, finding a desired approximate solutions of fractional differential equations is clearly vital [31]. In recent years, many effective methods have been proposed for finding approximate solution to fractional differential equations, such as Adomian decomposition method [14, 16], homotopy perturbation method [17, 20], homotopy analysis method [21], optimal homotopy asymptotic method [22-24], variational iteration method [25], generalized differential transform method [26], finite difference method [27], semi-disrete scheme and Chebyshev collocation method [28], Wavelet Operational [29] and other methods [30, 33]. In this paper, optimal homotopy asymptotic method is utilized to obtain an approximate solution of linear and nonlinear conformable fractional differential equations. Some conformable fractional differential equations, and nonlinear conformable fractional Bratu-type equations are solved, to illustrate the proposed method. The organization of this paper is as follows: In Section 2, the basic definitions alike conformable fractional derivative and integral, are described. In Section 3, the conformable fractional homotopy asymptotic method for specific second-order fractional differential equations, are presented. In Section 4, Some equations,

as illustrative examples, by means of the proposed approach are solved. Finally, conclusions are given in Section 5.

2. Basic definitions

The purpose of this section is to recall some preliminaries of the proposed method.

2.1. Conformable fractional derivative (CFD).

For a function $f : [0, \infty) \longrightarrow \mathbb{R}$ the conformable fractional derivative of f, of order α, is defined by

$$(T_\alpha f)(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}, \tag{2.1}$$

for all $x > 0$, $\alpha \in (0, 1)$. If f is $\alpha-$differentiable in some $(0, a)$, lets define $(T_\alpha f)(0) = \lim_{x \to 0^+} (T_\alpha f)(x)$, provided that $\lim_{x \to 0^+} (T_\alpha f)(x)$ exists. If the conformable derivative of f of order α exists, then we simply say that f is $\alpha-$differentiable (see [1, 2]). One can easily show that satisfies all the following properties (see [1]): Let $\alpha \in (0, 1]$ and f, g be $\alpha-$differentiable at a point $x > 0$, then

    A. For $a, b \in \mathbb{R}$, $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$,
    B. For all $p \in \mathbb{R}$, $T_\alpha(x^p) = px^{p-\alpha}$,
    C. For all constant functions $f(x) = \lambda$, $T_\alpha(\lambda) = 0$,
    D. $T_\alpha(f \cdot g) = g \cdot T_\alpha(f) + f \cdot T_\alpha(g)$,
    E. $T_\alpha \left(\dfrac{f}{g} \right) = \dfrac{g \cdot T_\alpha(f) - fT_\alpha(g)}{g^2}$,
    F. $T_\alpha(f) = x^{1-\alpha} \dfrac{df}{dx}$.

If $\alpha \in (n, n+1]$ and f is $n-$differentiable at $x > 0$, then the conformable fractional derivative of f of order is defined as follows

$$(T_\alpha f)(x) = \lim_{\varepsilon \to 0} \frac{f^{([\alpha]-1)}(x + \varepsilon x^{([\alpha]-\alpha)}) - f^{([\alpha]-1)}(x)}{\varepsilon}, \tag{2.2}$$

where $[\alpha]$ is the smallest integer greater than or equal to α. When f is $(n+1)-$differentiable at $x > 0$, as a consequence of (2.2), can be (see [1])

$$(T_\alpha f)(x) = x^{[\alpha]-\alpha} \frac{d^{[\alpha]} f(x)}{dx^{[\alpha]}}.$$

2.2. Conformable fractional integral (CFI).

Given a function $f : [\alpha, \infty) \to \mathbb{R}$, $\alpha \geq 0$ The conformable fractional integral of f, is defined by

$$I_\alpha^a(f)(x) = \int_a^x \frac{f(t)}{t^{1-\alpha}} dt, \tag{2.3}$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1)$ (see [1, 2]). For the sake of simplicity, lets consider $I_\alpha^0(f)(x) = I_\alpha(f)(x)$. One of the most useful results is the following statement (see [1]):

For all $x \geq a$ and any continuous function in the domain of I_α^a, we have $T_\alpha (I_\alpha^a f(x)) = f(x)$.

3. Conformable fractional optimal homotopy asymptotic method

Consider the general second-order fractional differential equations with initial value

$$T_\alpha T_\alpha u + G(t)T_\alpha u + F(t,u) = g(t), \quad u(0) = A, T_\alpha u(0) = B, \tag{3.1}$$

where F is a functional operator, and G, g are known function, and A, B are certain constant, and u is an unknown function.

According to optimal homotopy asymptotic technique, a homotopy $v(t,p) : \Omega \times [0,1] \to \mathbb{R}$ can be constructed which satisfies

$$(1-p)\big[T_\alpha T_\alpha\big(v(t,p)\big) + G(t)T_\alpha\big(v(t,p)\big)\big]$$
$$-H(p)\big[T_\alpha T_\alpha\big(v(t,p)\big) + G(t)T_\alpha\big(v(t,p)\big) + F\big(t,v(t,p)\big) - g(t)\big] = 0, \tag{3.2}$$

Where $p \in [0,1]$ is an embedding parameter, $H(p)$ is a nonzero auxiliary function for $p \neq 0$ and $H(0) = 0$, $v(t,p)$ is an unknown function. By substituting $p = 0$ and 1 in equation (3.2), we have $v(t,0) = u_0(t)$ and $v(t,1) = u(t)$, respectively. Thus as p is changing from zero to unity, the solution $v(t,p)$ varies continuously from $u_0(t)$ to the exact solution $u(t)$. By substituting $p = 0$ in Eq. (3.2) the initial approximation $u_0(t) = v(t,0)$ is obtained as the solution of conformable fractional equation,

$$T_\alpha T_\alpha\big(u_0(t)\big) + G(t)T_\alpha\big(u_0(t)\big) = 0, \quad u(0) = A, \ T_\alpha u(0) = B. \tag{3.3}$$

The auxiliary function $H(p)$ can be chose as the following

$$H(p) = c_1 p + c_2 p^2 + c_3 p^3 + \cdots, \tag{3.4}$$

where $c_1, c_2, c_3, \ldots$, are parameters, that is determined later. Expanding $v(t,p,c_1,c_2,\ldots)$, in a Taylor series of p, is

$$v(t,p,c_1,c_2,\ldots) = u_0(t) + \sum_{i=1}^{\infty} u_i(t,c_1,c_2,\ldots,c_i)p^i. \tag{3.5}$$

