Journal of Fractional Calculus and Applications
Vol. 10(1) Jan. 2019, pp. 68-84.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

ON THE INTEGRAL REPRESENTATIONS OF RELATIVE (p,q)-TH TYPE AND RELATIVE (p,q)-TH WEAK TYPE OF ENTIRE AND MEROMORPHIC FUNCTIONS

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Abstract. In this paper we wish to establish the integral representations of relative (p,q)-th type and relative (p,q)-th weak type of entire and meromorphic functions. We also investigate their equivalence relation under some certain condition.

1. Introduction

For any entire function f, $M_f(r)$, a function of r is defined as follows:

$$M(r) \equiv M_f(r) = \max_{|z|=r} |f(z)|.$$

If an entire function f is non-constant then $M_f(r)$ is strictly increasing and continuous and its inverse $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} M_f^{-1}(s) = \infty$.

Whenever f is meromorphic, one can define another function $T_f(r)$ known as Nevanlinna's characteristic function of f in the following manner which perform the same role as the maximum modulus function:

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r,a)$ $\left(\bar{N}_f(r,a)\right)$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + \bar{n}_f(0,a) \log r$$

$$\left(\bar{N}_f(r,a) = \int_0^r \frac{\bar{n}_f(t,a) - \bar{n}_f(0,a)}{t} dt + \bar{n}_f(0,a) \log r\right).$$

2010 *Mathematics Subject Classification.* 30D20, 30D30, 30D35.

Key words and phrases. Entire functions, meromorphic function, relative (p,q)-th order, relative (p,q)-th lower order, relative (p,q)-th type, relative (p,q)-th weak type.

Submitted Oct. 31, 2017. Revised March 9, 2018.

In addition we symbolize by $n_f(r,a)$ $\left(\bar{n}_f(r,a)\right)$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞-point is a pole of f. In many situations, $N_f(r,\infty)$ and $\bar{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\bar{N}_f(r)$, respectively. Also the function $m_f(r,\infty)$ alternatively symbolized by $m_f(r)$ known as the proximity function of f is defined in the following way:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta,$$

where

$$\log^+ x = \max\{\log x, 0\} \text{ for all } x \geqslant 0,$$

and some times one may denote $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r,a)$.

The term $m(r,a)$, which is defined to be the mean value of $\log^+\left|\frac{1}{f-a}\right|$ (or $\log^+|f|$ if $a=\infty$) on the circle $|z|=r$, receives a remarkable contribution only from those arcs on the circle where the functional values differ very little from the given value 'a'. The magnitude of the proximity function can thus be considered as a measure for the mean deviation on the circle $|z|=r$ of the functional value f from the value 'a'.

If f is an entire function, then the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as follows:

$$T_f(r) = m_f(r) \ .$$

Moreover, if f is non-constant entire then $T_f(r)$ is strictly increasing and continuous functions of r. Also its inverse $T_f^{-1} : (T_f(0), \infty) \to (0,\infty)$ exists and is such that $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$. Also the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of the Nevanlinna's characteristic functions of the meromorphic functions f and g.

The *order* and *lower order* of entire or meromorphic functions which are generally used in computational purpose are classical in complex analysis. Bernal [1, 2] introduced the *relative order* between two entire functions to avoid comparing growth just with $\exp z$. Extending the notion of *relative order,* Ruiz et al. [5] introduced the *relative* (p,q)*-th order* (respectively *relative lower* (p,q)*-th order)* where p and q are any two positive integers. Also for any two positive integers p and q, Debnath et al. [3] introduced the definition of *relative* (p,q)*-th order* and *relative lower* (p,q)*-th order* of a meromorphic function with respect to an entire function. Now to compare the growth of entire functions having the same *relative* (p,q)*-th order* or *relative lower* (p,q)*-th order*, we wish to introduce the definition of *relative* (p,q)*-th type* and *relative* (p,q)*-th weak type* of an entire function or meromorphic function with respect to an entire function and establish their integral representations. We also investigate their equivalence relations under certain conditions. We do not explain the standard definitions and notations in the theory of entire and meromorphic functions (see, e.g., [4, 6]) .

2. Preliminary remarks and definitions

We need the following notations.

$$\log^{[k]} r = \log\left(\log^{[k-1]} r\right) \text{ for } k = 1, 2, 3, \cdots ; \ \log^{[0]} r = r$$

and

$$\exp^{[k]} r = \exp\left(\exp^{[k-1]} r\right) \text{ for } k = 1, 2, 3, \cdots ; \ \exp^{[0]} r = r.$$

Taking this into account, let us denote

$$\rho_{\alpha}^{(p,q)}(\beta) = \limsup_{r\to\infty} \frac{\log^{[p]} \alpha^{-1}\beta(r)}{\log^{[q]} r} \text{ and } \lambda_{\alpha}^{(p,q)}(\beta) = \liminf_{r\to\infty} \frac{\log^{[p]} \alpha^{-1}\beta(r)}{\log^{[q]} r}$$

where p,q are any two positive integers and $\alpha(x), \beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions such that $\alpha^{-1}(x)$ exist.

If we consider $\alpha(x) = M_g(x)$ and $\beta(x) = M_f(x)$ where f and g are any two entire functions with index-pairs (m, q) and (m, p) respectively where p, q, m are positive integers such that $m \geq \max(p, q)$, then the above definition reduces to the definition of relative (p, q)-th order and relative (p, q)-th lower order of an entire function f with respect to another entire function g respectively as introduced by Ruiz et al. [5]. Similarly if we take $\alpha(x) = T_g(x)$ and $\beta(x) = T_f(x)$ where f is a meromorphic function and g be any entire function with index-pairs (m, q) and (m, p) respectively where p, q, m are positive integers such that $m \geq \max(p, q)$, then the above definition reduces to the definition of relative (p, q)-th order and relative (p, q)-th lower order of a meromorphic function f with respect to an entire function g respectively as introduced by Debnath et al. [3]. For details about index pair, one may see [3] and [5].

In order to refine the above growth scale, now we intend to introduce the definition of an another growth indicator, called *relative* (p, q)*-th type*, as follows:

Definition 1. Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions. The relative (p, q)-th type of $\beta(x)$ with respect to $\alpha(x)$ having finite positive relative (p, q)-th order $\rho_{\alpha}^{(p,q)}(\beta)$ $\left(a < \rho_{\alpha}^{(p,q)}(\beta) < \infty\right)$ where p and q are any two positive integers is defined by

$$\sigma_{\alpha}^{(p,q)}(\beta) = \limsup_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}} .$$

The above definition can alternatively be given in the following manner:

Definition 2. Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive *relative* (p, q)*-th order* $\rho_{\alpha}^{(p,q)}(\beta)$ $\left(a < \rho_{\alpha}^{(p,q)}(\beta) < \infty\right)$ where p and q are any two positive integers. Then the *relative* (p, q) *-th type* $\sigma_{\alpha}^{(p,q)}(\beta)$ of $\beta(x)$ with respect to $\alpha(x)$ is define as: The integral $\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}} dr \, (r_0 > 0)$ converges for $k > \sigma_{\alpha}^{(p,q)}(\beta)$ and diverges for $k < \sigma_{\alpha}^{(p,q)}(\beta)$.

Analogously, to determine the relative growth of two increasing functions having same non zero finite *relative* (p, q)*-th lower order*, one can introduce the definition of *relative* (p,q)*-th weak type* of finite positive *relative* (p,q)*-th lower order* $\lambda_{\alpha}^{(p,q)}(\beta)$ in the following way.

Definition 3. Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions having finite positive *relative* (p,q)*-th lower order* $\lambda_{\alpha}^{(p,q)}(\beta)$ $\left(a<\lambda_{\alpha}^{(p,q)}(\beta)<\infty\right)$ where p and q are any two positive integers.Then the *relative* (p,q)*-th weak type* of $\beta(x)$ with respect to $\alpha(x)$ is defined as

$$\tau_{\alpha}^{(p,q)}(\beta)=\liminf_{r\to\infty}\frac{\log^{[p-1]}\alpha^{-1}\beta(r)}{\left(\log^{[q-1]}r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}}\ .$$

The above definition can also alternatively be given

Definition 4. Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions having finite positive *relative* (p,q)*-th lower order* $\lambda_{\alpha}^{(p,q)}(\beta)$ $\left(a<\lambda_{\alpha}^{(p,q)}(\beta)<\infty\right)$ where p and q are any two positive integers. Then the *relative* (p,q)*-th weak type* $\tau_{\alpha}^{(p,q)}(\beta)$ of $\beta(x)$ with respect to $\alpha(x)$ is defined as

$$\int_{r_0}^{\infty}\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}}dr\ (r_0>0)$$

converges for $k>\tau_{\alpha}^{(p,q)}(\beta)$ and diverges for $k<\tau_{\alpha}^{(p,q)}(\beta)$.

Now a question may arise about the equivalence of the definitions of *relative* (p,q)*-th type* and *relative* (p,q)*-th weak type* with their integral representations. In this paper we would like to establish such equivalence of Definition 1 and Definition 2, and Definition 3 and Definition 4 and also investigate some growth properties related to *relative* (p,q)*-th type* and *relative* (p,q)*-th weak type* of $\beta(x)$ with respect to $\alpha(x)$.

3. **Lemma.**

In this section we present a lemma which will be needed in the sequel.