Substituting Eqs. (3.4) and (3.5) into equation (3.2) and setting to zero the coefficient of the same powers of p, then the zero order deformation equation is obtained as given in Eq. (3.3), and the other order deformation equations are given as follows

$$p^1; T_\alpha T_\alpha\big(u_1(t,c_1)\big) + G(t)T_\alpha\big(u_1(t,c_1)\big) - T_\alpha T_\alpha\big(u_0(t)\big) - G(t)T_\alpha\big(u_0(t)\big)$$
$$-c_1\big[T_\alpha T_\alpha\big(u_0(t)\big) + G(t)T_\alpha\big(u_0(t)\big) + F\big(t,u_0(t)\big) - g(t)\big] = 0, \quad u_1(0) = 0, \ T_\alpha u_1(0) = 0,$$
$$p^2; T_\alpha T_\alpha\big(u_2(t,c_1,c_2)\big) + G(t)T_\alpha\big(u_2(t,c_1,c_2)\big) - T_\alpha T_\alpha\big(u_1(t,c_1)\big) - G(t)T_\alpha\big(u_1(t,c_1)\big)$$
$$-c_2\big[T_\alpha T_\alpha\big(u_0(t)\big) + G(t)T_\alpha\big(u_0(t)\big) + F\big(t,u_0(t)\big) - g(t)\big]$$
$$-c_1\big[T_\alpha T_\alpha\big(u_1(t,c_1)\big) + G(t)T_\alpha\big(u_1(t,c_1)\big) - u_1(t,c_1)\frac{\partial F}{\partial u_0}\big(t,u_0(t)\big)\big] = 0, \ u_2(0) = T_\alpha u_2(0) = 0, \tag{3.6}$$
$$p^3; T_\alpha T_\alpha\big(u_3(t,c_1,c_2,c_3)\big) + G(t)T_\alpha\big(u_3(t,c_1,c_2,c_3)\big) - T_\alpha T_\alpha\big(u_2(t,c_1,c_2)\big) - G(t)T_\alpha\big(u_2(t,c_1,c_2)\big)$$
$$-c_3\big[T_\alpha T_\alpha\big(u_0(t)\big) + G(t)T_\alpha\big(u_0(t)\big) + F\big(t,u_0(t)\big) - g(t)\big]$$
$$-c_2\big[T_\alpha T_\alpha\big(u_1(t,c_1)\big) + G(t)T_\alpha\big(u_1(t,c_1)\big) - u_1(t,c_1)\frac{\partial F}{\partial u_0}\big(t,u_0(t)\big)\big]$$
$$-\frac{1}{2}c_1\big[2T_\alpha T_\alpha\big(u_2(t,c_1,c_2)\big) + 2G(t)T_\alpha\big(u_2(t,c_1,c_2)\big) + u_1^2(t,c_1)\frac{\partial^2 F}{\partial u_0^2}\big(t,u_0(t)\big)$$
$$+2u_2(t,c_1,c_2)\frac{\partial F}{\partial u_0}\big(t,u_0(t)\big)\big] = 0, \ u_3(0) = T_\alpha u_3(0) = 0,$$

$$\vdots$$

It should be noted that $u_1, u_2, u_3, \ldots$ are directed by linear conformable fractional equations (3.6), which this can be easily solved. The convergence of the series given in Eq. (3.5) depends upon the auxiliary parameters c_i for $i \geq 1$. If it converges at $p = 1$, we have

$$u(t, c_1, c_2, \ldots) = u_0(t) + \sum_{i=1}^{\infty} u_i(t, c_1, c_2, \ldots, c_i). \tag{3.7}$$

Generally, the th order approximate solution of Eq. (3.1), can be denoted as the following

$$u^m(t, c_1, c_2, \ldots, c_m) = u_0(t) + \sum_{i=1}^{m} u_i(t, c_1, c_2, \ldots, c_i). \tag{3.8}$$

By substitution of Eq. (3.8) into Eq. (3.1), the residual error can be expressed as follows

$$R(t, c_1, c_2, \ldots, c_m = \mathrm{T}_\alpha \mathrm{T}_\alpha \big(u^m(t, c_1, c_2, \ldots, c_m) \big) + G(t) \mathrm{T}_\alpha \big(u^m(t, c_1, c_2, \ldots, c_m) \big)$$
$$+ F\big(t, u^m(t, c_1, c_2, \ldots, c_m) \big) - g(t). \tag{3.9}$$

When $R(t, c_1, c_2, \ldots, c_m) = 0$, results that $u^m(t, c_1, c_2, \ldots, c_m)$ is an exact solution. Such a does not occur usually for nonlinear problems. In these cases, we can apply the least square approach:

$$J_m(c_1, c_2, \ldots, c_m) = \int_a^b R^2(t, c_1, c_2, \ldots, c_m) dt, \tag{3.10}$$

where the values a, b depend on the given problem. The unknown convergence control parameters $c_1, c_2, \ldots, c_m$ can be optimally identified from the following conditions

$$\frac{\partial J_m}{\partial c_i} = 0, \quad i = 1, 2, \ldots, m. \tag{3.11}$$

It is interesting to point out that when these parameters are determined, then the th order approximate solution given by Eq. (3.8) will be constructed.

4. Examples

In this section, to illustrate the proposed approach, some conformable fractional differential equations and conformable fractional Bratu-type differential equation will be solved.