Lemma 1. Let the integral $\int_{r_0}^{\infty}\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{A}\right)\right]^{k+1}}dr\ (r_0>0)$ converge where $0<A<\infty$. Then

$$\lim_{r\to\infty}\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{A}\right)\right]^{k}}=0\ .$$

Proof. Since the integral $\int_{r_0}^{\infty}\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{A}\right)\right]^{k+1}}dr\ (r_0>0)$ converges, then there exist $R(\varepsilon)>0$ such that

$$\int_{r_0}^{\infty}\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{A}\right)\right]^{k+1}}dr<\varepsilon,\ \text{if}\ r_0>R(\varepsilon)\ .$$

Therefore,

$$\int_{r_0}^{\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)+r_0} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{k+1}} dr < \varepsilon .$$

Since $\log^{[p-2]} \alpha^{-1}\beta(r)$ increases with r, so

$$\int_{r_0}^{\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)+r_0} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{k+1}} dr \geq$$

$$\frac{\log^{[p-2]} \alpha^{-1}\beta(r_0)}{\left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right]^{k+1}} \cdot \left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right] .$$

i.e., for all sufficiently large values of r,

$$\int_{r_0}^{\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)+r_0} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{k+1}} dr \geq$$

$$\frac{\log^{[p-2]} \alpha^{-1}\beta(r_0)}{\left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right]^{k}} ,$$

so that

$$\frac{\log^{[p-2]} \alpha^{-1}\beta(r_0)}{\left[\exp\left(\left(\log^{[q-1]} r_0\right)^A\right)\right]^{k}} < \varepsilon \text{ if } r_0 > R(\varepsilon) .$$

$$i.e., \lim_{r\to\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^A\right)\right]^{k}} = 0.$$

This proves the lemma.

4. **Main Results.**

In this section we state the main results of this chapter.

Theorem 1. Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive *relative* (p, q) *-th order* $\rho_\alpha^{(p,q)}(\beta)$ $\left(0 < \rho_\alpha^{(p,q)}(\beta) < \infty\right)$ and *relative* (p, q) *-th type* $\sigma_\alpha^{(p,q)}(\beta)$ where p and q are any two positive integers. Then Definition 1 and Definition 2 are equivalent.

Proof. Let us consider $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions such that $\rho_\alpha^{(p,q)}(\beta)$ $\left(0 < \rho_\alpha^{(p,q)}(\beta) < \infty\right)$ exists for any two positive integers p and q.

Case I. $\sigma_\alpha^{(p,q)}(\beta) = \infty$.

Definition 1 $\Rightarrow$ Definition 2.

As $\sigma_{\alpha}^{(p,q)}(\beta) = \infty$, from Definition 1 we have for arbitrary positive G and for a sequence of values of r tending to infinity that

$$\log^{[p-1]} \alpha^{-1}\beta(r) > G \cdot \left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}$$

$$i.e.,\ \log^{[p-2]} \alpha^{-1}\beta(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{G} . \tag{1}$$

If possible let the integral $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{G+1}} dr$ $(r_0 > 0)$ be converge.

Then by Lemma 1,

$$\limsup_{r\to\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{G}} = 0 .$$

So for all sufficiently large values of r,

$$\log^{[p-2]} \alpha^{-1}\beta(r) < \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{G} . \tag{2}$$

Therefore from (1) and (2) we arrive at a contradiction.

Hence $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{G+1}} dr$ $(r_0 > 0)$ diverges whenever G is finite, which is the Definition 2.

Definition 2 $\Rightarrow$ Definition 1.

Let G be any positive number. Since $\sigma_{\alpha}^{(p,q)}(\beta) = \infty$, from Definition 2, the divergence of the integral $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{G+1}} dr$ $(r_0 > 0)$ gives for arbitrary positive ε and for a sequence of values of r tending to infinity

$$\log^{[p-2]} \alpha^{-1}\beta(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{G-\varepsilon}$$

$$i.e.,\ \log^{[p-1]} \alpha^{-1}\beta(r) > (G-\varepsilon)\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)},$$

which implies that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}} \geq G - \varepsilon .$$

Since $G > 0$ is arbitrary, it follows that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}} = \infty .$$

Thus Definition 1 follows.

Case II. $0 \leq \sigma_{\alpha}^{(p,q)}(\beta) < \infty$.

Definition 1 $\Rightarrow$ Definition 2.

Subcase (A). $0 < \sigma_{\alpha}^{(p,q)}(\beta) < \infty$.

Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions such that $0 < \sigma_{\alpha}^{(p,q)}(\beta) < \infty$ exists for any two positive integers p and q. Then according to the Definition 1, for arbitrary positive ε and for all sufficiently large values of r, we obtain

$$\log^{[p-1]} \alpha^{-1}\beta(r) < \left(\sigma_{\alpha}^{(p,q)}(\beta) + \varepsilon\right)\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}$$

$$i.e.,\ \log^{[p-2]} \alpha^{-1}\beta(r) < \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\sigma_{\alpha}^{(p,q)}(\beta)+\varepsilon}$$

$$i.e.,\ \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k}} < \frac{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\sigma_{\alpha}^{(p,q)}(\beta)+\varepsilon}}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k}}$$

$$i.e.,\ \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k}} < \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k-\left(\sigma_{\alpha}^{(p,q)}(\beta)+\varepsilon\right)}}\ .$$

Therefore $\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}} dr\ (r_0 > 0)$ converges for $k > \sigma_{\alpha}^{(p,q)}(\beta)$.