Example 4.1. Consider the following linear fractional differential equation with initial value

$$\mathrm{T}_\alpha \mathrm{T}_\alpha u - 3\mathrm{T}_\alpha u + 2u = 2\left(\frac{1}{\alpha}t^\alpha\right)^2 + \left(\frac{1}{\alpha}t^\alpha\right) + 1, \quad u(0) = (\mathrm{T}_\alpha u)(0) = 1, \tag{4.1}$$

where $u(t) = \frac{5}{4}\exp\left(\frac{2}{\alpha}t^\alpha\right) - 5\exp\left(\frac{1}{\alpha}t^\alpha\right) + \left(\frac{1}{\alpha}t^\alpha\right)^2 + \frac{7}{2}\left(\frac{1}{\alpha}t^\alpha\right) + \frac{19}{4}.$

According to the proposed conformable fractional optimal homotopy asymptotic method, we have

$$(1 - p)\big[\mathrm{T}_\alpha \mathrm{T}_\alpha \big(v(t, p)\big) - 3\mathrm{T}_\alpha \big(v(t, p)\big)\big] - H(p)\big[\mathrm{T}_\alpha \mathrm{T}_\alpha \big(v(t, p)\big) - 3\mathrm{T}_\alpha \big(v(t, p)\big)$$
$$+ 2v(t, p) - 2\left(\frac{1}{\alpha}t^\alpha\right)^2 - \left(\frac{1}{\alpha}t^\alpha\right) - 1\big] = 0, \tag{4.2}$$

where

$$v(t, p, c_1, c_2, \ldots) = u_0(t) + \sum_{i=1}^{\infty} u_i(t, c_1, c_2, \ldots, c_i)p^i$$

$$H(p) = c_1 p + c_2 p^2 + c_3 p^3 + \cdots . \tag{4.3}$$

Substituting Eqs. (4.3) into Eq. (4.2) and setting to zero the coefficient of the same powers of p, we derive

$$p^0; \mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_0(t)\big) - 3\mathrm{T}_\alpha \big(u_0(t)\big) = 0, \ u_0(0) = (\mathrm{T}_\alpha u_0)(0) = 1,$$

$$p^1; \mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_1(t, c_1)\big) - 3\mathrm{T}_\alpha \big(u_1(t, c_1)\big) - 3\mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_0(t)\big) + 3\mathrm{T}_\alpha \big(u_0(t)\big) - c_1 \big[\mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_0(t)\big)$$

$$-3\mathrm{T}_\alpha \big(u_0(t)\big) + 2u_0(t) - 2\left(\frac{1}{\alpha}t^\alpha\right)^2 - \left(\frac{1}{\alpha}t^\alpha\right)^2 - 1\big] = 0, \ u_1(0) = (\mathrm{T}_\alpha u_1)(0) = 0$$

$$p^2; \mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_2(t, c_1, c_2)\big) - 3\mathrm{T}_\alpha \big(u_2(t, c_1, c_2)\big) - \mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_1(t, c_1)\big) + 3\mathrm{T}_\alpha \big(u_1(t, c_1)\big)$$

$$-c_2 \left[\mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_0(t)\big) - 3\mathrm{T}_\alpha \big(u_0(t)\big) + 2u_0(t) - 2\left(\frac{1}{\alpha}t^\alpha\right)^2 - \left(\frac{1}{\alpha}t^\alpha\right) - 1\right]$$

$$-c_1 \left[\mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_1(t, c_1)\big) - 3\mathrm{T}_\alpha \big(u_1(t, c_1)\big) + 2u_1(t, c_1)\right] = 0, \ u_2(0) = (\mathrm{T}_\alpha u_2)(0) = 0, \tag{4.4}$$

$$p^3; \mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_3(t, c_1, c_2, c_3)\big) - 3\mathrm{T}_\alpha \big(u_3(t, c_1, c_2, c_3)\big) - \mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_2(t, c_1, c_2)\big) + 3\mathrm{T}_\alpha \big(u_2(t, c_1, c_2)\big)$$

$$-c_3 \left[\mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_0(t)\big) - 3\mathrm{T}_\alpha \big(u_0(t)\big) + 2u_0(0) - 2\left(\frac{1}{\alpha}t^\alpha\right)^2 - \left(\frac{1}{\alpha}t^\alpha\right) - 1\right]$$

$$-c_2 \left[\mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_1(t, c_1)\big) - 3\mathrm{T}_\alpha \big(u_2(t, c_1)\big) + 2u_1(t, c_1)\right]$$

$$-\frac{1}{2}c_1 \left[2\mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_2(t, c_1, c_2)\big) - 6\mathrm{T}_\alpha \big(u_2(t, c_1, c_2)\big) + 4u_2(t, c_1, c_2)\right] = 0, \ u_3(0) = (\mathrm{T}_\alpha u_3)(0) = 0,$$

$$\vdots$$

Corresponding solution of conformable linear fractional differential equations (4.4), are as follows

$$u_0(t) = \frac{1}{3}\exp\left(\frac{3}{\alpha}t^\alpha\right) + \frac{2}{3},$$

$$u_1(t, c_1) = c_1 \left[\frac{2}{3\alpha}t^{3\alpha} + \frac{7}{9\alpha}t^{2\alpha} + \frac{4}{27\alpha}t^\alpha + \frac{10}{81} + \left(\frac{2}{9\alpha}t^\alpha - \frac{10}{81\alpha}\right)\exp\left(\frac{3}{\alpha}t^\alpha\right)\right]$$

$$\vdots$$

Therefore, third-terms approximation to the solution of Eq. (4.1), can be obtained as the following

$$u^3(t, c_1, c_2, c_3) = u_0(t) + u_1(t, c_1) + u_2(t, c_1, c_2) + u_3(t, c_1, c_2, c_3). \tag{4.5}$$

Table 4.1. Shows the optimal values of the convergence control constants c_1, c_2 and c_3 in $u^3(t, c_1, c_2, c_3)$ given in Eq. 4.5 for different values of α which can be obtained using the procedure mentioned in (3.9) up to (3.11).

Auxiliary parameters	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
c_1	-0.8379111029	-0.8379111029	-0.8379111029	-0.8379111029
c_2	0.03761658691	0.03761658691	0.03761658691	0.037616586919
c_3	0.01712665796	0.01712665796	0.01712665796	0.01712665796

TABLE 4.1. Values of auxiliary parameters for the third-order OHAM solution of Eq. (4.1) for different orders.

In Figure 4.1, the exact and approximate solutions of linear fractional equation for $\alpha = 0.4, 0.6, 0.8$ and 1.0, are plotted.
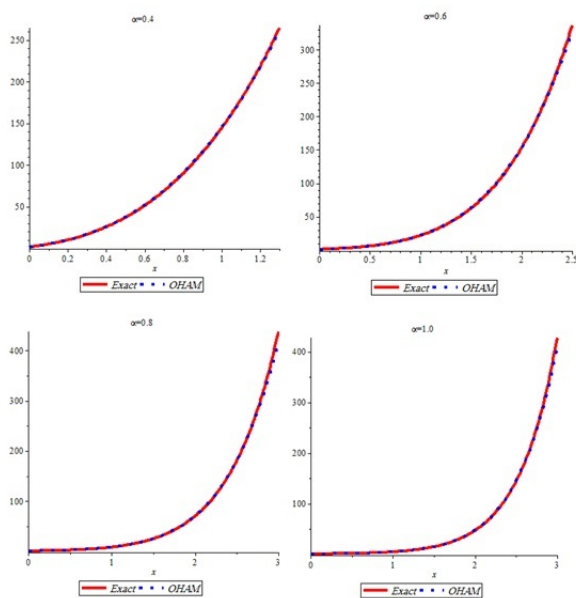

FIGURE 4.1. 3th-order approximation of OHAM and exact solution for Example 4.1.

Example 4.2. Consider the following fractional differential equation with initial value

$$T_\alpha T_\alpha u + 2T_\alpha u + u = \exp\left(-\frac{1}{\alpha}t^\alpha\right), \qquad u(0) = 0, \quad (T_\alpha u)(0) = 0. \qquad (4.6)$$

The exact solution of Eq. (4.6), is $u(x) = \left(\frac{1}{2}\left(\frac{1}{2}t^\alpha\right)^2 + \frac{1}{\alpha}t^\alpha\right)\exp\left(-\frac{1}{\alpha}t^\alpha\right).$

Conforming to the proposed conformable fractional homotopy asymptotic method, results in

$$(1-p)\big[T_\alpha T_\alpha\big(v(t,p)\big) + 2T_\alpha\big(v(t,p)\big)\big] - H(p)\big[T_\alpha T_\alpha\big(v(t,p)\big)$$
$$+2T_\alpha\big(v(t,p)\big) + v(t,p) - \exp\left(-\frac{1}{\alpha}t^\alpha\right)\big] = 0. \qquad (4.7)$$

Substituting Eqs. (4.3) into Eq. (4.7) and setting to zero the coefficient of like powers of p, we reads

$$p^0; T_\alpha T_\alpha \big(u_0(t)\big) + 2T_\alpha \big(u_0(t)\big) = 0, \ \ u_0(0) = (T_\alpha u_0)(0) = 1,$$

$$p^1; T_\alpha T_\alpha \big(u_1(t, c_1)\big) + 2T_\alpha \big(u_1(t, c_1)\big) - T_\alpha T_\alpha \big(u_0(t)\big) - 2T_\alpha \big(u_0(t)\big) - c_1 \big[T_\alpha T_\alpha \big(u_0(t)\big)$$

$$+2T_\alpha \big(u_0(t)\big) + u_0(t) - \exp\left(-\frac{1}{\alpha} t^\alpha\right)\big] = 0, \ \ u_1(0) = (T_\alpha u_1)(0) = 0,$$

$$p^2; T_\alpha T_\alpha \big(u_2(t, c_1, c_2)\big) + 2T_\alpha \big(u_2(t, c_1, c_2)\big) - T_\alpha T_\alpha \big(u_1(t, c_1)\big) - 2T_\alpha \big(u_1(t, c_1)\big)$$

$$-c_2 \left[T_\alpha T_\alpha \big(u_0(t)\big) + 2T_\alpha \big(u_0(t)\big) + u_0(t) - \exp\left(-\frac{1}{\alpha} t^\alpha\right)\right]$$

$$-c_1 \big[T_\alpha T_\alpha \big(u_1(t, c_1)\big) + 2T_\alpha \big(u_1(t, c_1)\big) + u_1(t, c_1)\big] = 0, \ \ u_2(0) = (T_\alpha u_2)(0) = 0, \qquad (4.8)$$

$$p^3; T_\alpha T_\alpha \big(u_3(t, c_1, c_2, c_3)\big) + 2T_\alpha \big(u_3(t, c_1, c_2, c_3)\big) - T_\alpha T_\alpha \big(u_2(t, c_1, c_2)\big) - 2T_\alpha \big(u_2(t, c_1, c_2)\big)$$

$$-c_3 \left[T_\alpha T_\alpha \big(u_0(t)\big) + 2T_\alpha \big(u_0(t)\big) + u_0(t) - \exp\left(-\frac{1}{\alpha} t^\alpha\right)^2\right]$$

$$-c_2 \big[T_\alpha T_\alpha \big(u_1(t, c_1)\big) + 2T_\alpha \big(u_1(t, c_1)\big) + u_1(t, c_1)\big]$$

$$-\frac{1}{2} c_1 \big[2T_\alpha T_\alpha \big(u_2(t, c_1, c_2)\big) + 4T_\alpha \big(u_2(t, c_1, c_2)\big) + 2u_2(t, c_1, c_2)\big] = 0, \ \ u_3(0) = (T_\alpha u_3)(0) = 0,$$

$$\vdots$$

Matching solution of fractional differential equations (4.8), are

$$u_0(t) = -\frac{1}{2} \exp\left(-\frac{2}{\alpha} t^\alpha\right) + \frac{1}{2},$$

$$u_1(t, c_1) = c_1 \left[\frac{1}{4\alpha} t^\alpha - \frac{3}{4} - \left(\frac{1}{2\alpha} t^\alpha + \frac{1}{4}\right) \exp\left(-\frac{2}{\alpha} t^\alpha\right) + \exp\left(-\frac{1}{\alpha} t^\alpha\right)\right],$$

$$\vdots$$

Third-terms approximation to the solution of Eq. (4.6), will be obtained as follows

$$u^3(t, c_1, c_2, c_3) = u_0(t) + u_1(t, c_1) + u_2(t, c_1, c_2) + u_3(t, c_1, c_2, c_3). \qquad (4.9)$$

Table 4.2. Shows the optimal values of the convergence control constants c_1, c_2 and c_3 in $u^3(t, c_1, c_2, c_3)$ given in Eq. 4.9 for different values of α which can be obtained using the procedure mentioned in (3.9) up to (3.11).