Again by Definition 1, we obtain for a sequence values of r tending to infinity that

$$\log^{[p-1]} \alpha^{-1}\beta(r) > \left(\sigma_{\alpha}^{(p,q)}(\beta) - \varepsilon\right)\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}$$

$$i.e.,\ \log^{[p-2]} \alpha^{-1}\beta(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\sigma_{\alpha}^{(p,q)}(\beta)-\varepsilon}. \quad (3)$$

So for $k < \sigma_{\alpha}^{(p,q)}(\beta)$, we get from (3) that

$$\frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k}} > \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k-\left(\sigma_{\alpha}^{(p,q)}(\beta)-\varepsilon\right)}}\ .$$

Therefore $\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}} dr\ (r_0 > 0)$ diverges for $k < \sigma_{\alpha}^{(p,q)}(\beta)$.

Hence $\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}} dr\ (r_0 > 0)$ converges for $k > \sigma_{\alpha}^{(p,q)}(\beta)$ and diverges for $k < \sigma_{\alpha}^{(p,q)}(\beta)$.

Subcase (B). $\sigma_{\alpha}^{(p,q)}(\beta) = 0$.

When $\sigma_{\alpha}^{(p,q)}(\beta) = 0$ for any two positive integers p and q , Definition 1 gives for all sufficiently large values of r that

$$\frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}} < \varepsilon \ .$$

Then as before we obtain that $\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}} dr \ (r_0 > 0)$ converges for $k > 0$ and diverges for $k < 0$.

Thus combining Subcase (A) and Subcase (B), Definition 2 follows.

Definition 2 ⇒ Definition 1.

From Definition 2 and for arbitrary positive ε the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\sigma_{\alpha}^{(p,q)}(\beta)+\varepsilon+1}} dr \ (r_0 > 0)$$

converges. Then by Lemma 1, we get

$$\limsup_{r\to\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\sigma_{\alpha}^{(p,q)}(\beta)+\varepsilon}} = 0 \ .$$

So we obtain all sufficiently large values of r that

$$\frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\sigma_{\alpha}^{(p,q)}(\beta)+\varepsilon}} < \varepsilon$$

$$i.e., \ \log^{[p-2]} \alpha^{-1}\beta(r) < \varepsilon \cdot \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\sigma_{\alpha}^{(p,q)}(\beta)+\varepsilon}$$

$$i.e., \ \log^{[p-1]} \alpha^{-1}\beta(r) < \log\varepsilon + \left(\sigma_{\alpha}^{(p,q)}(\beta) + \varepsilon\right)\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}$$

$$i.e., \ \limsup_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}} \leq \sigma_{\alpha}^{(p,q)}(\beta) + \varepsilon \ .$$

Since $\varepsilon > 0$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}} \leq \sigma_{\alpha}^{(p,q)}(\beta) \ . \tag{4}$$

On the other hand the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\sigma_{\alpha}^{(p,q)}(\beta)-\varepsilon+1}} dr \, (r_0 > 0)$$

implies that there exists a sequence of values of r tending to infinity such that

$$\frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\sigma_{\alpha}^{(p,q)}(\beta)-\varepsilon+1}} > \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{1+\varepsilon}}$$

$$i.e., \ \log^{[p-2]} \alpha^{-1}\beta(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\sigma_{\alpha}^{(p,q)}(\beta)-2\varepsilon}$$

$$i.e., \ \log^{[p-1]} \alpha^{-1}\beta(r) > \left(\sigma_{\alpha}^{(p,q)}(\beta) - 2\varepsilon\right)\left(\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)$$

$$i.e., \ \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}} > \left(\sigma_{\alpha}^{(p,q)}(\beta) - 2\varepsilon\right) .$$

As $\varepsilon > 0$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}} \geq \sigma_{\alpha}^{(p,q)}(\beta) . \tag{5}$$

So from (4) and (5) , we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}} = \sigma_{\alpha}^{(p,q)}(\beta) .$$

This proves the theorem.

Theorem 2. Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive *relative* (p,q) *-th lower order* $\lambda_{\alpha}^{(p,q)}(\beta)$ $\left(0 < \lambda_{\alpha}^{(p,q)}(\beta) < \infty\right)$ and *relative* (p,q) *-th weak type* $\tau_{\alpha}^{(p,q)}(\beta)$ where p and q are any two positive integers. Then Definition 3 and Definition 4 are equivalent.

Proof. Let us consider $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions such that $\lambda_{\alpha}^{(p,q)}(\beta)$ $\left(0 < \lambda_{\alpha}^{(p,q)}(\beta) < \infty\right)$ exists for any two positive integers p and q.

Case I. $\tau_{\alpha}^{(p,q)}(\beta) = \infty$.

Definition 3 $\Rightarrow$ Definition 4.