Auxiliary parameters	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
c_1	-0.9101798255	-0.9611597674	-0.9782159569	-0.9860817914
c_2	0.006410499989	0.001634668655	0.0005180834564	0.0002083294705
c_3	0.005526098232	0.0003140590347	0.00003842662077	0.000007306947318

TABLE 4.2. Values of auxiliary parameters for the third-order OHAM solution of Eq. (4.6) for different orders.

In Figure 4.2, the exact and approximate solutions of linear fractional equation for $\alpha = 0.4, 0.6, 0.8$ and 1.0, are plotted.
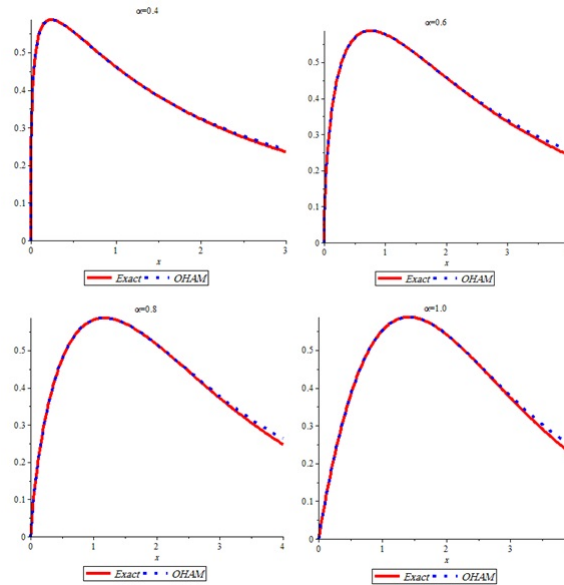
FIGURE 4.2. 3th-order approximation of OHAM and exact solution for Example 4.2.

Example 4.3. Consider the following fractional differential equation with initial value

$$\left(\frac{1}{\alpha}t^\alpha\right)T_\alpha T_\alpha u + 8T_\alpha u + \left(\frac{1}{\alpha}t^\alpha\right)^2 u = \left(\frac{1}{\alpha}t^\alpha\right)^6 - \left(\frac{1}{\alpha}t^\alpha\right)^5$$

$$+44\left(\frac{1}{\alpha}t^\alpha\right)^3 - 30\left(\frac{1}{\alpha}t^\alpha\right)^2, \quad u(0) = (T_\alpha)(0) = 0, \tag{4.10}$$

where $u(t) = \left(\frac{1}{\alpha}t^\alpha\right)^4 - \left(\frac{1}{\alpha}t^\alpha\right)^3$.

By the proposed conformable fractional OHAM approach, we get

$$(1-p)\left[T_\alpha T_\alpha\left(v(t,p)\right) + \frac{8\alpha}{t^\alpha}T_\alpha\left(v(t,p)\right)\right] - H(p)\left[T_\alpha T_\alpha\left(v(t,p)\right) + \frac{8\alpha}{t^\alpha}T_\alpha\left(v(t,p)\right) + \frac{1}{\alpha}t^\alpha v(t,p)\right.$$

$$\left. - \left(\frac{1}{\alpha}t^\alpha\right)^5 + \left(\frac{1}{\alpha}t^\alpha\right)^4 - 44\left(\frac{1}{\alpha}t^\alpha\right)^2 + 30\left(\frac{1}{\alpha}t^\alpha\right)\right] = 0, \tag{4.11}$$

Substituting into Eq. (4.11) and setting to zero the coefficient, we reads

$$p^0;\ T_\alpha T_\alpha\left(u_0(t)\right) + \frac{8\alpha}{t^\alpha}T_\alpha\left(u_0(t)\right) = 0,\ u_0(0) = (T_\alpha u_0)(0) = 0,$$

$$p^1;\ T_\alpha T_\alpha\left(u_1(t,c_1)\right) + \frac{8\alpha}{t^\alpha}T_\alpha T_\alpha\left(u_1(t,c_1)\right) - T_\alpha T_\alpha\left(u_0(t)\right) - \frac{8\alpha}{t^\alpha}T_\alpha T_\alpha\left(u_0(t)\right) - c_1\left[T_\alpha T_\alpha\left(u_0(t)\right)\right.$$

$$\left. + \frac{8\alpha}{t^\alpha}T_\alpha T_\alpha\left(u_0(t)\right) + \frac{1}{\alpha}t^\alpha u_0(t) - \left(\frac{1}{\alpha}t^\alpha\right)^5 + \left(\frac{1}{\alpha}t^\alpha\right)^4 - 44\left(\frac{1}{\alpha}t^\alpha\right)^2 + 30\left(\frac{1}{\alpha}t^\alpha\right)\right] = 0,\ u_1(0) = (T_\alpha u_1)(0) = 0,$$

$$p^2;\ T_\alpha T_\alpha\left(u_2(t,c_1,c_2)\right) + \frac{8\alpha}{t^\alpha}T_\alpha\left(u_2(t,c_1,c_2)\right) - T_\alpha T_\alpha\left(u_1(t,c_1)\right) - \frac{8\alpha}{t^\alpha}T_\alpha\left(u_1(t,c_1)\right) \tag{4.12}$$

$$-c_2\left[T_\alpha T_\alpha\left(u_0(t)\right) + \frac{8\alpha}{t^\alpha}T_\alpha\left(u_0(t)\right) + \frac{1}{\alpha}t^\alpha u_0(t) - \left(\frac{1}{\alpha}t^\alpha\right)^5 + \left(-\frac{1}{\alpha}t^\alpha\right)^4 - 44\left(-\frac{1}{\alpha}t^\alpha\right)^2 + 30\left(-\frac{1}{\alpha}t^\alpha\right)\right]$$

$$-c_1\left[T_\alpha T_\alpha\left(u_1(t,c_1)\right) + \frac{8\alpha}{t^\alpha}T_\alpha\left(u_1(t,c_1)\right) + \frac{1}{\alpha}t^\alpha u_1(t,c_1)\right] = 0,\ u_2(0) = (T_\alpha u_2)(0) = 0,$$

$$\vdots$$

Corresponding solution of conformable fractional equations (4.12), are as follows