As $\tau_{\alpha}^{(p,q)}(\beta) = \infty$, from Definition 3 we obtain for arbitrary positive G and for all sufficiently large values of r that

$$\log^{[p-1]}\alpha^{-1}\beta(r) > G\cdot\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}$$

$$i.e.,\ \log^{[p-2]}\alpha^{-1}\beta(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{G} . \tag{6}$$

Now if possible let the integral $\int\limits_{r_0}^{\infty}\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{G+1}}dr\ (r_0>0)$ be converge.

Then by Lemma 1,

$$\liminf_{r\to\infty}\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{G}} = 0 .$$

So for a sequence of values of r tending to infinity we get

$$\log^{[p-2]}\alpha^{-1}\beta(r) < \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{G} . \tag{7}$$

Therefore from (6) and (7), we arrive at a contradiction.

Hence $\int\limits_{r_0}^{\infty}\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{G+1}}dr\ (r_0>0)$ diverges whenever G is finite, which is the Definition 4.

Definition 4 ⇒ Definition 3.

Let G be any positive number. Since $\tau_{\alpha}^{(p,q)}(\beta)=\infty$, from Definition 4, the divergence of the integral

$$\int\limits_{r_0}^{\infty}\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{G+1}}dr\ (r_0>0)$$

gives for arbitrary positive ε and for all sufficiently large values of r that

$$\log^{[p-2]}\alpha^{-1}\beta(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{G-\varepsilon}$$

$$i.e.,\ \log^{[p-1]}\alpha^{-1}\beta(r) > (G-\varepsilon)\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)},$$

which implies that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}\alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}} \geq G-\varepsilon .$$

Since $G>0$ is arbitrary, it follows that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}\alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}} = \infty .$$

Thus Definition 3 follows.

Case II. $0\leq\tau_{\alpha}^{(p,q)}(\beta)<\infty$.

Definition 3 $\Rightarrow$ Definition 4.
Subcase (C). $0 < \tau_{\alpha}^{(p,q)}(\beta) < \infty$.

Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions such that $0 < \tau_{\alpha}^{(p,q)}(\beta) < \infty$ exists for any two positive integers p and q. Then according to the Definition 3, for a sequence of values of r tending to infinity we get

$$\log^{[p-1]} \alpha^{-1}\beta(r) < \left(\tau_{\alpha}^{(p,q)}(\beta) + \varepsilon\right) \left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}$$

$$i.e.,\ \log^{[p-2]} \alpha^{-1}\beta(r) < \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\tau_{\alpha}^{(p,q)}(\beta)+\varepsilon}$$

$$i.e.,\ \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k}} < \frac{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\tau_{\alpha}^{(p,q)}(\beta)+\varepsilon}}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k}}$$

$$i.e.,\ \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k}} < \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k-\left(\tau_{\alpha}^{(p,q)}(\beta)+\varepsilon\right)}}.$$

Therefore $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}} dr\ (r_0 > 0)$ converges for $k > \tau_{\alpha}^{(p,q)}(\beta)$.

Again by Definition 3, we obtain for all sufficiently large values of r that

$$\log^{[p-1]} \alpha^{-1}\beta(r) > \left(\tau_{\alpha}^{(p,q)}(\beta) - \varepsilon\right) \left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}$$

$$i.e.,\ \log^{[p-2]} \alpha^{-1}\beta(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\tau_{\alpha}^{(p,q)}(\beta)-\varepsilon}. \tag{8}$$

So for $k < \tau_{\alpha}^{(p,q)}(\beta)$, we get from (8) that

$$\frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k}} > \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k-\left(\tau_{\alpha}^{(p,q)}(\beta)-\varepsilon\right)}}.$$

Therefore $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}} dr\ (r_0 > 0)$ diverges for $k < \tau_{\alpha}^{(p,q)}(\beta)$.

Hence $\int\limits_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}} dr\ (r_0 > 0)$ converges for $k > \tau_{\alpha}^{(p,q)}(\beta)$ and diverges for $k < \tau_{\alpha}^{(p,q)}(\beta)$.

Subcase (D). $\tau_{\alpha}^{(p,q)}(\beta) = 0$.

When $\tau_{\alpha}^{(p,q)}(\beta)=0$ for any two positive integers p and q , Definition 3 gives for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-1]}\alpha^{-1}\beta(r)}{\left(\log^{[q-1]}r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}}<\varepsilon\ .$$

Then as before we obtain that $\int\limits_{r_0}^{\infty}\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}}dr\ (r_0>0)$ converges for $k>0$ and diverges for $k<0$.

Thus combining Subcase (C) and Subcase (D), Definition 4 follows.

Definition 4 $\Rightarrow$ Definition 3.

From Definition 4 and for arbitrary positive ε the integral

$$\int\limits_{r_0}^{\infty}\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\tau_{\alpha}^{(p,q)}(\beta)+\varepsilon+1}}dr\ (r_0>0)$$

converges. Then by Lemma 1, we get

$$\liminf_{r\to\infty}\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\tau_{\alpha}^{(p,q)}(\beta)+\varepsilon}}=0\ .$$

So we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\tau_{\alpha}^{(p,q)}(\beta)+\varepsilon}}<\varepsilon$$

$$i.e.,\ \log^{[p-2]}\alpha^{-1}\beta(r)<\varepsilon\cdot\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{\tau_{\alpha}^{(p,q)}(\beta)+\varepsilon}$$

$$i.e.,\ \log^{[p-1]}\alpha^{-1}\beta(r)<\log\varepsilon+\left(\tau_{\alpha}^{(p,q)}(\beta)+\varepsilon\right)\left(\log^{[q-1]}r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}$$

$$i.e.,\ \liminf_{r\to\infty}\frac{\log^{[p-1]}\alpha^{-1}\beta(r)}{\left(\log^{[q-1]}r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}}\leq\tau_{\alpha}^{(p,q)}(\beta)+\varepsilon\ .$$

Since $\varepsilon>0$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}\alpha^{-1}\beta(r)}{\left(\log^{[q-1]}r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}}\leq\tau_{\alpha}^{(p,q)}(\beta)\ . \tag{9}$$

On the other hand the divergence of the integral

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_\alpha^{(p,q)}(\beta)}\right)\right]^{\tau_\alpha^{(p,q)}(\beta)-\varepsilon+1}} dr \; (r_0 > 0)$$

implies for all sufficiently large values of r that

$$\frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_\alpha^{(p,q)}(\beta)}\right)\right]^{\tau_\alpha^{(p,q)}(\beta)-\varepsilon+1}} > \frac{1}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_\alpha^{(p,q)}(\beta)}\right)\right]^{1+\varepsilon}}$$

$$i.e., \; \log^{[p-2]} \alpha^{-1}\beta(r) > \left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_\alpha^{(p,q)}(\beta)}\right)\right]^{\tau_\alpha^{(p,q)}(\beta)-2\varepsilon}$$

$$i.e., \; \log^{[p-1]} \alpha^{-1}\beta(r) > \left(\tau_\alpha^{(p,q)}(\beta) - 2\varepsilon\right)\left(\left(\log^{[q-1]} r\right)^{\lambda_\alpha^{(p,q)}(\beta)}\right)$$

$$i.e., \; \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\lambda_\alpha^{(p,q)}(\beta)}} > \left(\tau_\alpha^{(p,q)}(\beta) - 2\varepsilon\right) .$$

As $\varepsilon > 0$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\lambda_\alpha^{(p,q)}(\beta)}} \geq \tau_\alpha^{(p,q)}(\beta) . \tag{10}$$

So from (9) and (10) we obtain

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\lambda_\alpha^{(p,q)}(\beta)}} = \tau_\alpha^{(p,q)}(\beta) .$$

This proves the theorem.

Next we introduce the following two relative growth indicators which will also help our subsequent study.

Definition 5. Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive relative (p,q)-th order $\rho_\alpha^{(p,q)}(\beta)$ $\left(a < \rho_\alpha^{(p,q)}(\beta) < \infty\right)$ where p and q are any two positive integers. Then the *relative (p,q)-th lower type* of $\beta(x)$ with respect to $\alpha(x)$ is defined as :

$$\overline{\sigma}_\alpha^{(p,q)}(\beta) = \liminf_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_\alpha^{(p,q)}(\beta)}} .$$

The above definition can alternatively be defined in the following manner:

Definition 6. Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive *relative (p,q)-th order* $\rho_\alpha^{(p,q)}(\beta)$ $\left(a < \rho_\alpha^{(p,q)}(\beta) < \infty\right)$ where p and q are any two positive integers. Then the *relative (p,q) -th lower type* $\overline{\sigma}_\alpha^{(p,q)}(\beta)$ of $\beta(x)$ with respect to $\alpha(x)$ is defined as: The integral $\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_\alpha^{(p,q)}(\beta)}\right)\right]^{k+1}} dr \; (r_0 > 0)$ converges for $k > \overline{\sigma}_\alpha^{(p,q)}(\beta)$ and diverges for $k < \overline{\sigma}_\alpha^{(p,q)}(\beta)$.

Definition 7. Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive relative (p,q)*-th lower order* $\lambda_\alpha^{(p,q)}(\beta)$ $\left(a < \lambda_\alpha^{(p,q)}(\beta) < \infty\right)$. Then the growth indicator $\overline{\tau}_\alpha^{(p,q)}(\beta)$ of $\beta(x)$ with respect to $\alpha(x)$ is defined as :

$$\overline{\tau}_\alpha^{(p,q)}(\beta) = \limsup_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\lambda_\alpha^{(p,q)}(\beta)}} .$$

The above definition can also alternatively defined as:

Definition 8. Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive *relative* (p,q) *-th lower order* $\lambda_\alpha^{(p,q)}(\beta)$ $\left(a < \lambda_\alpha^{(p,q)}(\beta) < \infty\right)$ where p and q are any two positive integers. Then the growth indicator $\overline{\tau}_\alpha^{(p,q)}(\beta)$ of $\beta(x)$ with respect to $\alpha(x)$ is defined as: The integral $\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_\alpha^{(p,q)}(\beta)}\right)\right]^{k+1}} dr \; (r_0 > 0)$ converges for $k > \overline{\tau}_g^{(p,q)}(f)$ and diverges for $k < \overline{\tau}_g^{(p,q)}(f)$.