$$u_0(t) = 0,$$

$$u_1(t, c_1) = c_1 \left[-\frac{1}{98} \left(\frac{1}{\alpha} t^\alpha \right)^7 + \frac{1}{78} \left(\frac{1}{\alpha} t^\alpha \right)^6 - \left(\frac{1}{\alpha} t^\alpha \right)^4 + \left(\frac{1}{\alpha} x^\alpha \right)^3 \right],$$

$$u_2(t, c_1, c_2) = c_1^2 \left[-\frac{1}{16660} \left(\frac{1}{\alpha} x^\alpha \right)^{10} + \frac{1}{11232} \left(\frac{1}{\alpha} x^\alpha \right)^9 \right.$$

$$\left. -\frac{1}{47} \left(\frac{1}{\alpha} x^\alpha \right)^7 + \frac{1}{39} \left(\frac{1}{\alpha} x^\alpha \right)^6 - \left(\frac{1}{\alpha} x^\alpha \right)^4 + \left(\frac{1}{\alpha} x^\alpha \right)^3 \right]$$

$$+(c_1 + c_2) \left[-\frac{1}{98} \left(\frac{1}{\alpha} x^\alpha \right)^7 + \frac{1}{78} \left(\frac{1}{\alpha} x^\alpha \right)^6 - \left(\frac{1}{\alpha} x^\alpha \right)^4 + \left(\frac{1}{\alpha} x^\alpha \right)^3 \right] + \frac{1}{30} \left(\frac{1}{\alpha} t^\alpha \right)^3,$$

$$\vdots$$

Consequently, two-terms approximation to the solution of Eq. (4.10), will be obtained as the following

$$u^2(t, c_1, c_2) = u_0(t) + u_1(t, c_1) + u_2(t, c_1, c_2). \tag{4.13}$$

Table 4.3. Shows the optimal values of the convergence control constants c_1 and c_2 in $u^3(t, c_1, c_2)$ given in Eq. 4.13 for different values of α which can be obtained using the procedure mentioned in (3.9) up to (3.11).

Auxiliary parameters	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1.0$
c_1	0.8872378786	0.8625071649	0.8625071649	0.8625071649
c_2	-3.527294218	-3.431935122	-3.431935122	-3.431935122

TABLE 4.3. Values of auxiliary parameters for the third-order OHAM solution of Eq. (4.10) for different orders.

In Figure 4.3. the exact solution and solution of conformable fractional OHAM of fractional equation, for $\alpha = 0.4, 0.6, 0.8$ and 1.0, are plotted.

Example 4.4. Consider the following fractional differential equation with initial value

$$\left(\frac{1}{\alpha} t^\alpha + 1 \right)^2 T_\alpha T_\alpha u + \left(\frac{1}{\alpha} t^\alpha + 1 \right) T_\alpha u + \left(\left(\frac{1}{\alpha} t^\alpha + 1 \right)^2 - \frac{1}{4} \right) u = 0, \quad u(0) = 1, \ (T_\alpha u)(0) = 0 \tag{4.14}$$

where $u(x) = \dfrac{1}{\sqrt{1 + \dfrac{1}{\alpha} t^\alpha}} \left[\dfrac{1}{2} \sin \left(\dfrac{1}{\alpha} t^\alpha \right) + \cos \left(\dfrac{1}{\alpha} t^\alpha \right) \right].$

According to the proposed conformable fractional OHPM approach, we obtain

$$(1 - p) \left[\left(\frac{1}{\alpha} t^\alpha + 1 \right)^2 T_\alpha T_\alpha \left(v(t, p) \right) + \left(\frac{1}{\alpha} t^\alpha + 1 \right) T_\alpha \left(v(t, p) \right) \right] - H(p) \left[\left(\frac{1}{\alpha} t^\alpha + 1 \right)^2 T_\alpha T_\alpha \left(v(t, p) \right) \right.$$

$$\left. + \left(\frac{1}{\alpha} t^\alpha + 1 \right) T_\alpha \left(v(t, p) \right) + \left(\left(\frac{1}{\alpha} t^\alpha + 1 \right)^2 - \frac{1}{4} \right) v(t, p) \right] = 0. \tag{4.15}$$
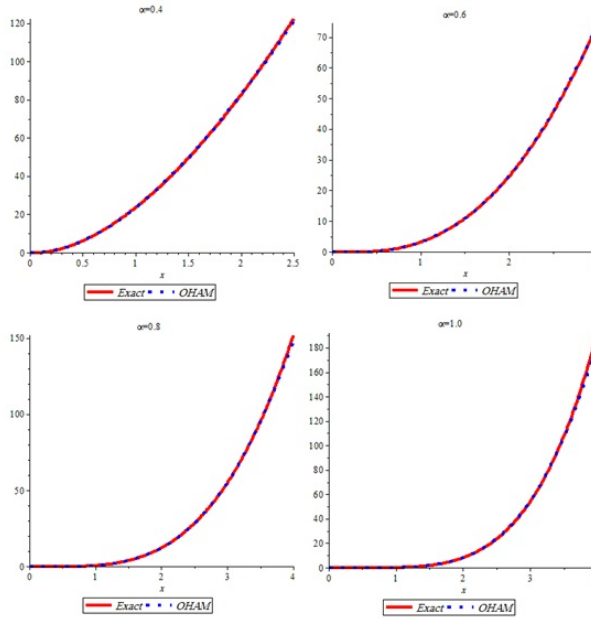
FIGURE 4.3. 2th-order approximation of OHAM and exact solution for Example 4.3.