Now we state the following two theorems without their proofs as those can easily be carried out with help of Lemma 1 and in the line of Theorem 1 and Theorem 2 respectively.

Theorem 3. Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive *relative* (p,q) *-th order* $\rho_\alpha^{(p,q)}(\beta)$ $\left(0 < \rho_\alpha^{(p,q)}(\beta) < \infty\right)$ and *relative* (p,q) *-th lower type* $\overline{\sigma}_\alpha^{(p,q)}(\beta)$ where p and q are any two positive integers. Then Definition 5 and Definition 6 are equivalent.

Theorem 4. $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions having finite positive *relative* (p,q) *-th lower order* $\lambda_\alpha^{(p,q)}(\beta)$ $\left(0 < \lambda_\alpha^{(p,q)}(\beta) < \infty\right)$ and the growth indicator $\overline{\tau}_\alpha^{(p,q)}(\beta)$ where p and q are any two positive integers. Then Definition 7 and Definition 8 are equivalent.

Theorem 5. $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0, +\infty)$ functions with $0 < \lambda_\alpha^{(p,q)}(\beta) \leq \rho_\alpha^{(p,q)}(\beta) < \infty$ where p and q are any two positive integers. Then

$$(i) \;\; \sigma_\alpha^{(p,q)}(\beta) = \limsup_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}(r)}{\left[\log^{[q-1]} \beta^{-1}(r)\right]^{\rho_\alpha^{(p,q)}(\beta)}},$$

$$(ii) \;\; \overline{\sigma}_\alpha^{(p,q)}(\beta) = \liminf_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}(r)}{\left[\log^{[q-1]} \beta^{-1}(r)\right]^{\rho_\alpha^{(p,q)}(\beta)}},$$

$$(iii) \;\; \tau_\alpha^{(p,q)}(\beta) = \liminf_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}(r)}{\left[\log^{[q-1]} \beta^{-1}(r)\right]^{\lambda_\alpha^{(p,q)}(\beta)}}$$

and

$$(iv)\ \ \overline{\tau}_{\alpha}^{(p,q)}(\beta)=\limsup_{r\to\infty}\frac{\log^{[p-1]}\alpha^{-1}(r)}{\left[\log^{[q-1]}\beta^{-1}(r)\right]^{\lambda_{\alpha}^{(p,q)}(\beta)}}\ .$$

Proof. Taking $\beta(r)=R$, theorem follows from the definitions of $\sigma_{\alpha}^{(p,q)}(\beta)$, $\overline{\sigma}_{\alpha}^{(p,q)}(\beta)$, $\tau_{\alpha}^{(p,q)}(\beta)$ and $\overline{\tau}_{\alpha}^{(p,q)}(\beta)$ respectively.

In the following theorem we obtain a relationship between $\sigma_{\alpha}^{(p,q)}(\beta)$, $\overline{\sigma}_{\alpha}^{(p,q)}(\beta)$, $\overline{\tau}_{\alpha}^{(p,q)}(\beta)$ and $\tau_{\alpha}^{(p,q)}(\beta)$.

Theorem 6. Let $\alpha(x)$ and $\beta(x)$ be any two positive continuous increasing to $+\infty$ on $[x_0,+\infty)$ functions such $\rho_{\alpha}^{(p,q)}(\beta)=\lambda_{\alpha}^{(p,q)}(\beta)\left(0<\lambda_{\alpha}^{(p,q)}(\beta)=\rho_{\alpha}^{(p,q)}(\beta)<\infty\right)$ where p and q are any two positive integers, then the following quantities

$$(i)\ \sigma_{\alpha}^{(p,q)}(\beta),\ \ (ii)\ \tau_{\alpha}^{(p,q)}(\beta),\ \ (iii)\ \overline{\sigma}_{\alpha}^{(p,q)}(\beta)\ \text{ and }\ \ (iv)\ \overline{\tau}_{\alpha}^{(p,q)}(\beta)$$

are all equivalent.

Proof. From Definition 4, it follows that the integral

$$\int_{r_0}^{\infty}\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}}dr\ (r_0>0)$$

converges for $k>\tau_{\alpha}^{(p,q)}(\beta)$ and diverges for $k<\tau_{\alpha}^{(p,q)}(\beta)$. On the other hand, Definition 2 implies that the integral $\int_{r_0}^{\infty}\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}}dr\ (r_0>0)$ converges for $k>\sigma_{\alpha}^{(p,q)}(\beta)$ and diverges for $k<\sigma_{\alpha}^{(p,q)}(\beta)$.

$(i)\Rightarrow(ii)$.