Substituting Eq. (4.3) into Eq. (4.15) and setting to zero the coefficient of the same powers of p, results in

$$p^0; \left(\frac{1}{\alpha}t^\alpha + 1\right)^2 T_\alpha T_\alpha\big(u_0(t)\big) + \left(\frac{1}{\alpha}t^\alpha + 1\right) T_\alpha\big(u_0(t)\big) = 0, \ u_0(0) = 1, \ (T_\alpha u_0)(0) = 0,$$

$$p^1; \left(\frac{1}{\alpha}t^\alpha + 1\right)^2 T_\alpha T_\alpha\big(u_1(t,c_1)\big) + \left(\frac{1}{\alpha}t^\alpha + 1\right) T_\alpha\big(u_1(t,c_1)\big) - \left(\frac{1}{\alpha}t^\alpha + 1\right)^2 T_\alpha T_\alpha\big(u_0(t)\big)$$

$$- \left(\frac{1}{\alpha}t^\alpha + 1\right) T_\alpha\big(u_0(t)\big) - c_1\left[\left(\frac{1}{\alpha}t^\alpha + 1\right)^2 T_\alpha T_\alpha\big(u_0(t)\big)\right.$$

$$\left. + \left(\frac{1}{\alpha}t^\alpha + 1\right) T_\alpha T_\alpha\big(u_0(t)\big) + \left(\left(\frac{1}{\alpha}t^\alpha + 1\right)^2 - \frac{1}{4}\right) u_0(t)\right] = 0, \ u_1(0) = (T_\alpha u_1)(0) = 0,$$

$$p^2; \left(\frac{1}{\alpha}t^\alpha + 1\right)^2 T_\alpha T_\alpha\big(u_2(t,c_1,c_2)\big) + \left(\frac{1}{\alpha}t^\alpha + 1\right) T_\alpha\big(u_2(t,c_1,c_2)\big) - \left(\frac{1}{\alpha}t^\alpha + 1\right)^2 T_\alpha T_\alpha\big(u_1(t,c_1)\big)$$

$$- \left(\frac{1}{\alpha}t^\alpha + 1\right) T_\alpha\big(u_1(t,c_1)\big) \tag{4.16}$$

$$-c_2\left[\left(\frac{1}{\alpha}t^\alpha + 1\right)^2 T_\alpha T_\alpha\big(u_0(t)\big) + \left(\frac{1}{\alpha}t^\alpha + 1\right) T_\alpha\big(u_0(t)\big) + \left(\left(\frac{1}{\alpha}t^\alpha + 1\right)^2 - \frac{1}{4}\right) u_0(t)\right]$$

$$-c_1\left[\left(\frac{1}{\alpha}t^\alpha + 1\right)^2 T_\alpha T_\alpha\big(u_1(t,c_1)\big) + \left(\frac{1}{\alpha}t^\alpha + 1\right) T_\alpha\big(u_1(t,c_1)\big)\right.$$

$$\left. + \left(\left(\frac{1}{\alpha}t^\alpha + 1\right)^2 - \frac{1}{4}\right) u_1(t,c_1) = 0, \ u_2(0) = (T_\alpha u_2)(0) = 0,\right.$$

$$\vdots$$

Corresponding solution of equations (4.16), are

$$u_0(t) = 1,$$

$$u_1(t,c_1) = c_1\left[\frac{1}{4}\left(\frac{1}{\alpha}t^\alpha\right)^2 + \frac{1}{2}\left(\frac{1}{\alpha}t^\alpha\right) - \frac{3}{4}\ln\left(1 + \frac{1}{\alpha}t^\alpha\right)\right],$$

$$\vdots$$

Two-terms approximation to the solution of Eq. (4.14), can be obtained as the following

$$u^2(t, c_1, c_2) = u_0(t) + u_1(t, c_1) + u_2(t, c_1, c_2). \tag{4.17}$$

The optimal values of the convergence control constants c_1 and c_2 in $u^2(t, c_1, c_2)$ given in Eq. (4.17) for different values of α which can be obtained using the procedure mentioned in (3.9) up to (3.11), that those are

$$c_1 = 0.9140123550, \qquad c_2 = 3.639201126.$$

In Figure 4.4, the exact solution and solution of conformable fractional OHAM of fractional equation, for $\alpha = 0.4, 0.6, 0.8$ and 1.0, are plotted.
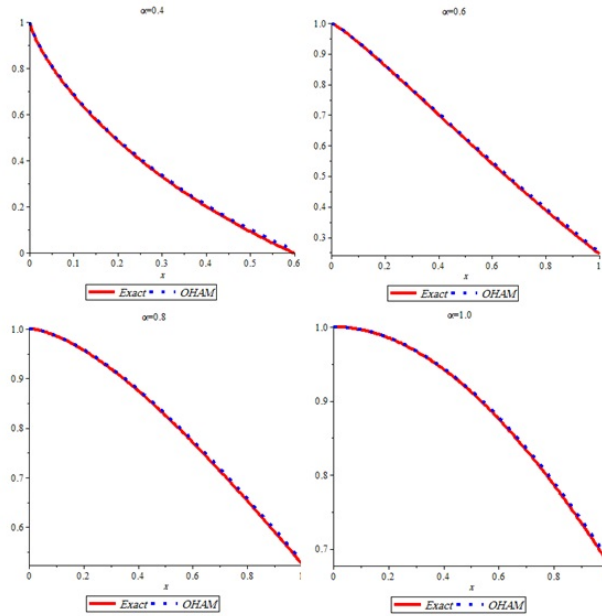

FIGURE 4.4. 2th-order approximation of OHAM and exact solution for Example 4.4.

Example 4.5. Consider the conformable Bratu-type equation with initial value

$$T_\alpha T_\alpha u + \pi^2 \exp(-u) = 0, \quad u(0) = 0, \ (T_\alpha u)(0) = \pi, \tag{4.18}$$

where $u(x) = \ln\left(1 + \sin\left(\frac{\pi}{\alpha} x^\alpha\right)\right)$.

Conforming to the proposed conformable fractional OHAM, we get

$$(1 - p)\big[T_\alpha T_\alpha\big)v(t, p))\big] - H(p)\big[T_\alpha T_\alpha\big(v(t, p)\big) + \pi^2\big(1 - v(t, p)\big)\big] = 0. \tag{4.19}$$