Now it is obvious that all the quantities in the expression

$$\left[\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}}-\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}}\right]$$

are of non negative type. So

$$\int_{r_0}^{\infty}\left[\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}}-\frac{\log^{[p-2]}\alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]}r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}\right)\right]^{k+1}}\right]dr\ (r_0>0)\geq 0$$

$$i.e., \int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\lambda_\alpha^{(p,q)}(\beta)}\right)\right]^{k+1}} dr \geq$$

$$\int_{r_0}^{\infty} \frac{\log^{[p-2]} \alpha^{-1}\beta(r)}{\left[\exp\left(\left(\log^{[q-1]} r\right)^{\rho_\alpha^{(p,q)}(\beta)}\right)\right]^{k+1}} dr \text{ for } r_0 > 0 .$$

$$i.e., \ \tau_\alpha^{(p,q)}(\beta) \geq \sigma_\alpha^{(p,q)}(\beta) . \tag{11}$$

Further as $\rho_\alpha^{(p,q)}(\beta) = \lambda_\alpha^{(p,q)}(\beta)$, therefore we get that

$$\begin{aligned} \sigma_\alpha^{(p,q)}(\beta) = & \limsup_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_\alpha^{(p,q)}(\beta)}} \\ \geq & \liminf_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_\alpha^{(p,q)}(\beta)}} = \liminf_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\lambda_\alpha^{(p,q)}(\beta)}} = \tau_\alpha^{(p,q)}(\beta) . \end{aligned} \tag{12}$$

Hence from (11) and (12) we obtain

$$\sigma_\alpha^{(p,q)}(\beta) = \tau_\alpha^{(p,q)}(\beta) . \tag{13}$$

$(ii) \Rightarrow (iii)$.

Since $\rho_\alpha^{(p,q)}(\beta) = \lambda_\alpha^{(p,q)}(\beta)$, we get

$$\tau_\alpha^{(p,q)}(\beta) = \liminf_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\lambda_\alpha^{(p,q)}(\beta)}} = \liminf_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_\alpha^{(p,q)}(\beta)}} = \overline{\sigma}_\alpha^{(p,q)}(\beta) .$$

$(iii) \Rightarrow (iv)$.

In view of (13) and the condition $\rho_\alpha^{(p,q)}(\beta) = \lambda_\alpha^{(p,q)}(\beta)$, it follows that

$$\overline{\sigma}_\alpha^{(p,q)}(\beta) = \liminf_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_\alpha^{(p,q)}(\beta)}}$$

$$i.e., \ \overline{\sigma}_\alpha^{(p,q)}(\beta) = \liminf_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\lambda_\alpha^{(p,q)}(\beta)}}$$

$$i.e., \ \overline{\sigma}_\alpha^{(p,q)}(\beta) = \tau_\alpha^{(p,q)}(\beta)$$

$$i.e., \ \overline{\sigma}_\alpha^{(p,q)}(\beta) = \sigma_\alpha^{(p,q)}(\beta)$$

$$i.e.,\ \overline{\sigma}_{\alpha}^{(p,q)}(\beta) = \limsup_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}}$$

$$i.e.,\ \overline{\sigma}_{\alpha}^{(p,q)}(\beta) = \limsup_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}}$$

$$i.e.,\ \overline{\sigma}_{\alpha}^{(p,q)}(\beta) = \overline{\tau}_{\alpha}^{(p,q)}(\beta)\ .$$

$(iv) \Rightarrow (i)$.

As $\rho_{\alpha}^{(p,q)}(\beta) = \lambda_{\alpha}^{(p,q)}(\beta)$, we obtain that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\lambda_{\alpha}^{(p,q)}(\beta)}} = \limsup_{r\to\infty} \frac{\log^{[p-1]} \alpha^{-1}\beta(r)}{\left(\log^{[q-1]} r\right)^{\rho_{\alpha}^{(p,q)}(\beta)}} = \sigma_{\alpha}^{(p,q)}(\beta)\ .$$

Thus the theorem follows.

Remark 1. If we consider $\alpha(x) = M_g(x)$ and $\beta(x) = M_f(x)$ where f and g are any two entire functions with index-pairs (m, q) and (m, p) respectively where p, q, m are positive integers such that $m \geq \max(p, q)$, then the above results reduces for the relative (p, q)-th growth indicators such as relative (p, q)-th type, relative (p, q)-th weak type etc. of an entire function f with respect to another entire function g.

Remark 2. If we take $\alpha(x) = T_g(x)$ and $\beta(x) = T_f(x)$ where f be a meromorphic function and g be any entire function with index-pairs (m, q) and (m, p) respectively where p, q, m are positive integers such that $m \geq \max(p, q)$, then the above theorems reduces for relative (p, q)-th growth indicators such as relative (p, q)-th type, relative (p, q)-th weak type etc. of a meromorphic function f with respect to an entire function g.

Acknowledgment

The author is extremely grateful to the anonymous learned referee for his keen reading, valuable suggestion and constructive comments for the improvement of the paper.

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