Substituting Eqs. (4.3) into Eq. (4.19) and setting to zero the coefficient of like powers of p, becomes

$$p^0; \mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_0(t)\big) = 0, \quad u_0(0) = 0, \quad (\mathrm{T}_\alpha u_0)(0) = \pi,$$

$$p^1; \mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_1(t,c_1)\big) - \mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_0(t)\big) - c_1 \big[\mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_0(t)\big) + \pi^2 \big(1 - u_0(t)\big)\big] = 0, \quad u_1(0) = (\mathrm{T}_\alpha u_1)(0) = 0,$$

$$p^2; \mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_2(t,c_1,c_2)\big) - \mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_1(t,c_1)\big) - c_2 \big[\mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_0(t)\big) + \pi^2 \big(1 - u_0(t)\big)\big] \qquad (4.20)$$

$$-c_1 \big[\mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_1(t,c_1)\big) - \pi^2 u_1(t,c_1)\big] = 0, \quad u_2(0) = (\mathrm{T}_\alpha u_2)(0) = 0,$$

$$p^3; \mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_3(t,c_1,c_2,c_3)\big) - \mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_2(t,c_1,c_2)\big) - c_3 \big[\mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_0(t)\big) + \pi^2 \big(1 - u_0(t)\big)\big]$$

$$-c_2 \big[\mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_1(t,c_1)\big) - \pi^2 u_1(t,c_1)\big] - c_1 \big[\mathrm{T}_\alpha \mathrm{T}_\alpha \big(u_2(t,c_1,c_2)\big) - \pi^2 u_2(t,c_1,c_2)\big] = 0, \quad u_3(0) = (\mathrm{T}_\alpha u_3)(0) = 0,$$

$$\vdots$$

Matching solution of conformable linear differential equations (4.20), are as follows

$$u_0(t) = \pi \left(\frac{1}{\alpha} t^\alpha\right),$$

$$u_1(t) = c_1 \pi^2 \left[\frac{1}{2} \left(\frac{1}{\alpha} t^\alpha\right)^2 - \frac{1}{6} \pi \left(\frac{1}{\alpha} t^\alpha\right)^3\right],$$

$$u_2(t) = \frac{1}{6} \pi^3 \left[\frac{1}{20} c_1^2 \pi^2 \left(\frac{1}{\alpha} t^\alpha\right)^5 - \frac{1}{4} \pi c_1^2 \left(\frac{1}{\alpha} t^\alpha\right)^4 - (c_1^2 + c_1 + c_2) \left(\frac{1}{\alpha} t^\alpha\right)^3\right]$$

$$+ \frac{1}{2} \left[c_1^2 \pi^2 + \pi^2 c_1 + \pi^2 c_2\right] \left(\frac{1}{\alpha} t^\alpha\right)^2,$$

$$\vdots$$

Third-terms approximation to the solution of Eq. (4.18), can be obtained as the following

$$u^3(t, c_1, c_2, c_3) = u_0(t) + u_1(t, c_1) + u_2(t, c_1, c_2) + u_3(t, c_1, c_2, c_3). \qquad (4.21)$$

Table 4.4. Shows the optimal values of the convergence control constants c_1, c_2 and c_3 in $u^3(t, c_1, c_2, c_3)$ given in Eq. 4.21 for different values of α which can be obtained using the procedure mentioned in (3.9) up to (3.11).

Auxiliary parameters	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$
c_1	-1.067087561	-1.012692289	-0.9963537959	-1.001497291
c_2	0.003188790034	0.0002156385026	-0.0001338547451	0.000002816029907
c_3	-0.005191026932	-0.00001861202016	0.000003592158757	$-1.798000219 \times 10^{-8}$

TABLE 4.4. Values of auxiliary parameters for the third-order OHAM solution of Eq. (4.18) for different orders.

In Figure 4.5, The exact solution and solution of conformable fractional OHAM of fractional equation, for $\alpha = 0.5, 0.7, 0.9$ and 1.0, are plotted.
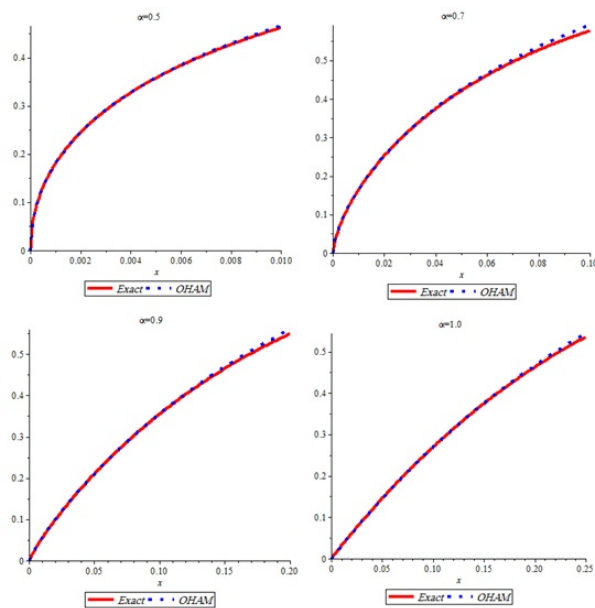
FIGURE 4.5. 3th-order approximation of OHAM and exact solution for Example 4.5.

5. Conclusion

In this paper, optimal homotopy asymptotic method is applied to obtain an approximate solution of fractional differential equations. Conformable fractional derivatives are used for fractional derivative in this study. The results can be expressed that CFD is a simple tool to obtain the approximate solution of linear and nonlinear fractional differential equations in comparison to the other definitions. What can one learn from the plots: approximate solution for different are larger convergence interval, when is closer to . To show the effectiveness of the method, some fractional differential equations and Bratu-type equation as an example have been solved by the conformable fractional optimal homotopy asymptotic method.

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MOUSA ILIE

DEPARTMENT OF MATHEMATICS, GUILAN SCIENCE AND RESEARCH BRANCH, ISLAMIC AZAD UNIVERSITY, RASHT, IRAN

DEPARTMENT OF MATHEMATICS, RASHT BRANCH, ISLAMIC AZAD UNIVERSITY, RASHT, IRAN

E-mail address: ilie@iaurasht.ac.ir

JAFAR BIAZAR

DEPARTMENT OF MATHEMATICS, RASHT BRANCH, ISLAMIC AZAD UNIVERSITY, RASHT, IRAN

DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCES, UNIVERSITY OF GUILAN, P.O. BOX. 41335-1914, GUILAN, RASHT, IRAN

E-mail address: biazar@guilan.ac.ir, biazar@iaurasht.ac.ir

ZAINAB AYATI

DEPARTMENT OF ENGINEERING SCIENCES, FACULTY OF TECHNOLOGY AND ENGINEERING EAST OF GUILAN, UNIVERSITY OF GUILAN, P.C. 44891-63157, RUDSAR-VAJARGAH, IRAN

E-mail address: Ayati.zainab@gmail.com